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C^* 環の正値可逆元の集合の間の THOMPSON 等距離写像について (阿部 敏一氏との共同研究)
THOMPSON ISOMETRIES ON POSITIVE INVERTIBLE ELEMENTS IN C^*-ALGEBRAS (JOINT WORK WITH TOSHIKAZU ABE)

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ABSTRACT. We introduce a notion of the generalized gyrovector spaces. We show a Mazur-Ulam theorem for the generalized gyrovector spaces. As an application we give an alternative proof for a result of Honma and Nogawa [8] on the Thompson's like isometries between the sets of positive invertible elements in unital C^*-algebras. This is an anouncement of the forthcoming paper [1] with Toshikazu Abe.

1. INTRUDUCTION

A Gyrogroup is a natural extension of a group to the nonassociative algebraic structure. In Einstein's theory of special relativity, the set of all admissible velocities is $\mathbb{R}_c^3 = \{\boldsymbol{u} \in \mathbb{R}^3 : \|\boldsymbol{u}\| < c\}$ for the speed of light in vacuum c. Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product and $\gamma_{\boldsymbol{u}}$ the Lorentz factor given by

$$\gamma_{\boldsymbol{u}} = (1 - \|\boldsymbol{u}\|^2/c^2)^{-\frac{1}{2}}. \tag{1}$$

The Einstein velocity addition $\oplus_E$ in $\mathbb{R}_c^3$ given by

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \langle \boldsymbol{u}, \boldsymbol{v} \rangle / c} \left\{ \boldsymbol{u} + \frac{1}{\gamma_{\boldsymbol{u}}} \boldsymbol{v} + \frac{1}{c^2} \frac{\gamma_{\boldsymbol{u}}}{1 + \gamma_{\boldsymbol{u}}} \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{u} \right\} \tag{2}$$

for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_c^3$ is not commutative nor associative, hence $(\mathbb{R}_c^3, \oplus_E)$ does not have a group structure. However, $(\mathbb{R}_c^3, \oplus_E)$ have a gyrocommutative gyrogroup structure and is called the Einstein gyrogroup. The (gyrocommutative) gyrogroup is the generalization of the (commutative) group, which is not necessarily (commutative nor) associative.

Certain gyrocommutative gyrogroup admits scalar multiplication, giving rise to a gyrovector space. Ungar initiated the study on a gyrogroup and a gyrovector space (cf. [20]). He describes that the hyperbolic geometry of Bolyai and Lobachevsky is now effectively regulated by gyrovector spaces just as Euclidean geometry is regulated by vector spaces. Any gyrovector space is equipped with the gyrometric, which is a measurement of the distance while it needs not be the metric exactly. Any (positive definite) real inner product space is a gyrovector space and the gyrometric is the metric induced by its norm. On the other hand, a real normed space need not be a gyrovector space. In this paper, we introduce a notion of a generalized gyrovector space which is a common generalization of the notion of a real normed space and that of a gyrovector space. We exhibit a Mazur-Ulam Theorem for the generalized gyrovector spaces; a bijection between the generalized gyrovector spaces which preserves the gyrometric also preserves the algebraic structure automatically. The celebrated Mazur-Ulam theorem states that a surjective isometry from a normed vector space onto a possibly different normed vector space is a real linear isomorphisms followed by a translation. A simple proof of the Mazur-Ulam

Theorem was given by Väisälä [21] by using the idea of Vogt [22]. Our proof of Theorem GMU emploies the same idea.

As an application of the Mazur-Ulam theorem for the generalized gyrovecgtor spaces we give an alternative proof of the representation theorem of Honma and Nogawa [8] on the Thompson's like isometries on the set of all positive invertibel elements in a C^*-algebras.

2. A_+^{-1} IS A GYROCOMMUTATIVE GYROGROUP

Definition 1 ((Gyrocommutative) Gyrogroup).
A groupoid $(G, \oplus)$ is a gyrogroup if there exists a point $\boldsymbol{e} \in G$ such that the following hold.

