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# Subnormal Toeplitz operators: A brief survey and open problems

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**Abstract**

In this note we attempt to set forth some of the recent developments that had taken place in subnormal Toeplitz operator theory. Moreover, we present some unsolved problems for the subnormality of Toeplitz operators.

## 1 Halmos's Problem 5: Subnormal Toeplitz operators

Throughout this note, let $\mathcal{H}$ denote a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators acting on $\mathcal{H}$. For an operator $T\in\mathcal{B}(\mathcal{H})$, $T^*$ denotes the adjoint of $T$. An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be *normal* if $T^*T=TT^*$, *hyponormal* if its self-commutator $[T^*,T]\equiv T^*T-TT^*$ is positive semi-definite. An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be *pure* if it has no nonzero reducing subspace on which it is normal. An operator $T\in\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N$ on $\mathcal{K}$ such that $N\mathcal{H}\subseteq\mathcal{H}$ and $T=N|_{\mathcal{H}}$. In this case, $N$ is called a *normal extension* of $T$. In general, it is quite difficult to examine whether such a normal extension exists for an operator. Of course, there are a couple of constructive methods for determining subnormality; one of them is the Bram-Halmos criterion of subnormality ([2], [4]), which states that an operator $T\in\mathcal{B}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j}(T^ix_j,T^jx_i)\geq 0$ for all finite collections $x_0,x_1,\cdots,x_k\in\mathcal{H}$. It is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k}\\ T & T^*T & \dots & T^{*k}T\\ \vdots & \vdots & \ddots & \vdots\\ T^k & T^*T^k & \dots & T^{*k}T^k\end{pmatrix}\geq 0 \quad (\text{all } k\geq 1). \tag{1}$$

Thus the Bram-Halmos criterion can be stated as follows: $T$ is subnormal if and only if the positivity condition (1) holds for all $k\geq 1$. The positivity condition (1) provides a measure of the gap between hyponormality and subnormality. In fact, condition (1) for $k=1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (1) for all $k\geq 1$. Recall ([cf. [11]) that for $k\geq 1$, an operator $T\in\mathcal{B}(\mathcal{H})$ is said to be *$k$-hyponormal* if $T$ satisfies the positivity condition (1) for a fixed $k$. Thus the Bram-Halmos criterion can be stated as: $T$ is subnormal if and only if $T$ is $k$-hyponormal for all $k\geq 1$.

The present note concerns the question: *Which Toeplitz operators are subnormal?* A Toeplitz operator $T_\varphi$ (with symbol $\varphi\in L^\infty\equiv L^\infty(\mathbb{T})$) is defined by the expression $T_\varphi f:=P(\varphi f)$ for each $f\in H^2\equiv H^2(\mathbb{T})$, where $P$ is the orthogonal projection from $L^2\equiv L^2(\mathbb{T})$ onto $H^2$. A Toeplitz operator $T_\varphi$ is called *analytic* if $\varphi\in H^\infty\equiv L^\infty\cap H^2$. Any analytic Toeplitz operator is easily seen to be subnormal: indeed, $M_\varphi$ is a normal extension of $T_\varphi$, where $M_\varphi$ is the normal operator of multiplication by $\varphi$ on $L^2$. P.R. Halmos raised the following problem, so-called the *Halmos's Problem 5* in his 1970 lectures "Ten Problems in Hilbert Space" [15], [16]:

Is every subnormal Toeplitz operator either normal or analytic ?

The question is natural because the two classes, the normal and analytic Toeplitz operators, are fairly well understood and are obviously subnormal. We begin with a brief survey of research related to P.R. Halmos's Problem 5.

In 1988, the hyponormality of Toeplitz operators $T_\varphi$ was completely characterized in terms of their symbols $\varphi$ via an elegant theorem of C. Cowen [6]. Cowen's method is to recast the operator-theoretic problem of hyponormality for Toeplitz operators into the problem of finding a solution with specified properties to a certain functional equation involving the symbol $\varphi$. Today, this theorem is referred as *Cowen's Theorem.*

**Cowen's Theorem** ([6], [17]). For $\varphi \in L^\infty$, write

$$\mathcal{E}(\varphi) := \left\{k \in H^\infty : \ ||k||_\infty \leq 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\right\}.$$

Then $T_\varphi$ is hyponormal if and only if $\mathcal{E}(\varphi)$ is nonempty.

