数理解析研究所講究録
第 1938 巻 2015 年 15-25

MODEL COMPLETE GENERIC GRAPHS I

HIROTAKA KIKYO
GRADUATE SCHOOL OF SYSTEM INFORMATICS
KOBE UNIVERSITY

1. INTRODUCTION

Generic structures constructed by the Hrushovski's amalgamation construction are known to have theories which are nearly model complete. If an amalgamation class has the full amalgamation property then its generic structure has a theory which is not model complete [2]. On the other hand, Hrushovski's strongly minimal structure constructed by the amalgamation construction which refuted a Zilber's conjecture has a model complete theory [4].

We have shown that the generic structure for $\mathbf{K}_f$ for 3-hypergraphs is model complete under some assumption on f [8]. In this case, the coefficient for the predimension function is 1.

In this paper, we show a similar result for binary graphs with coefficient 1/2 for the predimension function. This will be extended to any positive rational coefficient less than 1 in subsequent papers.

For a graph G, $V(G)$ will denote the set of vertices of G and $E(G)$ the set of edges of G. To see a graph G as a structure in the model theoretic sense, it is a structure in language $\{E\}$ where E is a binary relation symbol. $V(G)$ will be the universe, and $E(G)$ will be the interpretation of E.

We essentially use notation and terminology from Wagner [10].

For a set X, $[X]^n$ denotes the set of all n-element subsets of X.

For a set X, $|X|$ denotes the cardinality of X.

Suppose A is a graph. If $X \subseteq V(A)$, $A|X$ denotes the induced subgraph B of A such that $V(B) = X$. If there is no ambiguity, X denotes $A|X$. $B \subseteq A$ means that B is an induced subgraph of A. This means that B is a substructure of A in the model theoretic sense.

Definition 1.1. For a finite graph A, we define a predimension function δ as follows:

$$\delta(A) = |V(A)| - (1/2)|E(A)|.$$

Supported by JSPS KAKENHI Grant Number 25400203.

Suppose A,B,C are graphs such that $A,B \subseteq C$. AB denotes $C|(V(A) \cup V(B))$. Put

$$\delta(A/B) = \delta(AB) - \delta(B).$$

Definition 1.2. Assume that A, B are graphs such that $A \subseteq B$ and A is finite.

$A \leq B$ if whenever $A \subseteq X \subseteq B$ with X finite then $\delta(A) \leq \delta(X)$.

$A < B$ if whenever $A \subsetneq X \subseteq B$ with X finite then $\delta(A) < \delta(X)$. In this case, we say that A is *closed* in B if $A < B$. We also say that B is a *strong extension* of A.

Note that $\leq$ and $<$ are order relations.

Suppose $A < B$ and $A < C$. A graph embedding $g : B \to C$ is called a *closed* embedding of B into C over A if $g(B) < C$ and $g(x) = x$ for any $x \in A$.

With this notation, put

$$\mathbf{K}_{1/2} = \{A : \text{finite} \mid A > \emptyset\}.$$

Definition 1.3. Let $\mathbf{K} \subseteq \mathbf{K}_{1/2}$ be an infinite class. $\mathbf{K}$ has the *amalgamation property* if for any $A,B,C \in \mathbf{K}$, whenever $A < B$ and $A < C$ then there is $D \in \mathbf{K}$ such that there is a closed embedding of B into D over A and a closed embedding of C into D over A.

$\mathbf{K}$ has the *hereditary property* if for any finite graphs A,B, whenever $A \subseteq B \in \mathbf{K}$ then $A \in \mathbf{K}$.

$\mathbf{K}$ is called an *amalgamation class* if $\emptyset \in \mathbf{K}$ and $\mathbf{K}$ has the hereditary property and the amalgamation property.