(G1) $\forall \boldsymbol{a} \in G$
$$\boldsymbol{e} \oplus \boldsymbol{a} = \boldsymbol{a}, \quad .$$
(G2) $\forall \boldsymbol{a} \in G \;\; \exists \ominus \boldsymbol{a}$ s.t.
$$\ominus \boldsymbol{a} \oplus \boldsymbol{a} = \boldsymbol{e}.$$
(G3) $\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G \;\; \exists! \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c} \in G$ s.t.
$$\boldsymbol{a} \oplus (\boldsymbol{b} \oplus \boldsymbol{c}) = (\boldsymbol{a} \oplus \boldsymbol{b}) \oplus \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}]\boldsymbol{c}.$$
(G4) $\operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}]$ is an gyroautomorphism for $\forall \boldsymbol{a}, \boldsymbol{b} \in G$

(G5) $\forall \boldsymbol{a}, \boldsymbol{b} \in G$
$$\operatorname{gyr}[\boldsymbol{a} \oplus \boldsymbol{b}, \boldsymbol{b}] = \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}].$$
Gyrocommutative if the following (G6) is also satisfied.

(G6) $\forall \boldsymbol{a}, \boldsymbol{b} \in G$
$$\boldsymbol{a} \oplus \boldsymbol{b} = \operatorname{gyr}[\boldsymbol{a}, \boldsymbol{b}](\boldsymbol{b} \oplus \boldsymbol{a}).$$

Note that
$$e \oplus a = a \oplus e = a, \qquad a \in G,$$
$\ominus$ is unique for every $a \in G$ and
$$\ominus a \oplus a = a \oplus \ominus a = e, \quad a \in G.$$
To analyze the gyrogroups the coaddition is useful.

Definition 2. Let $(G, \oplus)$ is a gyrogroup. The gyrogroup coaddition $\boxplus$ is defined by
$$\boldsymbol{a} \oplus \operatorname{gyr}[\boldsymbol{a}, \ominus \boldsymbol{b}]\boldsymbol{b}$$
for all $\boldsymbol{a}, \boldsymbol{b} \in G$.

Note that the gyrogroup coaddition $\boxplus$ is commutative if and only if the gyrogroup $(G, \oplus)$ is gyrocommutative (cf. [20, Theorem 3.4]).

Theorem 3 (A_+^{-1} is a gyrocommutative gyrogroup, [1]). *Suppose that A_+^{-1} is the set of all positive invertible elemets in a unital C^*-algebra A. For $0 < t \in \mathbb{R}$, put*
$$a \oplus_t b = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}}, \quad a, b \in A_+^{-1}$$
Then $(A_+^{-1}, \oplus_t)$ is a gyrocommutative gyrogroup. The gyrogroup identity = the identity element e of A as the C^*-algebra. The inverse element $\ominus a$ is a^{-1}
For $a, b \in A_+^{-1}$ put
$$X = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}.$$
Then X is a unitary element in A and
$$\operatorname{gyr}[a, b]c = XcX^*, \quad a, b, c \in A_+^{-1}.$$

Proof.

$$e \oplus_t a = (e^{\frac{t}{2}} a^t e^{\frac{t}{2}})^{\frac{1}{t}} = a, \quad \forall a \in A_+^{-1}.$$

Hence (G1) holds.

For every $a \in A_+^{-1}$

$$a^{-1} \oplus_t a = ((a^{-1})^{\frac{t}{2}} a^t (a^{-1})^{\frac{t}{2}})^{\frac{1}{t}} = (a^{-\frac{t}{2}} a^t a^{-\frac{t}{2}})^{\frac{1}{t}} = e.$$

Hence (G2); $\ominus a = a^{-1}$.

Let $a, b, c \in A_+^{-1}$. Put

$$X = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}.$$

Then X is unitary since

$$\begin{aligned} XX^* &= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}\right) \left(b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}\right) \\ &= ((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}})(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}) = e, \end{aligned}$$

$$X^*X = \left(b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}\right) \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}\right) = b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-1} a^{\frac{t}{2}} b^{\frac{t}{2}} = e.$$

Put

$$\mathrm{gyr}[a,b]c = XcX^*.$$

Then

$$\begin{aligned} (3) \quad (a \oplus_t b) \oplus_t \mathrm{gyr}[a,b]c &= \left((a \oplus_t b)^{\frac{t}{2}} (\mathrm{gyr}[a,b]c)^t (a \oplus_t b)^{\frac{t}{2}}\right)^{\frac{1}{t}} \\ &= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}} c^t b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}\frac{t}{2}}\right)^{\frac{1}{t}} \\ &= \left(a^{\frac{t}{2}} b^{\frac{t}{2}} c^t b^{\frac{t}{2}} a^{\frac{t}{2}}\right)^{\frac{1}{t}} = \left(a^{\frac{t}{2}} ((b^{\frac{t}{2}} c^t b^{\frac{t}{2}})^{\frac{1}{t}})^t a^{\frac{t}{2}}\right)^{\frac{1}{t}} = a \oplus_t (b \oplus_t c) \end{aligned}$$

Hence (G3) holds.