A function $\varphi \in L^\infty$ is said to be of bounded type if there are bounded analytic functions $\psi_1, \psi_2 \in H^\infty$ such that $\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)}$ for almost all $z \in \mathbb{T}$. Evidently, rational functions are of bounded type. In 1976, M.B. Abrahamse showed that the answer is affirmative for Toeplitz operators with bounded type symbols ([1]):

**Theorem 1.1** (Abrahamse's Theorem). If

(i) $T_\varphi$ is hyponormal;

(ii) $\varphi$ or $\overline{\varphi}$ is of bounded type;

(iii) $\ker[T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$,

then $T_\varphi$ is normal or analytic.

*Proof.* See [Ab]. □

On the other hand, observe that if $S$ is a subnormal operator on $\mathcal{H}$ and if $N$ is the minimal normal extension of $S$ then

$$\ker[S^*, S] = \{f : < f, [S^*, S]f >= 0\} = \{f : \ ||S^*f|| = ||Sf||\} = \{f : \ N^*f \in \mathcal{H}\}.$$

Therefore, $S(\ker[S^*, S]) \subseteq \ker[S^*, S]$.

By Theorem 1.1 and the preceding remark we get:

**Corollary 1.2.** If $T_\varphi$ is subnormal and if $\varphi$ or $\overline{\varphi}$ is of bounded type, then $T_\varphi$ is normal or analytic.

The following lemma gives a criterion for a function to be of bounded type.

**Lemma 1.3.** [1] A function $\varphi$ is of bounded type if and only if $\ker H_\varphi \neq \{0\}$.

From Theorem 1.1 we can see that

$$\varphi = \frac{\psi}{\theta} \ (\theta, \psi \text{ inner}), \ T_\varphi \text{ subnormal } \Rightarrow \ T_\varphi \text{ normal or analytic} \tag{2}$$

The following proposition strengthen the conclusion of (2), whereas weakens the hypothesis of (2).

**Proposition 1.4.** [1] If $\varphi = \frac{\psi}{\theta}$ $(\theta, \psi$ inner) and if $T_\varphi$ is hyponormal, then $T_\varphi$ is analytic.

Thus we have:

**Proposition 1.5.** [1] If $A$ is a weighted shift with weights $a_0, a_1, a_2, \cdots$ such that

$$0 \leq a_0 \leq a_1 \leq \cdots < a_N = a_{N+1} = \cdots = 1,$$

then $A$ is not unitarily equivalent to any Toeplitz operator.

*Proof.* Note that $A$ is hyponormal, $||A|| = 1$ and $A$ attains its norm. If $A$ is unitarily equivalent to $T_\varphi$ then by a result of Brown and Douglas [3], $T_\varphi$ is hyponormal and $\varphi = \frac{\psi}{\theta}$ ($\theta, \psi$ inner). By Proposition 1.4, $T_\varphi \equiv T_\psi$ is an isometry, so $a_0 = 1$, a contradiction. $\square$

Recall that the Bergman shift (whose weights are given by $\sqrt{\frac{n+1}{n+2}}$ ) is subnormal. The following question arises naturally:

$$\text{Is the Bergman shift unitarily equivalent to a Toeplitz operator?} \tag{3}$$

An affirmative answer to the question (3) gives a negative answer to Halmos's Problem 5. To see this, assume that the Bergman shift $S$ is unitarily equivalent to $T_\varphi$, then

$$\mathfrak{R}(\varphi) \subseteq \sigma_e(T_\varphi) = \sigma_e(S) = \text{the unit circle } \mathbb{T}$$

(where $\mathfrak{R}(\cdot)$ denotes the essential range and $\sigma_e(\cdot)$ denotes the essential spectrum). Thus $\varphi$ is unimodular. Since $S$ is not an isometry it follows that $\varphi$ is not inner. Therefore $T_\varphi$ is not an analytic Toeplitz operator.

**Theorem 1.6** (Sun's Theorem). [18] Let $T$ be a weighted shift with a strictly increasing weight sequence $\{a_n\}_{n=0}^\infty$. If $T$ is unitarily equivalent to $T_\varphi$ then

$$a_n = \sqrt{1-\alpha^{2n+2}}\,||T_\varphi|| \quad (0 < \alpha < 1).$$

**Corollary 1.7.** [18] The Bergman shift is not unitarily equivalent to any Toeplitz operator.

*Proof.* $\frac{n+1}{n+2} \neq 1 - \alpha^{2n+2}$ for any $\alpha > 0$. $\square$

**Lemma 1.8.** [7] The weighted shift $T \equiv W_\alpha$ with weights $\alpha_n \equiv (1-\alpha^{2n+2})^{\frac{1}{2}}$ $(0 < \alpha < 1)$ is subnormal.

