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SELF-ADJOINTNESS AND SYMMETRICITY OF OPERATOR MEANS

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1. INTRODUCTION

We recall that a n-monotone function on $[0, \infty)$ is a function which preserves the order on the set of all $n \times n$ positive semi-definite matrices. Moreover, if f is n-monotone for all $n \in \mathbb{N}$, then f is called operator monotone.

In the theory of operator connections by Kubo and Ando it is well-known that there is an affine order isomorphism from the class of operator connections σ onto the class of nonnegative operator monotone functions f on $(0, \infty)$ by $f(t) = I\sigma t$. A connection σ is called mean if it satisfies the normalization condition $I\sigma I = I$, which is equivalent to that the representing function f of σ satisfies $f(1) = 1$. This theory has found a number of applications in operator theory and quantum information theory. Restricting the definition of operator connections on the set of positive semi-definite matrices of order n, we can consider matrix connections of positive matrices of order n (or matrix connections of order n).

Definition 1.1. A binary operation σ on M_n^+, $(A, B) \mapsto A\sigma B$ is called a *matrix connection of order n* (or *n-connection*) if it satisfies the following properties:

(I) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.
(II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(III) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$

where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \ldots$ and A_n converges strongly to A.

A *mean* is a normalized connection, i.e. $1\sigma 1 = 1$. An *operator connection* means a connection of every order. A *n-semi-connection* is a binary operation on M_n^+ satisfying the conditions (II) and (III).

Recall that a n-monotone function f is symmetric if $f(t) = tf\left(\frac{1}{t}\right)$ and f is self-adjoint if $f(t) = \dfrac{1}{f\left(\frac{1}{t}\right)}$.

A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called an *interpolation function of order n* ([1]) if for any $T, A \in M_n$ with $A > 0$ and $T^*T \leq 1$

$$T^*AT \leq A \quad \Longrightarrow \quad T^*f(A)T \leq f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n on $\mathbb{R}_+$.

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Remark 1.2. Let $P(\mathbb{R}_+)$ be a set of all Pick functions on $\mathbb{R}_+$, P' the set of all positive Pick functions on $\mathbb{R}_+$, i.e., functions of the form

$$h(s) = \int_{[0,\infty]} \frac{(1+t)s}{1+ts} d\rho(t), \quad s > 0,$$

where ρ is some positive Radon measure on $[0,\infty]$. For $n \in \mathbb{N}$ denote by P'_n the set of all strictly positive n-monotone functions. The following properties can be found in [1], [2],[3], [12], [17] or [4], :

(i) $P' = \cap_{n=1}^{\infty} P'_n$, $P' = \cap_{n=1}^{\infty} \mathcal{C}_n$;
(ii) $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$;
(iii) $P'_{n+1} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq P'_n$, $P'_n \subsetneq \mathcal{C}_n$
(iv) $\mathcal{C}_{2n} \subsetneq P'_n$ [20];
(v) A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if $\dfrac{t}{f(t)}$ belongs to $\mathcal{C}_n$ [4, Proposition 3.5].

The following useful characterization of a function in $\mathcal{C}_n$ is due to Donoghue (see [10], [9]), and to Ameur (see [1]).

Theorem 1.3. [4, Corollary 2.4] A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a positive Pick function h on $\mathbb{R}$, such that

$$f(\lambda_i) = h(\lambda_i) \quad \text{for} \quad i = 1, \ldots, n.$$

As a consequence, Ameur gave a 'local' integral representation of every function in $\mathcal{C}_n$ as follows.

Theorem 1.4. [2, Theorem 7.1] Let A be a positive definite matrix in M_n and $f \in \mathcal{C}_n$. Then there exists a positive Radon measure $\rho_{\sigma(A)}$ on $[0,\infty]$ such that

$$f(A) = \int_{[0,\infty]} A(1+s)(A+s)^{-1} d\rho_{\sigma(A)}(s), \tag{1}$$

where $\sigma(A)$ is the set of eigenvalues of A.