Let $a, b, c, d \in A_+^{-1}$. As X is unitary, $\mathrm{gyr}[a,b] : A_+^{-1} \to A_+^{-1}$ is bijection.

$$\begin{aligned} (4) \quad \mathrm{gyr}[a,b](c \oplus_t d) &= X(c \oplus_t d)X^* = X(c^{\frac{t}{2}} d^t c^{\frac{t}{2}})^{\frac{1}{t}} X^* = (Xc^{\frac{t}{2}} d^t c^{\frac{t}{2}} X)^{\frac{1}{t}} \\ &= \left((Xc^{\frac{t}{2}} X^*)(Xd^t X^*)(Xc^{\frac{t}{2}} X^*)\right)^{\frac{1}{t}} \\ &= \left((XcX^*)^{\frac{t}{2}} (XdX^*)^t (XcX^*)^{\frac{t}{2}}\right)^{\frac{1}{t}} \\ &= (\mathrm{gyr}[a,b]c \oplus_t (\mathrm{gyr}[a,b]d). \end{aligned}$$

Thus $\mathrm{gyr}[a,b]$ is a gyroautomorphism for any $a, b \in A_+^{-1}$; (G4) holds.

To prove (G5) we first show that

$$(5) \qquad \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}} = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}.$$

To prove it we compute

$$\begin{aligned} (6) \quad &\left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^2 \\ &= (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} \\ &= (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}}) a^{-\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} \end{aligned}$$

By (6) we get (5). The gyroautomorphism $\mathrm{gyr}[a\oplus_t b, b]$ is a unitary transformation defined by the unitary element

$$((a \oplus_t b)^{\frac{t}{2}} b^t (a \oplus_t b)^{\frac{t}{2}})^{-\frac{1}{2}} (a \oplus_t b)^{\frac{t}{2}} b^{\frac{t}{2}}.$$

We compute

$$\begin{aligned}
(7)\quad & \left((a \oplus_t b)^{\frac{t}{2}} b^t (a \oplus_t b)^{\frac{t}{2}}\right)^{-\frac{1}{2}} (a \oplus_t b)^{\frac{t}{2}} b^{\frac{t}{2}} \\
&= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}\frac{t}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}\frac{t}{2}}\right)^{-\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}\frac{t}{2}} b^{\frac{t}{2}} \\
&= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^t (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}}\right)^{-\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^{\frac{t}{2}} \\
&= (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{2}} b^{\frac{t}{2}} = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}}
\end{aligned}$$

The last term is the unitary element which corresponds to $\mathrm{gyr}[a, b]$ Hence $\mathrm{gyr}[a \oplus_t b, b] = \mathrm{gyr}[a, b]$; (G5) holds.

We compute

$$\mathrm{gyr}[a, b](b \oplus_t a) = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}} (b^{\frac{t}{2}} a^t b^{\frac{t}{2}})^{\frac{1}{t}} b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}},$$

since it is the unitary transform we have

$$\begin{aligned}
&= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^{\frac{t}{2}} (b^{\frac{t}{2}} a^t b^{\frac{t}{2}}) b^{\frac{t}{2}} a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}\right)^{\frac{1}{t}} \\
&= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} a^{\frac{t}{2}} b^t a^t b^t a^{\frac{t}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}\right)^{\frac{1}{t}} \\
&= \left((a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}} (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{-\frac{1}{2}}\right)^{\frac{1}{t}} = (a^{\frac{t}{2}} b^t a^{\frac{t}{2}})^{\frac{1}{t}} = a \oplus_t b.
\end{aligned}$$

Thus (G6) holds.

We conclued that $(A_+^{-1}, \oplus_t)$ is a gyrocommutative gyrogroup. □

3. A_+^{-1} IS A GGV

We define the generalized gyrovector space (GGV) on the given gyrocommutative gyrogroup. The gyrovector space are dfined by Ungar. The GGV is a generalization of the gyrovector space.