*Proof.* Write $r_n := \alpha_0^2\alpha_1^2 \cdots \alpha_{n-1}^2$ for the moment of $W$. Define a discrete measure $\mu$ on $[0,1]$ by

$$\mu(z) = \begin{cases} \Pi_{j=1}^\infty (1-\alpha^{2j}) & (z=0) \\ \Pi_{j=1}^\infty (1-\alpha^{2j}) \frac{\alpha^{2k}}{(1-\alpha^2)\cdots(1-\alpha^{2k})} & (z = \alpha^k; k = 1, 2, \cdots). \end{cases}$$

Then $r_n = \int_0^1 t^n d\mu$. By Berger's theorem, $T$ is subnormal. $\square$

By Theorem 1.6 and Lemma 1.8, we have:

**Corollary 1.9.** If $T_\varphi$ is unitarily equivalent to a weighted shift, then $T_\varphi$ is subnormal.

**Remark 1.10.** [7] If $T_\varphi$ is unitarily equivalent to a weighted shift, what is the form of $\varphi$ ? A careful analysis of the proof of Theorem 1.6 shows that

$$\psi = \varphi - \alpha\overline{\varphi} \in \mathbf{H}^\infty.$$

But

$$\begin{aligned} T_\psi = T_\varphi - \alpha T_\varphi^* &= \begin{pmatrix} 0 & -\alpha a_0 & & & \\ a_0 & 0 & -\alpha a_1 & & \\ & a_1 & 0 & -\alpha a_2 & \\ & & a_2 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\alpha & & & \\ 1 & 0 & -\alpha & & \\ & 1 & 0 & -\alpha & \\ & & 1 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} + K \quad (K \text{ compact}) \\ &\cong T_{z-\alpha\overline{z}} + K \quad (\text{where } \cong \text{ denotes the unitary equivalence}). \end{aligned}$$

Thus $\operatorname{ran}(\psi) = \sigma_e(T_\psi) = \sigma_e(T_{z-\alpha\bar{z}}) = \operatorname{ran}(z - \alpha\bar{z})$. Thus $\psi$ is a conformal mapping of $\mathbb{D}$ onto the interior of the ellipse with vertices $\pm i(1+\alpha)$ and passing through $\pm(1-\alpha)$. On the other hand, $\psi = \varphi - \alpha\overline{\varphi}$. So $\alpha\overline{\psi} = \alpha\overline{\varphi} - \alpha^2\varphi$, which implies

$$\varphi = \frac{1}{1-\alpha^2}(\psi + \alpha\overline{\psi}).$$

We now have:

**Theorem 1.11** (Cowen and Long Theorem). [7] For $0 < \alpha < 1$, let $\psi$ be a conformal map of $\mathbb{D}$ onto the interior of the ellipse with vertices $\pm i(1-\alpha)^{-1}$ and passing through $\pm(1+\alpha)^{-1}$. Then $T_{\psi+\alpha\overline{\psi}}$ is a subnormal weighted shift that is neither analytic nor normal.

**Corollary 1.12.** [7] If $\varphi = \psi + \alpha\overline{\psi}$ is as in Theorem 1.11, then neither $\varphi$ nor $\overline{\varphi}$ is of bounded type.

*Proof.* From Abrahamse's theorem and Theorem 1.11. □

**Problem 1.**

(1) For which $f \in H^\infty$, is there $\lambda$ $(0 < \lambda < 1)$ with $T_{f+\lambda\overline{f}}$ subnormal ?

(2) If $\psi$ is a Riemann map between simply connected domains, does it follow that $T_{\psi+\alpha\,\bar{\psi}}$ is subnormal for some $\alpha$ with $0 < \alpha < 1$?

(3) Conversely, if $T_{\psi+\alpha\,\bar{\psi}}$ is subnormal for some $\alpha$ with $0 < \alpha < 1$, does it follow that $\psi$ is a Riemann map between simply connected domains?

**Problem 2.** Suppose $\psi$ is as in Theorem 1.11. Are there $g \in H^\infty$, $g \neq \lambda\psi + c$, such that $T_{\psi+\overline{g}}$ is subnormal ?

We conjecture that if $T_\varphi$ is non-normal subnormal then $\mathcal{E}(\varphi) = \{\lambda\}$ with $|\lambda| < 1$. However we were unable to decide whether or not it is true. By comparison, if $T_\varphi$ is normal then $\mathcal{E}(\varphi) = \{e^{i\theta}\}$.