Definition 1.4. Suppose $\mathbf{K} \subseteq \mathbf{K}_{1/2}$. A countable graph M is a *generic graph* of $(\mathbf{K}, <)$ if the following conditions are satisfied:

(1) If $A \subseteq M$ and A is finite then there exists a finite graph $B \subseteq M$ such that $A \subseteq B < M$.
(2) If $A \subseteq M$ then $A \in \mathbf{K}$.
(3) For any $A, B \in \mathbf{K}$, if $A < M$ and $A < B$ then there is a closed embedding of B into M over A.

If $\mathbf{K}$ is an amalgamation class then there is a generic graph of $(\mathbf{K}, <)$.

There is a smallest B satisfying (1), written $\mathrm{cl}(A)$. We have $A \subseteq \mathrm{cl}(A) < M$ and if $A \subseteq B < M$ then $\mathrm{cl}(A) \subseteq B$. The set $\mathrm{cl}(A)$ is called a *closure* of A in M. Apparently, $\mathrm{cl}(A)$ is unique for given finite set A.

In general, if A and D are graphs and $A \subseteq D$, we write $\mathrm{cl}_D(A)$ for the smallest subgraph B such that $A \subseteq B < D$.

Definition 1.5. Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing concave (convex upward) unbounded function. Assume that $f(0) \leq 0$, and $f(1) \leq 1$. Define $\mathbf{K}_f$ as follows:

$$\mathbf{K}_f = \{A \in \mathbf{K}_1 \mid B \subseteq A \Rightarrow \delta(B) \geq f(|V(B)|)\}$$

Note that if $\mathbf{K}_f$ is an amalgamation class then the generic graph of $(\mathbf{K}_f, <)$ has a countably categorical theory.

In order to discuss if a given graph is in $\mathbf{K}_f$ or not, the following definition will be convenient.

Definition 1.6. Let B be a graph and c an integer. B is called *c-normal* to f if $\delta(B) \geq f(|V(B)|+c)$. B is called *normal* to f if B is c-normal to f for some $c \geq 0$. B is called *critical to f* if it is 0-normal but not 1-normal to f.

The following lemma is immediate from the definition above, but it is very important.

Lemma 1.7. *Suppose a graph B is critical to f. Then whenever $B \subseteq C$ with $C \in \mathbf{K}_f$ then $B < C$.*

$A \in \mathbf{K}_f$ if and only if every induced subgraph of A is normal to f. If A is c-normal, $A \subseteq B$, $\delta(B/A) = 0$, $|V(B) - V(A)| \leq c$ then B is normal.

Fact 1.8. *Let $(\mathbf{K}, <)$ be an amalgamation class and M the generic graph of $(\mathbf{K}, <)$. If A and B are finite subgraphs of M and $\sigma_0 : A \to B$ is a graph isomorphism then there is a graph automorphism σ of M extending σ_0 (i.e., $\sigma|A = \sigma_0$).*

Proof. □

The following is the main theorem.

Theorem 1.9. *Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing concave unbounded function. Assume that $f(0) \leq 0$, $f(1) \leq 1$, and $f(x) + 1/2 \geq f(2x)$ for any positive integer x.*

Then $(\mathbf{K}_f, <)$ has the free amalgamation property and the generic graph of $(\mathbf{K}_f, <)$ is model complete.

In the rest of the paper, we assume that the assumption of Theorem 1.9 holds:

Assumption 1.10. (1) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone increasing concave unbounded function.
(2) $f(0) \leq 0, f(1) \leq 1$.
(3) $f(x) + 1/2 \geq f(2x)$ for any positive integer x.

Definition 1.11. Suppose X, Y are sets and $j : X \to Y$ a map. For $Z \subseteq [X]^m$, put $j(Z) = \{\{j(x_1), \dots, j(x_m)\} \mid \{x_1, \dots, x_m\} \in Z\}$.

Let B, C are graphs and assume that $X \subseteq V(B) \cap V(C)$. Let D be a graph. We write $D = B \rtimes_X C$ if the following hold:

(1) There is a 1-1 map $f : V(B) \to V(D)$.
(2) There is a 1-1 map $g : V(C) \to V(D)$.