Definition 4 (Generalized gyrovector space; GGV). A gyrocommutative gyrogroup $(G, \oplus)$ is a called a GGV $(G, \oplus, \otimes, \phi)$ if $\otimes : \mathbb{R} \times G \to G$, and an injection $\phi : G \to (\mathbb{V}, \|\cdot\|)$ for a real normed space $(\mathbb{V}, \|\cdot\|)$ are defined and the following conditions are satisfied.

(GGV0) $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$

$$\|\phi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a})\| = \|\phi(\boldsymbol{a})\|;$$

(GGV1) $\forall \boldsymbol{a} \in G$

$$1 \otimes \boldsymbol{a} = \boldsymbol{a};$$

(GGV2) $\forall \boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$

$$(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$$

(GGV3) $\forall \boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$

$$(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a});$$

(GGV4) $\forall \boldsymbol{a} \in G \setminus \{\boldsymbol{e}\},\ r \in \mathbb{R} \setminus \{0\}$

$$\phi(|r| \otimes \boldsymbol{a})/\|\phi(r \otimes \boldsymbol{a})\| = \phi(\boldsymbol{a})/\|\phi(\boldsymbol{a})\|;$$

(GGV5) $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G,\ r \in \mathbb{R}$

$$\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a};$$

(GGV6) $\forall \boldsymbol{v} \in G,\ r_1, r_2 \in \mathbb{R}$

$$\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G;$$

(GGVV) $\{\pm\|\phi(\boldsymbol{a})\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;

(GGV7) $\forall \boldsymbol{a} \in G, r \in R$

$$\|\phi(r \otimes \boldsymbol{a})\| = |r| \otimes' \|\phi(\boldsymbol{a})\|;$$

(GGV8) $\forall \boldsymbol{a}, \boldsymbol{b} \in G$

$$\|\phi(\boldsymbol{a} \oplus \boldsymbol{b})\| \leq \|\phi(\boldsymbol{a})\| \oplus' \|\phi(\boldsymbol{b})\|$$

A gyrovector space $(G, \oplus, \otimes)$ is defined by Ungar; a gyrocommutative gyrogroup $(G, \oplus)$ with $G \subset W$ for a real inner product space W such that the exotic scalar multiplication $\otimes : \mathbb{R} \times G \to G$ is defined and that the conditions through (GGV1) to (GGV8) with ϕ being the identy map, and the condition

(V0) $\langle \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}, \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b}\rangle = \langle \boldsymbol{a}, \boldsymbol{b}\rangle$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b} \in G$

instead of the condition (GGV0) are satisfied, where $\langle \cdot, \cdot \rangle$ is the inner product on W. In short a gyrovector space is a subset of an inner product space while a GGV is an inverse image of ϕ of a normed vector space. Any gyrovector space is a GGV since the condition (V0) implies the condition (GGV0) with *phi* being the identity map (cf. [20, Definition6.2]). In addition, any real normed space is a GGV. Let V be a real normed space with the addition $+$ and the scalar multiplication $\cdot$, then $(V, +, \cdot)$ is a GGV with the map ϕ being the identity map on V).

We now define the gyromidpoint and the gyrometric for a GGV. They are the generalization for the corresponding concepts for a gyrovector space.

Definition 5. Let $(G, \oplus, \otimes, \phi)$ is a GGV. The gyromidpoint $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$ of $\boldsymbol{a}, \boldsymbol{b} \in (G, \oplus, \otimes)$ is defined as $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{2} \otimes (\boldsymbol{a} \boxplus \boldsymbol{b})$, where $\boxplus$ is the gyrogroup coaddition of the gyrogroup $(G, \oplus)$.

Note that $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{p}(\boldsymbol{b}, \boldsymbol{a})$ as $\boxplus$ is commutative. In addition we have

$$\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} \oplus \frac{1}{2} \otimes (\ominus \boldsymbol{a} \oplus \boldsymbol{b}) \tag{8}$$

(cf. [20, Definition 6.32 and Theorem 6.34]). In particular, $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ if the GGV $(G, \oplus, \otimes)$ is indeed a real normed space $(\mathbb{V}, +, \cdot)$.

Definition 6. Let $(G, \oplus, \otimes, \phi)$ is a GGV. Put

$$\varrho(\boldsymbol{a}, \boldsymbol{b}) = \|\phi(\boldsymbol{a} \ominus \boldsymbol{b})\|, \boldsymbol{a}, \boldsymbol{b} \in G,$$

where $\boldsymbol{a} \ominus \boldsymbol{b}$ is the abbreviation of $\boldsymbol{a} \oplus (\ominus \boldsymbol{b})$. We call ϱ the gyrometric on G.