Applying this representation, we give a 'local' integral formula for a connection of order n corresponding to a n-monotone function on $(0,\infty)$ Furthermore, this 'local' formula also establishes, for each interpolation function f of order $2n$, a connection σ of order n corresponding to the given interpolation function f. Therefore, it shows that the map from the n-connections to the interpolation functions of order n is injective with the range containing the interpolation functions of order $2n$.

In this note we present two topics as follows:

(1) For each $n \in \mathbb{N}$ there is an affine isomorphism from the set of matrix symmetric connections of order n onto the class of matrix symmetric n-monotone functions, which is based on [D. T. Hoa, T. M. Ho, H. Osaka, Interpolation classes and matrix means, Banach Journal of Mathematical Analysis, 9(2015), no. 3, 140-152].
(2) We characterize a class of non-selfadjoint operator means and a class of non-symmetric operator means between the harmonic mean ! and the arithmetic mean ∇ which is based on the joint work with Shuhei Wada.

2. From n-connections to P_n'

For any n-connection σ, the matrix $I_n\sigma(tI_n)$ is a scalar by [13, Theorem 3.2], and so we can define a function f on $(0,\infty)$ by

$$f(t)I_n = I_n\sigma(tI_n),$$

where I_n is the identity in M_n. Then $f \in P_n' \subsetneq C_n$. Moreover, this correspondence is injective.

Let f be a function belonging to C_n. We can define a binary operation σ on positive definite matrices in M_n by:

$$A\sigma B = A^{\frac{1}{2}} f[A^{\frac{-1}{2}} B A^{\frac{-1}{2}}] A^{\frac{1}{2}}, \quad \forall A, B > 0. \tag{2}$$

This operation satisfies the property (III) of the definition of connection.

Lemma 2.1. Let f be a positive function on $(0,\infty)$ belonging to C_n. Then there is a semi-connection of order n, σ, such that $f(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. (i.e.,a binary operation σ satisfying the axiom (II) and (III) in Definition 1.1).

Proof. We can define a binary σ by the formula (2). Because of the continuity of f (see Remark 2.2 below), we imply that σ has the property (III) in the definition. By Theorem 1.4, there exists a Radon measure ρ such that

$$A\sigma B = \int_{[0,\infty]} \frac{1+s}{s}\{(sA) : B\} d\rho(s)$$

For any positive definite matrix C of order n,

$$\begin{aligned} C(A\sigma B)C &= \int_{[0,\infty]} \frac{1+s}{s} C\{(sA) : B\} C d\rho(s) \\ &= \int_{[0,\infty]} \frac{1+s}{s}\{(sCAC) : CBC\} d\rho(s) \\ &= (CAC)\sigma(CBC). \end{aligned}$$

∎

In the proof above, we need the continuity of $f \in C_n$. Actually, we follow the definition of interpolation function in [4] and the continuity is the prior assumption for any function. However, even if we did not assume the continuity of the functions under consideration, we have

Remark 2.2. If $f \in C_n(I)$ for $n > 2$ then f is continuous on I.

Now we can state the main theorem of this section.

Theorem 2.3. For any natural number n there is an injective map Σ from the set of matrix connections of order n to $P_n' \supset C_{2n}$ associating each connection σ to the function f_σ such that $f_\sigma(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. Furthermore, the range of this map contains C_{2n}.