(3) $f(x) = g(x)$ for any $x \in X$.
(4) $V(D) = f(B) \cup g(C)$.
(5) $f(B) \cap g(C) = f(X) = g(X)$.
(6) $E(D) = f(E(B)) \cup g(E(C) - E(C|X))$.

f is a graph isomorphism from B to $D|f(V(B))$ but C and $D|g(V(C))$ are not necessarily isomorphic as graphs.

If $C|X = \emptyset$, then $B \rtimes_X C$ is a graph obtained by attaching C to B at points in X. We have $\delta(B \rtimes_X C) = \delta(B) + \delta(C) - \delta(C|X)$.

In case that $B|X = C|X$, we write $B \otimes_X C$ for $B \rtimes_X C$. If $A = B|X = C|X$, then we also write $B \otimes_A C$ instead of $B \otimes_{V(A)} C$.

The following lemma is immediate.

Lemma 1.12. *Suppose $D = B \rtimes_X C$ where $X \subseteq V(B) \cap V(C)$.*

(1) *If $C|X < C$ then $B < D$.*
(2) *If $C|X \leq C$ then $B \leq D$.*

Definition 1.13. Suppose $\mathbf{K} \subseteq \mathbf{K}_{1/2}$. K has the *free amalgamation property* if whenever $A, B, C \in \mathbf{K}$ with $A < B,\ A < C$ then $B \otimes_A C \in \mathbf{K}$.

Fact 1.14. *If a class $\mathbf{K} \subseteq \mathbf{K}_{1/2}$ has the free amalgamation property then it has the amalgamation property.*

Lemma 1.15. *Suppose A, B, C are graphs such that $A \subseteq B$, $A \subseteq C$, $\delta(A) < \delta(B)$ and $\delta(A) < \delta(C)$. If B and C are normal to f then $B \otimes_A C$ is normal to f.*

Proof. Put $D = B \otimes_A C$. By symmetry, we can assume that $|V(C)| \leq |V(B)|$. Thus, $|V(D)| \leq 2|V(B)|$. Also, $\delta(D) = \delta(B) + \delta(C) - \delta(A) > \delta(B)$ since $\delta(C) - \delta(A) > 0$.

$$\begin{aligned} \delta(D) &\geq \delta(B) + 1/2 \\ &\geq f(2(|V(B)|)) \\ &\geq f(|V(D)|). \end{aligned}$$

Therefore, D is normal to f. □

Proposition 1.16. $(\mathbf{K}_f, <)$ *has the free amalgamation property.*

Proof. Suppose $A, B, C \in \mathbf{K}_f$, $A < B$, and $A < C$. Put $D = B \otimes_A C$. We can assume that $B \subseteq D,\ \ C \subseteq D,\ \ B \cap C = A$.

Suppose $U \subseteq D$. If $U \subseteq B$ or $U \subseteq C$ then $U \in \mathbf{K}_f$ since $B, C \in \mathbf{K}_f$.

Now, suppose that $U \not\subseteq B$ and $U \not\subseteq C$. Then $U = (U \cap B) \otimes_{U \cap A} (U \cap C)$, $\delta(U \cap B) > \delta(U \cap A)$, and $\delta(U \cap C) > \delta(U \cap A)$. $U \cap B$ and $U \cap C$ are normal to f since B and C are in $\mathbf{K}_f$. U is normal to f by Lemma 1.15.

Therefore, $D \in \mathbf{K}_f$. □

2. 0-EXTENSIONS

Definition 2.1. Suppose A, B are graphs such that $A \subseteq B$. B is a *0-extension of A* if $A \leq B$ and $\delta(B/A) = 0$. B is a *minimal 0-extension of A* if B is a minimal graph D such that $A \leq D$ and $\delta(D/A) = 0$. In this case, $A \subseteq U \subsetneq B$ implies $A < U$.