The gyrometric ϱ satisfies the equation

$$\varrho(\boldsymbol{a}, \boldsymbol{b}) = \varrho(\ominus \boldsymbol{a}, \ominus \boldsymbol{b}) = \varrho(\boldsymbol{b}, \boldsymbol{a}) \tag{9}$$

as

$$\begin{aligned}\varrho(\boldsymbol{a},\boldsymbol{b}) &= \|\phi(\boldsymbol{a}\ominus\boldsymbol{b})\|\\ &= \|\phi(\ominus(\boldsymbol{a}\ominus\boldsymbol{b}))\|\\ &= \|\phi(\ominus\boldsymbol{a}\oplus\boldsymbol{b})\| = \varrho(\ominus\boldsymbol{a},\ominus\boldsymbol{b})\\ &= \|\phi(\mathrm{gyr}[\ominus\boldsymbol{a},\boldsymbol{b}](\boldsymbol{b}\ominus\boldsymbol{a}))\|\\ &= \|\phi(\boldsymbol{b}\ominus\boldsymbol{a})\| = \varrho(\boldsymbol{b},\boldsymbol{a}).\end{aligned}$$

In particular, if $(G,\oplus,\otimes)$ is a real normed space, then gyrometric is a metric induced by its norm.

We show a gyrocommutative gyrogroup of the positive invertible elements in a C^*-algebra is indeed a GGV.

Theorem 7 (A_+^{-1} is a GGV, [1]). *Suppose that A_+^{-1} is the gyrocommutative gyrogroup of the set of all positive invertibel elements in a unital C^*-algebra A with*

$$a\oplus_t b = (a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}}, \qquad a,b\in A_+^{-1}.$$

Put

$$r\otimes a = a^r, \quad \phi(a)=\log a, \qquad a\in A_+^{-1},\ r\in\mathbb{R},$$
$$(\pm\|\phi(A_+^{-1})\|,\oplus',\otimes') = (\mathbb{R},+,\times).$$

Then $(A_+^{-1},\oplus_t,\otimes,\log)$ is a GGV with $\|\cdot\|$.

Proof. As $\mathrm{gyr}[u,v]$ is the unitary transformation by the unitary element $X=(u^{\frac{t}{2}}v^t u^{\frac{t}{2}})^{-\frac{1}{2}}u^{\frac{t}{2}}v^{\frac{t}{2}}$ we have

$$\|\log(\mathrm{gyr}[u,v]a)\| = \|\log XaX^*\| = \|X\log aX^*\| = \|\log a\|.$$

Thus (GGV0) holds.

For every $a\in A_+^{-1}$ we have

$$1\otimes a = a^1 = 1;$$

(GGV1) holds.

For $r_1,r_2\in\mathbb{R}$ and $a\in A_+^{-1}$ we have

$$(r_1+r_2)\otimes a = a^{r_1+r_2}$$

and

$$(r_1\otimes a)\oplus_t(r_2\otimes a) = a^{r_1}\oplus_t a^{r_2} = ((a^{r_1})^{\frac{t}{2}}(a^{r_2})(a^{r_1})^{\frac{t}{2}})^{\frac{1}{t}} = a^{r_1+r_2}.$$

Hence $(r_1+r_2)\otimes a = (r_1\otimes a)\oplus_t(r_2\otimes a)$; (GGV2) holds.

(GGV3) is easy by

$$(r_1r_2)\otimes a = a^{r_1r_2} = (a^{r_2})^{r_1} = r_1\otimes(r_2\otimes a).$$

(GGV4) is also easy by

$$\frac{\log(|r|\otimes a)}{\|\log(r\otimes a)\|} = \frac{\log a^{|r|}}{\|\log a^r\|} = \frac{\log a}{\|\log a\|}.$$

Letting $X=(u^{\frac{t}{2}}v^t u^{\frac{t}{2}})^{-\frac{1}{2}}u^{\frac{t}{2}}v^{\frac{t}{2}}$ we have

$$\mathrm{gyr}[u,v](r\otimes a) = Xa^rX^* = (XaX^*)^r = r\otimes\mathrm{gyr}[u,v]a$$

for every $a\in A_+^{-1}$; (GGV5) holds.