**Problem 3.** If $T_\varphi$ is non-normal subnormal, does it follow that $\mathcal{E}(\varphi) = \{\lambda\}$ with $|\lambda| < 1$?

If the answer to Problem 4 is affirmative, i.e., the Cowen's remark is true then for $\varphi = \overline{g} + f$,

$$T_\varphi \text{ is subnormal } \Longrightarrow \overline{g} - \lambda\overline{f} \in H^2 \text{ with } |\lambda| < 1 \Longrightarrow g = \overline{\lambda}f + c \ (c \text{ a constant}),$$

which says that the answer to Problem 3 is negative.

When $\psi$ is as in Theorem 1.11, we examine the question: For which $\lambda$, is $T_{\psi+\lambda\psi}$ subnormal ? We then have:

**Theorem 1.13.** [5] Let $\lambda \in \mathbb{C}$ and $0 < \alpha < 1$. Let $\psi$ be the conformal map of the disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ passing through $\pm(1-\alpha)$. For $\varphi = \psi + \lambda\overline{\psi}$, $T_\varphi$ is subnormal if and only if $\lambda = \alpha$ or $\lambda = \frac{\alpha^k e^{i\theta} + \alpha}{1+\alpha^{k+1}e^{i\theta}}$ $(-\pi < \theta \leq \pi)$.

To prove Theorem 1.13, we need an auxiliary lemma:

**Proposition 1.14.** [6] Let $T$ be the weighted shift with weights

$$w_n^2 = \sum_{j=0}^{n} \alpha^{2j}.$$

Then $T + \mu T^*$ is subnormal if and only if $\mu = 0$ or $|\mu| = \alpha^k$ $(k = 0, 1, 2, \cdots)$.

*Proof.* See [CoL]. □

*Proof of Theorem 1.13.* By Theorem 1.11, $T_{\psi+\alpha\overline{\psi}} \cong (1-\alpha^2)^{\frac{3}{2}}T$, where $T$ is a weighted shift of Proposition 1.14. Thus $T_\psi \cong (1-\alpha^2)^{\frac{1}{2}}(T-\alpha T^*)$, so

$$T_\varphi = T_\psi + \lambda T_\psi^* \cong (1-\alpha^2)^{\frac{1}{2}}(1-\lambda\alpha)\left(T + \frac{\lambda-\alpha}{1-\lambda\alpha}T^*\right).$$

Applying Proposition 1.14 with $\frac{\lambda-\alpha}{1-\lambda\alpha}$ in place of $\mu$ gives that for $k = 0, 1, 2, \cdots$,

$$\begin{aligned} \left|\frac{\lambda-\alpha}{1-\lambda\alpha}\right| = \alpha^k &\iff \frac{\lambda-\alpha}{1-\lambda\alpha} = \alpha^k e^{i\theta} \\ &\iff \lambda - \alpha = \alpha^k e^{i\theta} - \lambda\alpha^{k+1}e^{i\theta} \\ &\iff \lambda(1+\alpha^{k+1}e^{i\theta}) = \alpha + \alpha^k e^{i\theta} \\ &\iff \lambda = \frac{\alpha + \alpha^k e^{i\theta}}{1+\alpha^{k+1}e^{i\theta}} \;(-\pi < \theta \le \pi) \end{aligned}$$

□

## 2 Block Toeplitz operators

We review (block) Toeplitz operators and (block) Hankel operators (cf. [12], [13]). For $\mathcal{X}$ a Hilbert space, let $L^2_{\mathcal{X}} \equiv L^2_{\mathcal{X}}(\mathbb{T})$ be the Hilbert space of $\mathcal{X}$-valued norm square-integrable measurable functions on $\mathbb{T}$, and let $H^2_{\mathcal{X}} \equiv H^2_{\mathcal{X}}(\mathbb{T})$ and $H^\infty_{\mathcal{X}} \equiv H^\infty_{\mathcal{X}}(\mathbb{T})$ be the corresponding Hardy spaces. Let $M_{m\times n} \equiv M_{m\times n}(\mathbb{C})$ denote the set of $m \times n$ complex matrices and write $M_n := M_{n\times n}$. If $\Phi$ is a matrix-valued function in $L^\infty_{M_n}$, then the (*block*) *Toeplitz operator* $T_\Phi$ and the (*block*) *Hankel operator* $H_\Phi$ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f := P(\Phi f) \quad \text{and} \quad H_\Phi f := JP^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}), \tag{4}$$

where $P$ and $P^\perp$ denote the orthogonal projections that map $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively, and $J$ denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $(Jg)(z) := \overline{z}I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ ($I_n :=$ the $n \times n$ identity matrix). For $\Phi \in L^\infty_{M_{m\times n}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{5}$$

In 2006, Gu, Hendricks and Rutherford [14] extended Cowen's Theorem to block Toeplitz operators. Their characterization for hyponormality of block Toeplitz operators resembles Cowen's Theorem except for an additional condition - the normality of the symbol.

