数理解析研究所講究録
第 1935 巻 2015 年 38-44

Hypercontractions on Banach space

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1 Introduction

The operator T on a Hilbert space H is an n-hypercontraction for some positive integer n as in Agler [2], if for all $1 \leq m \leq n$,

$$\beta_m(T) := \sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k}T^k \geq 0$$

or equivalently, for all $1 \leq m \leq n$,

$$\langle \beta_m(T)h, h\rangle = \sum_{k=0}^{m}(-1)^k \binom{m}{k} \left\|T^k h\right\|^2 \geq 0 \text{ for all } h \in H.$$

Inspired by the above definition of n-hypercontractions and the work of m-isometries on Hilbert spaces [3] [4] and recent work on (m,p)-isometries on a Banach space X [8] [6] [14] [13], we introduce (m,p)-hypercontractions on X. Let $p \in [1,\infty)$ and let $B(X)$ be the algebra of all bounded linear operators on X. An operator $T \in B(X)$ is called an (m,p)-contraction if

$$\beta_{(m,p)}(T,x) := \sum_{k=0}^{m}(-1)^k \binom{m}{k} \left\|T^k x\right\|^p \geq 0 \text{ for all } x \in X. \tag{1}$$

We say T is an (n,p)-hypercontraction if T is an (m,p)-contraction for all $1 \leq m \leq n$. An operator T is an (m,p)-isometry if $\beta_{(m,p)}(T,x) = 0$ for all $x \in X$. We note that an (m,p)-isometry is automatically an $(m+1,p)$-isometry, see

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formula (4) below. But an (m,p)-hypercontraction is in general not an $(m+1,p)$-hypercontraction. When $n=1$, the power p is irrelevant and a $(1,p)$-contraction is just a contraction. When $n>1$, the power p is highly relevant. For example, it was proved in [8] that there is no $(2,2)$-isometric weighted shifts on l_p for $p \neq 2$. A characterization of (m,q)-isometric weighted shifts on l_p spaces is given by one of the authors in [13]. One of the results in [13] states that if a weighted shift on l_p is an (m,q)-isometry, then $q=pk$ for some integer k.

The following result is well-known [10] [16] [9].

THEOREM A. *Let S denote the unilateral (unweighted) shift of multiplicity one and let $S^{*(\infty)}$ be the backward shift of infinite multiplicity. Let $T \in B(H)$. Then T is unitarily equivalent to a part of $S^{*(\infty)}$ if and only if $\|T\| \leq 1$ and $T^k \to 0$ strongly.*

Agler in [1] developed a C^*-algbera method for operator models and proved an analog of Theorem A with S replaced by Bergman shift B.

THEOREM B. *Let $T \in B(H)$. Then T is unitarily equivalent to a part of $B^{*(\infty)}$ if and only if $I - 2T^*T + T^{*2}T \geq 0$ and $T^k \to 0$ strongly.*

To state the more general result in Agler [2], we need to introduce some notations. Let n be a fixed positive integer.

$$M_n = \left\{ f(z) = \sum_{i=0}^{\infty} \widehat{f}(i) z^i : \|f(z)\|_n^2 = \sum_{i=0}^{\infty} (w_{n,i})^{-1} \left| \widehat{f}(i) \right|^2 < \infty \right\},$$

where $w_{n,i}$ is defined by

$$w_{n,i} = \binom{n-1+i}{n-1} \text{ so that } (1-z)^{-n} = \sum_{i=0}^{\infty} w_{n,i} z^i, |z| < 1. \tag{2}$$

M_n is the Hilbert space of analytic functions on the open unit disc D with the reproducing kernel $k_w(z) = (1 - \overline{w}z)^{-n}$. Let S_n be the operator on M_n defined by

$$S_n(f)(z) = zf(z), f \in M_n.$$

Thus S_1 is the unilateral shift on the Hardy space, S_2 is the Bergman shift on the Bergman space and S_n is a weighted shift.

THEOREM C. *Let $T \in B(H)$. Then T is unitarily equivalent to a part of $S_n^{*(\infty)}$ if and only if $\beta_n(T) \geq 0$ and $T^k \to 0$ strongly.*

In this paper, we extend Theorem C to Banach spaces. Recall $l_p(X)$ denote the Banach space defined by

$$l_p(X) = \left\{ f = \{x_i\}_{i=0}^{\infty} : \|f\|^p = \sum_{i=0}^{\infty} \|x_i\|^p < \infty, x_i \in X \text{ for } i \geq 0 \right\}.$$

