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COHOMOLOGY OF BIANCHI GROUPS

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1. Introduction

This is a report on the paper titled "On the Dimension of the Cohomology of Bianchi Groups" by Mehmet H. Şengün and Seyfi Türkelli [10].

Bianchi groups are groups of the form $\mathrm{SL}_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of an imaginary quadratic field. Just as the cohomology of the classical modular group $\mathrm{SL}_2(\mathbb{Z})$ is central to the theory of classical modular forms, the cohomology of Bianchi groups is central to the study of Bianchi modular forms, that is, modular forms over imaginary quadratic fields.

Understanding the behavior of the dimension of the cohomology of Bianchi groups and their congruence subgroups is a long open problem. Up to date, there is no explicit dimension formula of any sort. Utilizing the compactification theory of Borel-Serre (which basically amounts to closing the cusps of the 3-folds associated to Bianchi groups with 2-tori), we can decompose the cohomology into two parts: the cuspidal part and the Eisenstein part. While it is easy to compute the dimension of the Eisenstein part, understanding the dimension of the cuspidal part is very hard.

In 1984 Rohlfs, developing an idea that goes back to Harder (see the end of [4]), provided in [7] an explicit lower bound for (the cuspidal part of) the first cohomology with trivial complex coefficients of Bianchi groups. Around the same time, Krämer, mainly using techniques developed by Rohlfs, made these lower bounds sharper. In their recent paper [3], Finis, Grunewald and Tirao provided explicit lower bounds for the cuspidal part of the first cohomology with certain non-trivial coefficient systems of Bianchi groups.

There has been significant recent developments in understanding the behavior of the dimension asymptotically. In [2] Calegari and Emerton, using techniques from non-commutative Iwasawa theory, provided asymptotic upper-bounds for the first cohomology, with a fixed coefficient system, as one goes down in a tower of principal congruence subgroups of prime-power level of a fixed Bianchi group. In a complementary direction, Marshall proved in [5], using the approach of Calegari and Emerton, an asymptotic upper-bound for the first cohomology of a congruence subgroup of a Bianchi group as the coefficient system varies.

The authors' goal in this paper is two-fold: first, they give an exposition of the techniques developed by Rohlfs and later generalized by Krämer and Blume-Nienhaus in their Bonn PhD theses, second they provide asymptotic lower-bounds for the cohomology of congruence subgroups of Bianchi groups using these techniques.

1.1. Summary of Results.

Fix a square-free negative integer $d \neq -1, -3$, let K be the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ with class number h and ring of integers $\mathcal{O}$. Let G be the associated Bianchi group $\mathrm{SL}_2(\mathcal{O})$ and Γ be a finite index subgroup of G. Given a nonnegative integer k, let E_k be the space of homogeneous polynomials over $\mathbb{C}$ in two variables of degree

k with the following Γ-action: given a polynomial $p(x,y) \in E_k$,

$$p(x,y) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = p(ax+by, cx+dy).$$

Let $E_{k,k} := E_k \otimes_{\mathbb{C}} \overline{E_k}$ is a Γ-module where the action of Γ on the second component is twisted by the conjugation.

The group G acts discontinuously as isometries on the hyperbolic 3-space $\mathbb{H} \simeq \mathbb{C} \times \mathbb{R}^+$ and the quotient $Y_\Gamma := \Gamma \backslash \mathbb{H}$ has the structure of an hyperbolic 3-fold. Let $\mathcal{E}$ be the local system on Y_Γ induced by some complex finite-dimensional Γ-representation E. It is well known that Y_Γ is an Eilenberg-MacLane space for Γ and so

$$H^n(\Gamma, E) \cong H^n(Y_\Gamma, \mathcal{E}).$$

Let $\sigma \in G(K/\mathbb{Q})$ be the only nontrivial element; that is, the complex conjugation. Suppose that σ acts on E and Γ in a compatible way so that it induces an action on the cohomology $H^i(\Gamma, E)$. Since σ is an involution, the eigenvalues of this action is ± 1 and so the trace $\mathrm{tr}(\sigma \mid H^i(\Gamma, E))$ is an integer.

One defines the *Lefschetz number* of σ as the following integer

$$L(\sigma, \Gamma, E) = \sum_i (-1)^i \mathrm{tr}(\sigma \mid H^i(\Gamma, E)).$$

These Lefschetz numbers were first considered by Harder in [4] where he computed them to give lower bounds for the cohomology of certain types of principal congruence subgroups Γ with $E = \mathbb{C}$. In his 1976 Bonn Habilitation Rohlfs developed tools to compute these Lefschetz numbers for general arithmetic groups. In 1984, Rohlfs used these tools to provide lower bounds for the Lefschetz number for the case $\Gamma = \mathrm{SL}_2(\mathcal{O})$ and $E = \mathbb{C}$. Later that year, in his Bonn Ph.D. thesis, Krämer gave a closed formula for the Lefschetz number for the same case. The following is a generalization of their results to higher level and weight.