For $r_1,r_2\in\mathbb{R}$ and $v\in A_+^{-1}$ we have

$$\mathrm{gyr}[r_1\otimes v, r_2\otimes v] = \mathrm{gyr}[v^{r_1},v^{r_2}]$$

is a unitary transformation defined by

$$\left((v^{r_1})^{\frac{t}{2}}(v^{r_2})^t(v^{r_1})^{\frac{t}{2}}\right)^{-\frac{1}{2}}(v^{r_1})^{\frac{t}{2}}(v^{r_2})^{\frac{t}{2}} = e.$$

Hence (GGV6) holds.

It is easy to see

$$\{\pm\|\log a\| : a \in A_+^{-1}\} = \mathbb{R}.$$

Let $\oplus_t'$ be the usual addition in $\mathbb{R}$ and $\otimes'$ is the scalar multiplication. Then

$$(\{\pm\|\log a\| : a \in A_+^{-1}\}, \oplus', \otimes')$$

is the usual real linear space of dimension 1; (GGVV) holds.

It is also easy to have

$$\|\log(r \otimes a)\| = \|\log a^r\| = |r|\|\log a\| = |r| \otimes' \|\log a\|;$$

(GGV7) holds.

To prove (GGV8) we need some calculation. Let $a, b \in A_+^{-1}$. We denote the spectrum by $\sigma(\cdot)$. Let

$$\Lambda_a = \max\{\lambda : \lambda \in \sigma(a)\}, \quad \lambda_a = \min\{\lambda : \lambda \in \sigma(a)\}$$

and

$$\Lambda_b = \max\{\lambda : \lambda \in \sigma(b)\}, \quad \lambda_b = \min\{\lambda : \lambda \in \sigma(b)\}.$$

Then

$$\lambda_a e \le a \le \Lambda_a e, \quad \lambda_b e \le b \le \Lambda_b e.$$

Since b and e commute we have

$$\lambda_b^t e \le b^t \le \Lambda_b^t e, \quad \lambda_a^t \le a^t \le \Lambda_a^t.$$

Then

$$0 \le a^{\frac{t}{2}}(\Lambda_b^t e - b^t)a^{\frac{t}{2}} = \Lambda_b^t a^t - a^{\frac{t}{2}} b^t a^{\frac{t}{2}} \le \Lambda_b^t \Lambda_a^t e - a^{\frac{t}{2}} b^t a^{\frac{t}{2}},$$

hence

$$a^{\frac{t}{2}} b^t a^{\frac{t}{2}} \le (\Lambda_a \Lambda_b)^t e.$$

So we have

$$\max\{\lambda : \lambda \in \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\} \le (\Lambda_a \Lambda_b)^t. \tag{10}$$

In a similar way we have

$$\min\{\lambda : \lambda \in \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\} \ge (\lambda_a \lambda_b)^t. \tag{11}$$

On the other hand

$$\begin{aligned} \|\log a^{\frac{t}{2}} b^t a^{\frac{t}{2}}\| &= \max\{|\lambda| : \lambda \in \sigma(\log a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\} \\ &= \max\{|\lambda| : \lambda \in \log \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\} \\ &= \max\{|\log \max\{\lambda \in \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\}|, |\log \min\{\lambda : \lambda \in \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\}|\} \end{aligned}$$

We have by (10), (11)

$$\log \max\{\lambda \in \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\} \le \log(\Lambda_a \Lambda_b)^t = t(\log \Lambda_a + \log \Lambda_b),$$

$$\log \min\{\lambda \in \sigma(a^{\frac{t}{2}} b^t a^{\frac{t}{2}})\} \ge t(\log \lambda_a + \log \lambda_b).$$

It follows that

$$\begin{aligned}
&\|\log a^{\frac{t}{2}}b^t a^{\frac{t}{2}}\| \\
&\quad = \max\{|\log\max\{\lambda\in\sigma(a^{\frac{t}{2}}b^t a^{\frac{t}{2}})\}|, |\log\min\{\lambda:\lambda\in\sigma(a^{\frac{t}{2}}b^t a^{\frac{t}{2}})\}|\} \\
&\qquad \le t\max\{|\log\Lambda_a+\log\Lambda_b|, |\log\lambda_a+\log\lambda_b|\} \\
&\qquad \le t\max\{|\log\Lambda_a|+|\log\Lambda_b|, |\log\lambda_a|+|\log\lambda_b|\} \\
&\qquad \le t(\max\{|\log\Lambda_a|, |\log\lambda_a|\}+\max\{|\log\Lambda_b|, |\log\lambda_b|\} \\
&\qquad = t(\|\log a\|+\|\log b\|)
\end{aligned}$$