**Lemma 2.1.** (Hyponormality of Block Toeplitz Operators) [14] For each $\Phi \in L^\infty_{M_n}$, let

$$\mathcal{E}(\Phi) := \left\{K \in H^\infty_{M_n} : \, ||K||_\infty \le 1 \text{ and } \Phi - K\Phi^* \in H^\infty_{M_n}\right\}.$$

Then $T_\Phi$ is hyponormal if and only if $\Phi$ is normal and $\mathcal{E}(\Phi)$ is nonempty.

T. Nakazi and K. Takahashi [17] have shown that if $\varphi \in L^\infty$ is such that $T_\varphi$ is a hyponormal operator whose self-commutator $[T_\varphi^*, T_\varphi]$ is of finite rank then there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$ such that

$$\deg(b) = \operatorname{rank}[T_\varphi^*, T_\varphi].$$

What is the matrix-valued version of Nakazi and Takahashi's Theorem? A candidate is as follows: If $\Phi \in L^\infty_{M_n}$ is such that $T_\Phi$ is a hyponormal operator whose self-commutator $[T_\Phi^*, T_\Phi]$ is of finite rank then there exists a finite Blaschke-Potapov product $B \in \mathcal{E}(\Phi)$ such that $\deg(B) = \operatorname{rank}[T_\Phi^*, T_\Phi]$. We note that the degree of the finite Blaschke-Potapov product $B$ is defined by

$$\deg(B) := \deg(\det B). \tag{6}$$

Thus we have:

**Problem 4.** If $\Phi \in L^\infty_{M_n}$ is such that $T_\Phi$ is a hyponormal operator whose self-commutator $[T^*_\Phi, T_\Phi]$ is of finite rank, does there exist a finite Blaschke-Potapov product $B \in \mathcal{E}(\Phi)$ such that rank $[T^*_\Phi, T_\Phi] = \deg\left(\det B\right)$.

On the other hand, in [17], it was shown that if $\varphi \in L^\infty$ is such that $T_\varphi$ is subnormal and $\varphi = q\overline{\varphi}$, where $q$ is a finite Blaschke product then $T_\varphi$ is normal or analytic. We thus pose its matrix-valued version:

**Problem 5.** If $\Phi \in L^\infty_{M_n}$ is such that $T_\Phi$ is subnormal and $\Phi = B\Phi^*$, where $B$ is a a finite Blaschke-Potapov product, does it follow that $T_\Phi$ is normal or analytic ?

We recall (cf. [9]) that for $\Psi \in L^\infty_{M_n}$ such that $\Psi^*$ is of bounded type, write $\Psi = \Theta_2 B^* = B^*\Theta_2$. Let $\Omega$ be the greatest common left inner divisor of $B$ and $\Theta_2$. Then $B = \Omega B_\ell$ and $\Theta_2 = \Omega\Omega_2$ for some $B_\ell \in H^2_{M_n}$ and some inner matrix $\Omega_2$. Therefore we can write

$$\Psi = B_\ell^*\Omega_2, \quad \text{where } B_\ell \text{ and } \Omega_2 \text{ are left coprime:} \tag{7}$$

in this case, $B_\ell^*\Omega_2$ is called a *left coprime factorization* of $\Psi$. Similarly,

$$\Psi = \Delta_2 B_r^*, \quad \text{where } B_r \text{ and } \Delta_2 \text{ are right coprime:} \tag{8}$$

in this case, $\Delta_2 B_r^*$ is called a *right coprime factorization* of $\Psi$.

As a first inquiry in the matrix-valued version of Halmos's Problem 5 the following question can be raised (cf. [8], [9], [10]):

Is Abrahamse's Theorem valid for block Toeplitz operators?

Related this question, the following theorem was proven:

**Theorem 2.2.** ([9, Theorem 4.5]) Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. Then we may write

$$\Phi_- = B^*\Theta,$$

where $B \in H^2_{M_n}$ and $\Theta := \theta I_n$ with a finite Blaschke product $\theta$. Suppose $B$ and $\Theta$ are coprime. If both $T_\Phi$ and $T^2_\Phi$ are hyponormal then $T_\Phi$ is either normal or analytic.