Proposition 1.1. [10, Proposition 1.3] *Let $N > 2$ be a positive integer*

$$L(\sigma, \Gamma(N), E_{k,k}) = \begin{cases} (A+2B)\dfrac{-N^3}{12}\displaystyle\prod_{p \mid N}(1-p^{-2}) \cdot (k+1) & \textit{if } N \textit{ is even} \\[2ex] (A+3B)\dfrac{-N^3}{12}\displaystyle\prod_{p \mid N}(1-p^{-2}) \cdot (k+1) & \textit{if } N \textit{ is odd.} \end{cases}$$

where A, B are explicit constants depending on the ramification data of $K/\mathbb{Q}$.

The constants A and B are in fact certain powers of 2 and they were computed by Rohlfs in [6]. These constants vary depending on the ramification data of the imaginary quadratic field K and the integer N.

Corollary 1.2. [10, Corollary 3.5] *Let p be an odd rational prime that is unramified in $K = \mathbb{Q}(\sqrt{-d})$ and t be the number of distinct prime divisors of the discriminant of $K/\mathbb{Q}$. Then, for $n > 0$*

$$L(\sigma, \Gamma(p^n), E_{k,k}) = \begin{cases} -2^t \cdot \dfrac{p^{3n} - p^{3n-2}}{12} \cdot (k+1) & \textit{if } d \equiv 1 \mod 4 \\[2ex] -5 \cdot 2^{t-1} \cdot \dfrac{p^{3n} - p^{3n-2}}{12} \cdot (k+1) & \textit{else.} \end{cases}$$

The assumption in Proposition 1.1 that $N > 2$ is important. Because, in this case $\Gamma(N)$ has no torsion and so they can use a Lefschetz fixed theorem, due to Rohlfs, to calculate the Lefschetz number in question.

When the arithmetic group Γ has torsion, for example $\Gamma = \mathrm{SL}_2(\mathcal{O})$, there is a generalized Lefschetz fixed point theorem, due to Blume-Nienhaus, which is more involved because of the "contributions" of the torsion elements. Nevertheless, the tools they need to calculate the Lefschetz number for $\Gamma = \mathrm{SL}_2(\mathcal{O})$ are laid out by Rohlfs and also by Blume-Nienhaus. One of the main results of the paper is this following:

Theorem 1.3. [10, Theorem 3.2] *Let D be the discriminant of $K/\mathbb{Q}$ with D_2 its 2-part. Let ρ represent either τ or σ. Also, put $q = 1$ or $q = -1$ depending on whether $\rho = \tau$ or $\rho = \sigma$, respectively.*

$$\begin{aligned}(-1)^k L(\rho, \Gamma, E_{k,k}) &= \tfrac{-q}{12} \prod_{\substack{p\mid D\\ p\neq 2}} \left(p + \left(\tfrac{q}{p}\right)\right) \prod_{\substack{p\mid D\\ p=2}} (D_2 + (q|2)) \cdot (k+1) \\ &+ \tfrac{q}{12} \prod_{\substack{p\mid D\\ p\neq 2}} \left(1 + \left(\tfrac{-q}{p}\right)\right) \prod_{\substack{p\mid D\\ p=2}} (4 + (-q|2)) \cdot (-1)^k (k+1) \\ &+ \tfrac{1}{2} \prod_{\substack{p\mid D\\ p\neq 2}} \left(1 + \left(\tfrac{-2q}{p}\right)\right) \cdot \left(\tfrac{k+1}{4}\right) \\ &+ \tfrac{1}{3} \left(\prod_{\substack{p\mid D\\ p\neq 3}} (1 + (-3q|p)) + (-1)^k \prod_{p\mid D} (1 + (-q|p)) \right) \cdot \left(\tfrac{k+1}{3}\right).\end{aligned}$$

Here products over empty sets are understood to be equal to 1.

Following Harder, the authors use the trace of the involution σ on $H^i(\Gamma, E)$ to bound the dimension of this cohomology space from below. In order to carry this idea out, one needs to calculate the trace of σ on the Eisenstein part of the cohomology as well. The following theorem generalizes a part of the results announced by Harder at the very end of [4].