Proof. We have only to prove that the range of the map Σ contains C_{2n}. For any $f \in C_{2n}$, since $C_{2n} \subset C_n$, by Lemma 2.1 there is a semi-connection σ_f defined by the formula (2) and $f(t)I_n = I_n\sigma_f(tI_n)$ on $(0,\infty)$. Since $f \in C_{2n}$, by Theorem 1.4

we have that for any $0 < A \le C$ and $0 < B \le D$ there exists a Radon measure ρ on $\sigma(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) \cup \sigma(C^{\frac{-1}{2}}DC^{\frac{-1}{2}})$ such that

$$A\sigma_f B = \int_{[0,\infty]} \frac{1+s}{s}\{(sA) : B\}d\rho(s),$$

$$C\sigma_f D = \int_{[0,\infty]} \frac{1+s}{s}\{(sC) : D\}d\rho(s).$$

Since $\{(sA) : B\} \le \{(sC) : D\}$, the condition (I) satisfies. Hence σ_f is a connection of order n. Since $\Sigma(\sigma_f)(t)I_n = I_n\sigma_f(tI_n) = f(t)I_n$ for any $t \in \mathbb{R}^+$, we are done.
∎

3. Symmetric connections

As the same in [13], we can recall some notations and properties of connections as follows. Let σ be a n-connection. The *transpose* σ', the *adjoint* σ^* and the *dual* $\sigma^\perp$ of σ are defined by

$$A\sigma' B = B\sigma A, \quad A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}, \quad \sigma^\perp = \sigma'^*.$$

A connection is called *symmetric* if it equals to its transpose. Denoted by Σ_n^{sym} the set of n-monotone representing functions of symmetric n-connections, i.e., Σ_n^{sym} is the image of the set of all symmetric n-connections via the canonical map in Theorem 2.3. Then, using the same argument as in [13], we can state the following properties for any n-connection:

(1) $\sigma + \sigma'$ and $\sigma(:)\sigma'$ are symmetric.
(2) $\omega_l(\sigma)\omega_r = \sigma$; $\omega_r(\sigma)\omega_l = \sigma'$, where $A\omega_l B = A$ and $A\omega_r B = B$.
(3) The n-monotone representing function of the n-connection $\sigma(\tau)\rho$ is $f(x)g[h(x)/f(x)]$, where f, g, h are the representing functions of σ, τ, ρ in Theorem 2.3, respectively.
(4) σ is symmetric if and only if its n-monotone representing function f is *symmetric*, that is, $f(x) = xf(x^{-1})$.

Each n-connection corresponds to a positive n-monotone function belonging to Σ_n by Theorem 2.3. Therefore, combining with the observation above, we get the following.

Proposition 3.1. Let $f(x), g(x), h(x)$ belong to Σ_n. Then the following statements hold true:

(i) $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $f(x)g[h(x)/f(x)]$, $af(x) + bg(x)$ all belong to Σ_n;
(ii) $f(x) + k(x)$, $\frac{f(x)k(x)}{f(x)+k(x)}$ all belong to Σ_n^{sym}.

Corollary 3.2.

$$\mathcal{C}_{2n} \subseteq \Sigma_n \subsetneq P'_n.$$

But if we restrict our attention to the class of the symmetric, we get the following equality.

Theorem 3.3.

$$\Sigma_n^{sym} = P_n'^{sym},$$

where $P_n'^{sym}$ is the set of all symmetric functions in P'_n.

Proof. The inclusion $\Sigma_n^{sym} \subset P_n'^{sym}$ is trivial by Theorem 2.3.

Let f be a symmetric function in P_n'. We can define a binary operation on positive definite matrices of order n by

$$A\sigma B = A^{\frac{1}{2}} f[A^{\frac{-1}{2}} B A^{\frac{-1}{2}}] A^{\frac{1}{2}}.$$

For any $B \leq D$, then $A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \leq A^{\frac{-1}{2}} D A^{\frac{-1}{2}}$. Since f is n-monotone and the conjugate action preserves the order on self-adjoint matrices, we obtain

$$A^{\frac{1}{2}} f[A^{\frac{-1}{2}} B A^{\frac{-1}{2}}] A^{\frac{1}{2}} \leq A^{\frac{1}{2}} f[A^{\frac{-1}{2}} D A^{\frac{-1}{2}}] A^{\frac{1}{2}}.$$