Definition 2.2. Let $n \geq 3$ be an integer. Let $B = \{b_0, b_1, \ldots, b_{n-1}\}$ and $F = \{c_0, c_1, \ldots, c_{n-1}\}$ be two disjoint sets of size n. A *jellyfish* is a graph J such that $V(J) = B \cup F$ and

$$E(J) = \{b_i b_{(i+1) \bmod n} \mid i = 0, 1, \ldots, n-1\} \cup \{b_i c_i \mid i = 0, 1, \ldots, n-1\}.$$

n will be called the *length* of the jellyfish. B will be called the body of the jellyfish and F the set of feet of the jellyfish. Each edge $b_i c_i$ will be called a *leg*.

For a subgraph $D \subseteq J$, put $D_B = \{x \in V(D) \mid x \in B\}$, and $D_F = \{x \in V(D) \mid x \in F\}$.

By abuse of notation, $D|D_B$ and $D|D_F$ will also be denoted by D_B and D_F respectively.

Definition 2.3. A graph W such that $V(W) = \{c_0, c_1, b\}, E(W) = \{bc_0, bc_1\}$ is called a *wedge*. We call $\{c_0, c_1\}$ the set of feet and $\{b\}$ the body. We call each edge a leg.

Definition 2.4. When we can write $C = A \rtimes_X B$, we call C an *extension of A by B*. When B is a named graph like a jellyfish or a wedge, we also call C an "extension of A by a jellyfish" or an "extension of A by a wedge."

Lemma 2.5. *Let J be a jellyfish. Suppose $D \subseteq J$. Let k be the number of vertices v in D_B such that v is not adjacent to any vertices in D_F.*

The following hold:

(1) *If $D_B = J_B$ then $\delta(D/D_F) = (1/2)k$.*
(2) *If $D_B \neq J_B$ then $\delta(D/D_F) \geq (1/2)k + 1/2$.*
(3) *If D is a proper induced subgraph of J then $D_F < D$.*
(4) *J is a minimal 0-extension of J_F.*

Proof. (1) Suppose $D_B = J_B$. Since $\delta(J/J_F) = 0$, by considering the number of deleted legs, we have $\delta(D/D_F) = \delta(D_B/D_F) = (1/2)k$.

(2) Suppose $D_B \neq J_B$. D_B is not a cycle. If D_B is connected in D and every vertices in D_B is adjacent to some vertex in D_F, then $\delta(D/D_F) = 1/2$. In general, D has at most k less edges than that of D described just above. Therefore, $\delta(D/D_F) \geq (1/2)k + 1/2$.

(3) follows from (1) and (2).

(4) follows from $\delta(J/J_F) = 0$ and (3). □

Lemma 2.6. *Suppose A is normal to f. Let D be a proper induced subgraph of a jellyfish J such that $D_F \subseteq V(A)$ and $D_B \neq \emptyset$. Put $G = A \rtimes_{D_F} D$. Then the following hold:*

(1) $\delta(A) < \delta(G)$.
(2) *Suppose $D_B = J_B$. If there are at least 2 vertices in D_B which are not adjacent to any vertices in D_F then G is normal to f.*
(3) *If $D_B \neq J_B$ then G is normal to f.*
(4) *If A is c-normal to f for some $c \geq 1$ then G is c-normal.*

Proof. (1) By Lemma 2.5 (3), $D_F < D$. Hence, $A < A \rtimes_{D_F} D$.

For the rest of the proof, let k be the number of $x \in D_B$ such that x is not adjacent in D to any $y \in D_F$, and l the number of $x \in D_B$ such that x is adjacent in D to some $y \in D_F$. We have $D_B = l + k$ and $l \leq |D_F|$. Since $D_F \subseteq V(A)$, we have $l \leq |V(A)|$. Hence, $|V(G)| = |V(A)| + |D_B| \leq 2|V(A)| + k$.