Theorem 1.4. [10, Theorem 1.2] *Let t be the number of distinct prime divisors of the discriminant of $K/\mathbb{Q}$. Let $N = p_1^{n_1} \dots p_r^{n_r}$ be a positive integer whose prime divisors p_i are unramified in K and let $\Gamma = \Gamma(N)$ be the associated principal congruence subgroup of the Bianchi group $\mathrm{SL}_2(\mathcal{O})$.*

We have

$$\mathrm{tr}(\sigma \mid H^2_{Eis}(\Gamma, E_{k,k})) = -2^{t-1} \cdot \prod_{i=1}^{r} (p_i^{2n_i} - p_i^{2(n_i-1)}) + \delta(0,k),$$

where δ is the Kronecker δ-function, in other words, $\delta(0,k) = 0$ unless $k = 0$ in which case $\delta(0,k) = 1$. In particular,

$$\mathrm{tr}(\sigma \mid H^2_{Eis}(\mathrm{SL}_2(\mathcal{O}), E_{k,k})) = -2^{t-1} + \delta(0,k).$$

Computing the trace on the Eisenstein part of the first cohomology is more challenging as reported by Harder in [4]. He does not provide a proof but informs us that he uses the adelic setting and representation theoretic approach for his computations and his final result depends on certain factors in the functional equation of associated Hecke L-series. We provide a partial generalization of Harder's result, using an elementary approach which employs the cocycles of Sczech, see [9] which are defined by means of certain elliptic analogues of classical Dedekind sums.

Theorem 1.5. [10, Theorem 1.1] *Assume that K is of class number one and let p be a rational prime that is inert in K. Then,*

$$\operatorname{tr}(\sigma \mid H^1_{Eis}(\Gamma(p^n),\mathbb{C})) = \begin{cases} -(p^2+1), & \text{if } n=1 \\ -(p^{2n}-p^{2n-2}), & \text{if } n>1. \end{cases}$$

The authors hope to generalize this result to higher class numbers in the near future. Our results so far allow us to get explicit lower bounds for the cuspidal cohomology of certain principal congruence subgroups that are stablized by complex conjugation. These explicit lower bounds yield the following asymptotic bounds. For a related result, see the article [8] of Rohlfs and Speh.

Corollary 1.6. [10, Corollary 1.4] *Let p be a rational prime that is unramified in K and let $\Gamma(p^n)$ denote the principal congruence subgroup of level (p^n) of a Bianchi group $\mathrm{SL}_2(\mathcal{O})$. Then, as k increases and n is fixed*

$$\dim H^1_{cusp}(\Gamma(p^n), E_{k,k}) \gg k$$

where the implicit constant depends on the level $\Gamma(p^n)$ and the field K. Assume further that K is of class number one and that p is inert in K. Then, as n increases

$$\dim H^1_{cusp}(\Gamma(p^n), \mathbb{C}) \gg p^{3n}$$

where the implicit constant depends on the field K.

They also investigate the Lefschetz numbers and the Eisenstein traces for the involution given by the $\mathrm{GL}_2/\mathrm{SL}_2$-twist of complex conjugation. The results, when combined with those about complex conjugation, give a closed formula for the trace of σ on the first cohomology of $\mathrm{GL}_2(\mathcal{O})$. This implies the following asymptotics for the cohomology of $\mathrm{GL}_2(\mathcal{O})$.

Corollary 1.7. [10, Corollary 1.5] *Let D be the discriminant of $K/\mathbb{Q}$ and $\mathcal{O}_K$ be its ring of integers. As $K/\mathbb{Q}$ is fixed and $k \to \infty$, we have*

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}_K), E_{k,k}) \gg k$$

where the implicit constant depends on the discriminant D. As k is fixed and $|D| \to \infty$, we have

$$\dim H^1(\mathrm{GL}_2(\mathcal{O}_K), E_{k,k}) \gg \varphi(D)$$

where φ is the Euler φ function and the implicit constant depends on the weight k.

As $H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k})$ embeds into $H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), E_{k,k})$, the asymptotic lower bound as $|D| \to \infty$ of the above corollary also applies to $H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), E_{k,k})$. Rohlfs showed in [7] that $H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), \mathbb{C}) \gg \varphi(D)$ as $|D| \to \infty$, yielding the same asymptotic as ours. The results for $E_{k,k} = \mathbb{C}$ of the above Corollary can also be derived from the main results of thesis of Krämer. Note that Krämer also produces the upper bound

$$\dim H^1_{cusp}(\mathrm{SL}_2(\mathcal{O}), \mathbb{C}) \ll |D|^{3/2}.$$

Finally, let $H^1_{bc}(\mathrm{GL}_2(\mathcal{O}), E_{k,k})$ denote the subspace of $H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k})$ which corresponds to those cuspidal Bianchi modular forms which arise from classical cuspidal modular forms via base-change or arise from a quadratic extension of K via automorphic induction (see [3] for these notions). Using the results above, they give a "computational criteria" determining when the space $H^1(\mathrm{GL}_2(\mathcal{O}), E_{k,k})$ is exhausted by base-change classes, and, as an application, they get the following result, which is not included in [10].

Theorem 1.8. *Let K be an imaginary quadratic field with ring of integers $\mathcal{O}$ and discriminant ≥ -260. Then*

$$H^1(\mathrm{GL}_2(\mathcal{O}), \mathbb{C}) = H^1_{bc}(\mathrm{GL}_2(\mathcal{O}), \mathbb{C}).$$

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