More generally, we define weighted Banach space $l_{(n,p)}(X)$ by using weight sequences $\{w_{n,i}\}_{i=0}^{\infty}$ as in (2),

$$l_{(n,p)}(X) = \left\{ f = \{x_i\}_{i=0}^{\infty} : \|f\|_n^p = \sum_{i=0}^{\infty} w_{n,i} \|x_i\|^p < \infty, x_i \in X \text{ for } i \geq 0 \right\}.$$

Note $l_p(X) = l_{(1,p)}(X)$. Let B_n be the (unweighted) backward shift on $l_{(n,p)}(X)$ defined by

$$B_n(x_0, x_1, x_2, \cdots) = (x_1, x_2, \cdots), \{x_i\}_{i=0}^{\infty} \in l_{(n,p)}(X).$$

It is clear that B_1 can be extended to be an invertible bilateral shift defined on two sided $l_p(X)$ space. It is not clear how to extend B_n for $n > 1$. Let $T \in B(X)$. We say T is unitarily equivalent to a part of B_n if there is an isometry W_n from X into $l_{(n,p)}(X)$ such that

$$W_n T = B_n W_n. \tag{3}$$

Note that B_n is invariant on the range $W_n(X)$ and hence one may write

$$T = W_n^{-1} B_n W_n.$$

Now we state the main theorem of this paper.

Theorem 1 *Let $T \in B(X)$. Then T is unitarily equivalent to a part of B_n if and only if T is an (n,p)-contraction and $T^k \to 0$ strongly.*

Instead of working with weighted Banach space $l_{(n,p)}(X)$, we could just work on $l_p(X)$. The trade-off would be that we use weighted backward shift D_n on $l_p(X)$ instead of unweighted backward shift B_n on $l_{(n,p)}(X)$. The operator D_n on $l_p(X)$ is defined by

$$D_n(x_0, x_1, x_2, \cdots, x_i, \cdots) = (c_1 x_1, c_2 x_2, \cdots, c_i x_i, \cdots)$$

where $c_i = (w_{n,i-1}/w_{n,i})^{1/p}, i \geq 1$. The operator D_n is a contraction since $c_i \leq 1$ for all $i \geq 1$. Then Theorem 1 can be reformulated as the following: There is an isometry W_n from X into $l_p(X)$ such that $W_n T = D_n W_n$ if and only if T is an (n,p)-contraction and $T^k \to 0$ strongly.

2 Proof of Theorem 1

The proof of Theorem 1 needs several lemmas which we stated below. Here we will only give the proof of "if" part of Theorem 1 which is short.

We first state a lemma proved on page 2143 in [6].

Lemma 2 *Let $T \in B(X)$, $N \geq n \geq 1$ and $x \in X$. Then*

$$\beta_{(n,p)}(T, x) = \beta_{(n-1,p)}(T, x) - \beta_{(n-1,p)}(T, Tx). \tag{4}$$

We also need the following lemma.

Lemma 3 *Let $T \in B(X)$. If T is an (n,p)-contraction and $T^k \to 0$ strongly, then T is an (n,p)-hypercontraction. Furthermore, for each $x \in X$ and all $0 \leq m \leq n$,*

$$k^m \beta_{(m,p)}(T, T^k x) \to 0 \text{ as } k \to \infty. \tag{5}$$

Lemma 4 *Let $T \in B(X)$, $N \geq n \geq 1$ and $x \in X$. Then*

$$\sum_{k=0}^{N} w_{n,k} \beta_{(n,p)}(T, T^k x) + \sum_{l=0}^{n-1} w_{l+1,N} \beta_{(l,p)}(T, T^{N+1} x) = \|x\|^p \tag{6}$$

The proof of "if" part of Theorem 1. Let $T \in B(X)$ be such that $\beta_{(n,p)}(T, x) \geq 0$ for all $x \in X$ and $T^k \to 0$ strongly. We define W_n from X into $l_{(n,p)}(X)$ as

$$W_n x = \left\{ \beta^{1/p}_{(n,p)}(T, T^i x) \frac{T^i x}{\|T^i x\|} \right\}_{i=0}^{\infty}$$

with the understanding that if $T^i x = 0$ for a specific i, then $\beta^{1/p}_{(n,p)}(T, T^i x) \frac{T^i x}{\|T^i x\|} = 0$. We now show W_n is well-defined and is an isometry. We need to show $\|W_n x\|^p_n = \sum_{i=0}^{\infty} w_{n,i} \beta_{(n,p)}(T, T^i x)$ converges to $\|x\|^p$. By Lemma 4, for $N > n$,

$$\sum_{i=0}^{N} w_{n,i} \beta_{(n,p)}(T, T^i x) = \|x\|^p - \sum_{l=0}^{n-1} w_{l+1,N} \beta_{(l,p)}(T, T^{N+1} x) \leq \|x\|^p .$$

Furthermore for each $0 \leq l \leq n-1$, by Lemma 3 and $\frac{w_{l+1,N}}{(N+1)^l} \to 1$, we have

$$w_{l+1,N} \beta_{(l,p)}(T, T^{N+1} x) = \frac{w_{l+1,N}}{(N+1)^l} (N+1)^l \beta_{(l,p)}(T, T^{N+1} x) \to 0$$

as $N \to \infty$. The proof is complete.