(2) Suppose $D_B = J_B$. By Lemma 2.5 (1), $\delta(D/D_F) = (1/2)k$. Hence, $\delta(G/A) = \delta(D/D_F) = (1/2)k$. Since $k \geq 2$,

$$\begin{aligned} \delta(G) &= \delta(A) + (1/2)k \\ &\geq f(|V(A)|) + (1/2)k \\ &\geq f(2^k|V(A)|) \\ &= f(2|V(A)| + (2^k - 2)|V(A)|) \end{aligned}$$

$2^k - 2 \geq k$ by $k \geq 2$. Hence, $\delta(G) \geq f(2|V(A)| + k) \geq f(|V(G)|)$.

(3) Suppose $D_B \subsetneq J_B$. By Lemma 2.5 (2), $\delta(D/D_F) \geq (1/2)k + 1/2$.

$$\begin{aligned} \delta(G) &= \delta(A) + (1/2)(k+1) \\ &\geq f(|V(A)|) + (1/2)(k+1) \\ &\geq f(2^{(k+1)}|V(A)|) \\ &= f(2|V(A)| + (2^{k+1} - 2)|V(A)|) \\ &\geq f(2|V(A)| + k) \\ &\geq f(|V(G)|). \end{aligned}$$

(4) Suppose $\delta(A) \geq f(|V(A)| + c)$ $(c \geq 1)$.
Case $D_B = J_B$.

Since D is a proper induced subgraph of J, By Lemma 2.5 (1), we have $\delta(G) \geq \delta(A) + (1/2)k$ with $k \geq 1$.

$$\begin{aligned}
\delta(G) &\geq \delta(A) + (1/2)k \\
&\geq f(|V(A)| + c) + (1/2)k \\
&\geq f(2^k|V(A)| + 2^k c) \\
&\geq f(2|V(A)| + (2^k - 2)|V(A)| + (2^k - 1)c + c) \\
&\geq f(2|V(A)| + (k-1) + 1 + c) \\
&\geq f((2|V(A)| + k) + c) \\
&\geq f(|V(G)| + c).
\end{aligned}$$

Case $D_B \neq J_B$.

We have $\delta(G) \geq \delta(A) + (1/2)(k+1)$ with $k \geq 0$. Similar argument to that for Case $D_B = J_B$ shows the same inequality. □

Lemma 2.7. *Let A be a graph with at least one vertex which is normal to f. Let P_1 be a graph with one vertex, and $P_2 = P_1 \otimes P_1$. Then $A \otimes P_1$ is $(3|V(A)| - 1)$-normal to f, and $A \otimes P_2$ is $(15|V(A)| - 2)$-normal to f. Especially, $A \otimes P_1$ is $|V(A \otimes P_1)|$-normal to f, and $A \otimes P_2$ is $|V(A \otimes P_2)|$-normal to f.*

Proof.

$$\begin{aligned}
\delta(A \otimes P_1) &= \delta(A) + 1 \\
&\geq f(|V(A)|) + 1 \\
&\geq f(2^2|V(A)|) \\
&\geq f((|V(A)| + 1) + (3|V(A)| - 1)).
\end{aligned}$$

$$\begin{aligned}
\delta(A \otimes P_2) &= \delta(A) + 2 \\
&\geq f(2^4|V(A)|) \\
&\geq f((|V(A)| + 2) + (15|V(A)| - 2)).
\end{aligned}$$

□

Lemma 2.8. *Suppose $A_1 \in \mathbf{K}_f$, $A \subseteq A_1$, $P_2 \subseteq A_1$, $A_1 = A \otimes P_2$ where P_2 is a graph with 2 vertices and no edge. Let J be a jellyfish such that $J_F = V(A_0)$ with $P_2 \subseteq A_0 \subseteq A_1$. Put $G = A_1 \rtimes_{V(A_0)} J$. Then the following hold.*