This means $A\sigma B \leq A\sigma D$. Since f is symmetric, we also have

$$A\sigma D = D^{\frac{1}{2}} f[D^{\frac{-1}{2}} A D^{\frac{-1}{2}}] D^{\frac{1}{2}}.$$

Using this identity, we can also show that $A\sigma D \leq C\sigma D$ whenever $A \leq C$. Thus, $A\sigma B \leq A\sigma D \leq C\sigma D$ for any positive matrices A, B, C, D with $A \leq C$ and $B \leq D$. ∎

Remark 3.4. We would like to mention that even $P_{n+1}' \subsetneq P_n'$, but we still do not know whether $P_{n+1}'^{sym} \subsetneq P_n'^{sym}$ holds or not. As the first thought, we can obtain a symmetric function from the polynomial in P_{n+1}' but not in P_n' and such a function is a candidate to show $P_{n+1}'^{sym} \subsetneq P_n'^{sym}$. Unfortunately, this is not true as the following example.

4. Non-symmetric operator means

In [13] any symmetric operator mean σ satisfies $! \leq \sigma \leq \nabla$. In this section we show that there are many non-symmetric operator means σ such that $! \leq \sigma \leq \nabla$.

4.1. **Barbour transform.** In [14] for any strictly positive continuous functions on $(0, \infty)$ the Barbour path function $\phi_{\alpha,\beta,\gamma} : [0, 1] \to OM_+^1$ introduced by

$$\phi_{\alpha,\beta,\gamma}(x) = \frac{\alpha x + \beta(1-x)}{x + \gamma(1-x)}$$

and the basic properties are studied in [14], [18]. In [7] Barbour studied a function $F_x(1,t) = \phi_{t,\sqrt{t},\sqrt{t}}(x)$ which is an approximation of the exponential function t^x. We will denote a Barbour path $\phi_{\alpha,\beta,\gamma}(= \phi)$ such that $\phi(0) = f$, $\phi(\frac{1}{2}) = g$, $\phi(1) = h$ by the triple $[f, g, h]$.

Proposition 4.1. ([14]) For $f \in OM_+$ the Barbour path $[1, \frac{t+f}{1+f}, t]$ exists on OM_+^1.

The transform $\hat{}\ : OM_+ \to OM_+^1$ by $f \mapsto \frac{t+f}{1+f}$ plays an important role in the analysis of OM_+ and we call this transform the Barbour transform.

Proposition 4.2. ([14])

(1) The Barbour transform is injective and $\widehat{OM_+} = OM_+^1 \backslash \{1, t\}$.
(2) $\{f \in OM_+^1 \mid ! \leq f \leq \nabla\} = \widehat{OM_+^1}$, where $! \leq f$ means that $! \leq \sigma_f$, that is, for any positive operators A and B $A!B \leq A\sigma_f B$.

For $g \in OM_+^1$ we can define the inverse map ˇ of the Barbour transform by

$$\check{g}(t) = \frac{t-g}{g-1},$$

then $\check{g} \in OM_+$.

Using the Barbour transform we can characterize the self-adjointness and the symmetricity in OM_+.

Theorem 4.3. Let f be a positive cntinuous function on $(0,\infty)$. The folowings are equivalent.

(1) $f \in OM_+^1 \backslash \{1,t\}$ and $f = f^*$.
(2) There exists an operator monotone function $g \in OM_+$ such that $f = \sqrt{gg^*}$.
(3) There exixts an operator monotone function $g \in OM_+$ such that

$$f = \frac{t+g+g'}{1+g+g'}.$$

Remark 4.4. In [13] they asked existence of self-adjoint operator means except trivial means ω_l, ω_r, the geometric mean $\sharp$, and σ_{t^p} $(p \in [0,1])$, where $A\omega_l B = A$, $A\omega_r B = B$, $A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$ for any positive operators A and B. Using Theorem 4.3 we can construct many examples. For example, if $g(t) = \log(t+1)$, then corresponding operator means of functions $\sqrt{\log(t+1)/\log(t^{-1}+1)}$ and $\dfrac{t+\log(t+1)+t\log(t^{-1}+1)}{1+\log(t+1)+t\log(t^{-1}+1)}$ are self-adjoint. ∎

Theorem 4.5. Let f be a positive cntinuous function on $(0,\infty)$. The folowings are equivalent.