3 A similarity model on Banach space

Theorem 1 gives a characterization of an operator unitarily equivalent to a part of the (n,p)-hypercontraction B_n. What is a characterization of an operator similar to a part of B_n? This question has not even been discussed on Hilbert spaces for $n > 1$. For $n = 1$, the following model theorem of Rota [17] predates Theorem A and is the first example of a universal operator. Let $r(T)$ denote the spectral radius of a bounded operator T.

THEOREM D. *Let* $T \in B(H)$. If $r(T) < 1$, then T is similar to a part of $S^{*(\infty)}$.

The proof of Theorem D and some of its late generalizations (see the book [15]) can be adapted to Banach spaces. Thus some of the results below might be known to experts. Let $T \in B(X)$, we say T is similar to a part of B_n, the backward shift on $l_{(n,p)}(X)$, if there is an bounded operator W_n from X into $l_{(n,p)}(X)$ such that W_n is bounded below and $W_nT = B_nW_n$. The following result is inspired by Proposition 6.6 from [15] and the proof is also similar. However, Proposition 6.6 from [15] only deals with the case $n = 1$.

Theorem 5 *Let* $T \in B(X)$. *The following statements are equivalent.*
(a) There exist constants $\beta \geq \alpha > 0$ *and* $Q \in B(X)$, *such that for all* $x \in X$,

$$\alpha \|x\|^p \leq \sum_{k=0}^{\infty} w_{n,k} \left\|QT^k x\right\|^p \leq \beta \|x\|^p . \tag{7}$$

(b) T *is similar to a part of* B_n *on* $l_{(n,p)}(X)$.

Proof. The proof is adapted from the proof of Proposition 6.6 in [15]. Assume (a) holds. We define W_n from X into $l_{(n,p)}(X)$ by

$$W_n x = \left\{QT^k x\right\}_{k=0}^{\infty} .$$

Then assumption (7) is the same as $\alpha \|x\|^p \leq \|Wx\|_n^p \leq \beta \|x\|^p$. Therefore the range of W_n, denoted by $R(W_n)$, is a closed subspace of $l_{(n,p)}(X)$ and W_n from X onto $R(W_n)$ is invertible. It is also clear that $W_nT = B_nW_n$, so $R(W_n)$ is invariant for B_n and

$$T = W_n^{-1}(B_n|R(W_n))W_n.$$

That is, T is similar to the restriction of B_n to $R(W_n)$.

Now assume (b) holds. Let W_n from X into $l_{(n,p)}(X)$ be such that W_n is bounded below and $W_nT = B_nW_n$. Let P_k be the projection from $l_{(n,p)}(X)$ onto its k-th component, $P_k\{x_i\}_{i=0}^{\infty} = x_ke_k$. Let $Q_k = P_kW_n, k \geq 0$. Then for $x \in X$,

$$W_nx = \{Q_kx\}_{k=0}^{\infty}.$$

The relation $W_nT = B_nW_n$ means $Q_kTx = Q_{k+1}x$. Thus $Q_{k+1} = Q_kT$. Set $Q = Q_0$, we have

$$Wx = \{Q_kx\}_{k=0}^{\infty} = \left\{QT^kx\right\}_{k=0}^{\infty}.$$

Now W_n from X into $l_{(n,p)}(X)$ is bounded and bounded below is the same as (7). The proof is complete. ∎

The following is the analogue of Theorem D on Banach spaces.

Corollary 6 *Let $T \in B(X)$. If $r(T) < 1$, then T is similar to a part of B_1. Furthermore T is similar to a strict contraction.*

Proof. If $r(T) < 1$, the condition (7) (with $w_{1k} = 1$) holds for by taking Q to be the identity operator. Thus T is similar to a part of B_1. To prove T is similar to a strict contraction, we use the scaling as in [17]. Let ε be such that $r(T) < \varepsilon < 1$. By what we just proved, T/ε is similar to a part of B_1. Therefore T is similar to a part of εB_1. We will show below that $B_1|R(W_1)$ is in fact a strict contraction. ∎

Acknowledgement The research of Zhengli Chen was partially supported by the National Natural Science Foundation of China(No.11371012), the IFGP of Shaanxi Normal University (No.2013CXB012) and the FRF for the Central Universities(GK 201002006).

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