Hence we have (GGV8);

$$\|\log(a^{\frac{t}{2}}b^t a^{\frac{t}{2}})^{\frac{1}{t}}\| \le \|\log a\| \oplus' \|\log b\|, \quad a,b\in A_+^{-1}.$$

□

Remark 8. For the GGV $(A_+^{-1}, \oplus_t, \otimes, \log)$ of the set of all positive invertible elements in a unital C^*-algebra A with $t>0$ the gyromidpoint $\boldsymbol{p}(a,b)$ for $a,b\in A_+^{-1}$ is easily calculated by the definition as

$$\boldsymbol{p}(a,b) = (a^{\frac{t}{2}}(a^{-\frac{t}{2}}b^t a^{-\frac{t}{2}})^{\frac{1}{2}}a^{\frac{t}{2}})^{\frac{1}{t}}.$$

For $t=1$ it is the geometric mean of a and b.

4. A Mazur-Ulam theorem for GGV's

In this section we exhibit a Mazur-Ulam theorem for GGV's.

Definition 9. Suppose that $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be GGV's. Let ϱ_1 and ϱ_2 be gyrometrics of G_1 and G_2, respectively. We say that a map $T: G_1 \to G_2$ is gyrometric preserving if the equality

$$\varrho_2(T\boldsymbol{a}, T\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b})$$

holds for every pair $\boldsymbol{a}, \boldsymbol{b} \in G_1$.

Theorem 10 ([1]). *Let $(G_j, \oplus_j, \otimes_j, \phi_j)$ be a GGV and ϱ_j the gyrometric of $(G_j, \oplus_j, \otimes_j, \phi_j)$ for $j=1,2$. Suppose that $T: G_1 \to G_2$ is a gyrometric preserving surjection. Then T preserves the gyromidpoints;*

$$\boldsymbol{p}(T\boldsymbol{a}, T\boldsymbol{b}) = T\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{b})$$

for any pair $\boldsymbol{a}, \boldsymbol{b} \in G_1$.

The following corollary asserts that a surjective gyrometric preserving map preserves the algebraic structure followed by the left gyrotranslations. It follows that two GGV's which have a same gyrometric structure have the same GGV structure essentially.

Corollary 11 ([1]). *Let $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be GGV's. Let ϱ_1 and ϱ_2 be gyrometrics of G_1 and G_2, respectively. Suppose that a surjection $T: G_1 \to G_2$ satisfies*

$$\varrho_2(T\boldsymbol{a}, T\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b})$$

for any pair $\boldsymbol{a}, \boldsymbol{b} \in G_1$. Then T is of the form $T = T(\boldsymbol{e}) \oplus_2 T_0$, where T_0 is an isometrical isomorphism in the sense that the equalities

$$T_0(\boldsymbol{a} \oplus_1 \boldsymbol{b}) = T_0(\boldsymbol{a}) \oplus_2 T_0(\boldsymbol{b}); \tag{12}$$

$$T_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 T_0(\boldsymbol{a}); \tag{13}$$

$$\varrho_2(T_0\boldsymbol{a}, T_0\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b}). \tag{14}$$

for every $\boldsymbol{a}, \boldsymbol{b} \in G_1$ and $\alpha \in \mathbb{R}$ hold.

We can prove Theorem 10 by applying the similar argument given by Väisälä [21] of a simple proof of the celebrated Mazur-Ulam theorem. Precise proofs of Theorem 10 and Corollary 11 are given in [1] and they are omited in this paper.

5. Applications

Theorem 12 was proved by Honma and Nogawa [8, Theorem 8] and the case of $t=1$ is exhibited as Theorem 9 in [7]. The original proofs employ a non-commutative Mazur-Ulam theorem (cf. [6]). In this section we give a a little bit simpler proof by applying Corollary 11. Recall that a Jordan *-isomorphis from a C^*-algebra onto another one is a complex linear bijection which preserves * and the square of the elements.