(1) $f \in OM_+^1 \backslash \{1,t\}$ and $f = f'$.
(2) There exists an operator function $g \in OM_+$ such that

$$f = g + g'.$$

(3) There exists an operator monotone functions $g \in OM_+$ such that

$$f = \frac{t-\sqrt{gg^*}}{\sqrt{gg^*}-1}.$$

Proposition 4.6. Let f be a positive continuous function on $(0,\infty)$. The followings are equivalent.

(1) $f \in OM_+^1 \backslash \{1,t\}$ and $f = f'$.
(2) There exists an operator monotone function $g \in OM_+$ such that

$$f = \frac{t+\sqrt{gg^*}}{1+\sqrt{gg^*}}$$

Proof. This follows from the same argument in Theorem 4.3 using the formula $(\hat{h})' = \hat{h}^*$ for $h \in OM_+$. ∎

5. Non self-adjoint operator means

In [13] any symmetric operator mean σ satisfies $! \leq \sigma \leq \nabla$. In this section we consider the converse problem and show that there are many non self-adjoint operator means σ such that $! \leq \sigma \leq \nabla$.

Lemma 5.1. Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous function. The followings are equivalent.

(1) $f \in OM_+$ and $f \geq f_\nabla$, that is $f(t) \geq \frac{1+t}{2}$ for $t \in (0,\infty)$.
(2) There exists an operator monotone $g \in OM_+$ and nonnegative real number $a, b \geq \frac{1}{2}$ such that $\lim_{t\to 0} g(t) = 0$, $\lim_{n\to\infty} \frac{g(t)}{t} = 0$, and

$$f(t) = a + bt + g(t) \ (t \in (0,\infty)).$$

Lemma 5.2. Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous function. The followings are equivalent.

(1) $f \in OM_+$ and $f \leq f_!$, that is, $f(t) \leq \frac{2t}{1+t}$ $(t \in (0,\infty))$.
(2) There exists an operator monotone $g \in OM_+$ and nonnegative real number $a, b \geq \frac{1}{2}$ such that $\lim_{t\to 0} g(t) = 0$, $\lim_{n\to\infty} \frac{g(t)}{t} = 0$, and

$$f(t) = \frac{t}{a + bt + g(t)} \ (t \in (0,\infty)).$$

Corollary 5.3. If $f \in OM_+^1$ and $f \leq f_!$, then $f = f_!$.

Corollary 5.4. If $f \in OM_+^1$ and $f \geq f_\nabla$, then $f = f_\nabla$.

Proposition 5.5. Suppose that $f \in OM_+$ and $f < f_!$. Then $f_! \leq \hat{f} \leq f_\nabla$ and $\hat{f}$ is not self-adjoint.

Corollary 5.6. Let a, b be nonnegative real number greater than $\frac{1}{2}$ and $g \in OM_+$ satisfying the condition (2) in Lemma 5.2. Define a function $f\colon (0,\infty) \to (0,\infty)$ by $f(t) = \frac{t}{a+bt+g(t)}$ $(t \in (0,\infty))$. Then $f \in OM_+$, $f_! \leq \hat{f} \leq f_\nabla$, and $\hat{f}$ is not self-adjoint.

Lemma 5.7. If a symmetric operator mean is self-adjoint, then $\sigma = \sharp$.