In Theorem 2.2, the "coprime" condition is essential. To see this, let

$$T_\Phi := \begin{pmatrix} T_b + T_b^* & 0 \\ 0 & T_b \end{pmatrix} \quad (b \text{ is a finite Blaschke product}).$$

Since $T_b + T_b^*$ is normal and $T_b$ is analytic, it follows that $T_\Phi$ and $T^2_\Phi$ are both hyponormal. Obviously, $T_\Phi$ is neither normal nor analytic. Note that $\Phi_- \equiv \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)^* \cdot I_b$, where $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $I_b$ are not coprime. However we note that the above example is a direct sum of a normal Toeplitz operator and an analytic Toeplitz operator. Based on this observation, we have:

**Problem 6.** Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function. If $T_\Phi$ and $T^2_\Phi$ are hyponormal, but $T_\Phi$ is neither normal nor analytic, does it follow that $T_\Phi$ is of the form

$$T_\Phi = \begin{pmatrix} T_A & 0 \\ 0 & T_B \end{pmatrix} \quad (\text{where } T_A \text{ is normal and } T_B \text{ is analytic})?$$

It is well-known that if $T \in \mathcal{B}(\mathcal{H})$ is subnormal then $\ker[T^*, T]$ is invariant under $T$. Thus we might be tempted to guess that if the condition "$T_\Phi$ and $T^2_\Phi$ are hyponormal" is replaced by "$T_\Phi$ is hyponormal and $\ker[T^*_\Phi, T_\Phi]$ is invariant under $T_\Phi$," then the answer to Problem 7 is affirmative. But this is not the case. Indeed, consider

$$T_\Phi = \begin{pmatrix} 2U + U^* & U^* \\ U^* & 2U + U^* \end{pmatrix}.$$

Then a straightforward calculation shows that $T_\Phi$ is hyponormal and $\ker[T^*_\Phi, T_\Phi]$ is invariant under $T_\Phi$, but $T_\Phi$ is never normal (cf. [9, Remark 3.9]). However, if the condition "$T_\Phi$ and $T^2_\Phi$ are hyponormal" is strengthened to "$T_\Phi$ is subnormal", what conclusion do you draw?

## References


[1] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43**(1976), 597–604.

[2] J. Bram, *Subnormal operators*, Duke Math. J. **22**(1955), 75–94.

[3] A. Brown and R.G. Douglas, *Partially isometric Toeplitz operators*, Proc. Amer. Math. Soc. **16**(1965), 681–682.

[4] J.B. Conway, The Theory of Subnormal Operators, Math. Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[5] C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367**(1986), 215–219.

[6] C. Cowen, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103**(1988), 809–812.

[7] C.C. Cowen and J.J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351**(1984), 216–220.

[8] R.E. Curto, I.S. Hwang, D. Kang and W.Y. Lee, *Subnormal and quasinormal Toeplitz operators with matrix-valued rational symbols*, Adv. Math. **255**(2014), 561–585.

[9] R.E. Curto, I.S. Hwang and W.Y. Lee, *Hyponormality and subnormality of block Toeplitz operators*, Adv. Math. **230**(2012), 2094-2151.

[10] R.E. Curto, I.S. Hwang and W.Y. Lee, *Which subnormal Toeplitz operators are either normal or analytic ?*, J. Funct. Anal. **263(8)**(2012), 2333-2354.

[11] R.E. Curto and W.Y. Lee, *Towards a model theory for 2-hyponormal operators*, Integral Equations Operator Theory, **44**(2002), 290–315.

[12] R.G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

[13] R.G. Douglas, *Banach algebra techniques in the theory of Toeplitz operators*, CBMS Reg. Conf. Ser. Math., vol. 15, Amer. Math. Soc., Providence, RI, 1973.

[14] C. Gu, J. Hendricks and D. Rutherford, *Hyponormality of block Toeplitz operators*, Pacific J. Math. **223** (2006), 95–111.

[15] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc., **76**(1970), 887–933.

[16] P.R. Halmos, *Ten years in Hilbert space*, Integral Equations Operator Theory, **2**(1979), 529–564.

[17] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces* Trans. Amer. Math. Soc. **338**(1993), 753–767.

[18] S. Sun, *Bergman shift is not unitarily equivalent to a Toeplitz operator*, Kexue Tongbao(English Ed.) **28**(1983), 1027–1030.



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