Theorem 12 ([8]). *Let A and B be unital C^*-algebras and t a positive real number. Put $d_A(a,b)$(resp. $d_B(a,b)$) $= \|\log(a^{-\frac{t}{2}}b^ta^{-\frac{t}{2}})^{\frac{1}{t}}\|$, $a,b\in A_+^{-1}$ (resp. B_+^{-1}). Suppose that $T: A_+^{-1}\to B_+^{-1}$ is a surjective isometry; $\|\log(a^{-\frac{t}{2}}b^ta^{-\frac{t}{2}})^{\frac{1}{t}}\| = \|\log(T(a)^{-\frac{t}{2}}T(b)^tT(a)^{-\frac{t}{2}})^{\frac{1}{t}}\|$, $a,b\in A_+^{-1}$. Then there exists a Jordan *-isomorphism from A onto B and a central projection $p\in B$ such that T has the form*

$$T(a) = (T(e)^{\frac{t}{2}}(pJ(a)+(e-p)J(a^{-1}))^tT(e)^{\frac{t}{2}})^{\frac{1}{t}}, \quad a\in A_+^{-1}.$$

Conversely if T has the form as the above, then

$$\|\log(a^{-\frac{t}{2}}b^ta^{-\frac{t}{2}})^{\frac{1}{t}}\| = \|\log(T(a)^{-\frac{t}{2}}T(b)^tT(a)^{-\frac{t}{2}})^{\frac{1}{t}}\|, \quad a,b\in A_+^{-1}.$$

Proof. By Corollary 11 we have that $T(a) = T(e)\oplus_t T_0(a)$, $a\in A_+^{-1}$ for an isometrical isomorphism T_0:

$$T_0((a^{\frac{t}{2}}b^ta^{\frac{t}{2}})^{\frac{1}{t}}) = (T_0(a)^{\frac{t}{2}}T_0(b)^tT_0(a)^{\frac{t}{2}})^{\frac{1}{t}}, \quad a,b\in A_+^{-1}; \tag{15}$$

and

$$\|\log(a^{-\frac{t}{2}}b^ta^{-\frac{t}{2}})^{\frac{1}{t}}\| = \|\log(T_0(a)^{-\frac{t}{2}}T_0(b)^tT_0(a)^{-\frac{t}{2}})^{\frac{1}{t}}\|, \quad a,b\in A_+^{-1}. \tag{16}$$

By (13) in Corollary 11 we have

$$T_0(a^{\frac{1}{n}}) = T_0(a)^{\frac{1}{n}} \tag{17}$$

for every positive integer n. The rest of the proof is similar to that of [7, Theorem 9]. □

Corollary 13 ([1]). *Let A and B be unital C^*-algebras and t a positive real number. Suppose that $T: A_+^{-1}\to B_+^{-1}$ is an isomorphism between the gyrocommutative gyrogroup $(A_+^{-1},\oplus_t,\otimes,\log)$ and $(B_+^{-1},\oplus_t,\otimes,\log)$. Suppose that T preserves the spectrum; $\sigma(a)=\sigma(T(a))$ for every $a\in A_+^{-1}$, where $\sigma(\cdot)$ denotes the spectrum. Then T is extended to a Jordan $*$-isomorphism from A onto B.*

Proof. Since T is an automorphism and preserves the spectrum we have

$$\sigma(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}}) = \sigma(T(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})) = \sigma(T(a)^{\frac{t}{2}}T(b)^{-t}T(a)^{\frac{t}{2}}), \quad a,b\in A_+^{-1}.$$

By the spectrum mapping theorem we have

$$\sigma(\log(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})) = \sigma(\log(T(a)^{\frac{t}{2}}T(b)^{-t}T(a)^{\frac{t}{2}})), \quad a,b\in A_+^{-1}.$$

Hence

$$\|\log(a^{\frac{t}{2}}b^{-t}a^{\frac{t}{2}})\| = \|\log(T(a)^{\frac{t}{2}}T(b)^{-t}T(a)^{\frac{t}{2}})\|, \quad a,b\in A_+^{-1}.$$

As a positive element $T(e)$ satisfies $\sigma(T(e)) = \sigma(e) = \{1\}$, we infer that $T(e) = e$. By Theorem 12 there exists a Jordan *-isomorphism J from A_+^{-1} onto $\mathscr{B}$ and a central projection $p \in \mathscr{B}$ with

$$T(a) = pJ(a) + (e-p)J(a^{-1}), \quad a \in A_+^{-1}.$$

Letting $a = e/2$ we have $T(e/2) = pJ(e/2) + (e-p)J(2e) = pe/2 + 2(e-p)e$. As $\sigma(T(e/2)) = \sigma(e/2) = \{1/2\}$ we infer that $p = e$. Therefore $T(a) = J(a)$ for every $a \in A_+^{-1}$. □

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