Proof. Let f be a corresponding operator monotone function to σ. Then

$$f(t) = tf(\frac{1}{t}) = \frac{1}{f(\frac{1}{t})}.$$

Hence, $f(t) = \sqrt{t}$, and $\sigma = \sharp$. ∎

Remark 5.8. From Lemma 5.7 we know that all operator means of Arithmetic mean, logarithmic mean, Harmonic mean, Heinz mean, Petz-Hasegawa mean, Lehmer mean, and Power difference mean, are non-self-adjoint.

5.1. **Non-symmetric operator means.** In this section we present an algorizum for making non-symmetric means σ such that $! \leq \sigma \leq \nabla$.

Lemma 5.9. Let f be a positive operator monotone function on $(0, \infty)$ with $f(1) = 1$. The followings are equivalent:

(1) $\sigma_{\hat{f}}$ is non-symmetric and $! \leq \sigma_{\hat{f}} \leq \nabla$,
(2) f is non-self-adjoint.

Proof. (2) $\rightarrow$ (1): Since $(\hat{f})' = \widehat{f^*}$, if f is non-self-adjoint operator monotone, $\hat{f}$ is non-symmetric, that is, $\sigma_{\hat{f}}$ is non-symmetric. We have, then, $! \leq \sigma_{\hat{f}} \leq \nabla$ by Proposition 4.2 (2).

(1) $\rightarrow$ (2): If f is self-adjoint, then $\hat{f}$ is symmetric, and a contradiction. ∎

Hence we have the following result.

Proposition 5.10.

$$\begin{aligned} &\{f \mid f: \text{ non-symmetric}, f_! \leq f \leq f_\nabla\} \\ &= \left\{\hat{f} \mid f: \text{ non-self-adjoint}\right\} \\ &= \left\{\hat{\hat{f}} \mid f: \text{ non-symmetric}\right\} \\ &\supset \left\{\hat{f} \mid f: \text{ symmetric}\right\} \setminus \{\sharp\} \end{aligned}$$

Remark 5.11. From Proposition 5.10 a non-self-adjoint positive monotone functions f with $f(1) = 1$ give non-symmetric operator mean such that $! \leq \sigma_{\hat{f}} \leq \nabla$. For examples, let $-1 \leq p \leq 2$ and ALG_p be the corresponding function to the power diffrence mean defined by

$$\mathrm{ALG}_p(t) = \begin{cases} \dfrac{p-1}{p} \dfrac{1-t^p}{1-t^{p-1}} & t \neq 1 \\ 1 & t = 1 \end{cases}$$

and the Petz-Hasegawa function f_p which is defined by

$$f_p(t) = p(p-1)\frac{(t-1)^2}{(t^p - 1)(t^{1-p} - 1)}$$

are non-self-adjoint. Hence, $\sigma_{\widehat{ALG_p}}$ and $\sigma_{\widehat{f_p}}$ are non-symmetric operator means between ! and ∇.

Using Lemmas 5.1 and 5.2 we can give non-symmetic operator means between ! and ∇.

The following should be well-known.

Corollary 5.12. Let $f \in OM_+$ such that $\sigma_f \geq \nabla$ and let $g \in OM_+$ such that $f(t) = a + bt + g(t)$ in Lemma 5.1 $(a, b \geq \frac{1}{2})$. Suppose that g is symmetric and $a \neq b$. Then $\hat{\hat{f}}$ is not symmetric and $! \leq \sigma_{\hat{\hat{f}}} \leq \nabla$.

Proof. Since g is symmetric,

$$\begin{aligned} tf(\frac{1}{t}) &= t(a + b\frac{1}{t} + g(\frac{1}{t})) \\ &= ta + b + tg(\frac{1}{t}) \\ &= ta + b + g(t). \end{aligned}$$

Hence we know that f is not symmetric because that $a \neq b$.

Therefore, by Proposition 5.10 $\hat{\hat{f}}$ is not symmetric and $! \leq \sigma_{\hat{\hat{f}}} \leq \nabla$. ∎

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