(1) *G is a 0-extension of A_1.*

(2) *If $U \subseteq G$, $U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$ then $A_0 \subseteq U$.*

(3) *$A < G$.*

(4) *$G \in \mathbf{K}_f$.*

Proof. (1) By Lemma 2.5 (4) and Lemma 1.12. (2) Suppose $U \subseteq G, U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$. If $A_0 \not\subseteq U$ then $\delta(U/U \cap A_1) > 0$ by Lemma 2.5 (3). Hence, $P_2 \subseteq A_0 \subseteq U \cap A_1$.

(3) Suppose $A \subsetneq U \subseteq G$. Note that $V(G) = V(A_1) \cup J_B$. Put $U_0 = U - A_1 \subseteq J_B$ and $U_1 = U \cap A_1$. Then

$$\delta(U/A) = \delta(U_0U_1/A) = \delta(U_0/AU_1) + \delta(U_1/A).$$

Since $A_1 \leq G$, we have $\delta(U_0/AU_1) \geq \delta(U_0/A_1) \geq 0$. Since $A < A_1$, we have $\delta(U_1/A) \geq 0$.

If $\delta(U_0/AU_1) > 0$ then $\delta(U/A) > 0$.

If $\delta(U_0/AU_1) = 0$ then $P_2 \subseteq U_1$ by (1). Therefore, $\delta(U_1/A) = \delta(P_2/A) > 0$.

(4) Suppose $U \subseteq G$.

Case $V(J) \subseteq V(U)$.

In this case, $U = (U \cap A_1) \rtimes_{V(A_0)} J$. We have $U \cap A_1 = (U \cap A) \otimes P_2$. Since a length of a jellyfish is at least 3, we have $U \cap A \neq \emptyset$. By Lemma 2.7, $U \cap A_1$ is $|U \cap A_1|$-normal. U is a 0-extension of $U \cap A_1$ and $|V(U) - V(U \cap A_1)| = |J_B| = |J_F| = |A_0| \leq |U \cap A_1|$. Hence, U is normal to f.

Case $J \not\subseteq U$.

In this case, U is an extension of $U \cap A_1$ by a proper induced subgraph D of J. If $D_B = \emptyset$ then $U \subseteq A_1 \in \mathbf{K}_f$, and thus $U \in \mathbf{K}_f$.

If $D_B \neq J_B$, U is normal by Lemma 2.6 (3).

Suppose $D_B = J_B$. If $U \cap P_2 \neq \emptyset$ then U is 1-normal to f. By Lemma 2.6 (4), U is also normal. If $U \cap P_2 = \emptyset$ then more than 2 vertices in $D_B = J_B$ are not adjacent to any vertices in D_F. U is normal to f by Lemma 2.6 (2).

Now, we have $G \in \mathbf{K}_f$. □

Lemma 2.9. *Suppose $A_1 \in \mathbf{K}_f$, $A \subseteq A_1$, $P_2 \subseteq A_1$, $A_1 = A \otimes P_2$ where P_2 is a graph with 2 vertices and no edge. Let W be a jellyfish such that $W_F = V(P_2)$. Put $G = (A_1 \rtimes_{W_F} W) \rtimes_{W_F} W$. Then the following hold.*

(1) *G is a 0-extension of A_1.*
(2) *$U \subseteq G$, $U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$ then $P_2 \subseteq U$.*
(3) *$A < G$.*
(4) *$G \in \mathbf{K}_f$.*

Lemma 2.10. *Suppose $A_1 = A \otimes P_2$ and $P_2 \subseteq A_0 \subseteq A_1$ where $A \subseteq A_1$ and P_2 is a graph with 2 vertices and no edge. Suppose further that $A_1 \subseteq B \in \mathbf{K}_f$ and B is a 0-extension of A_1. Assume also that if $U \subseteq B$, $U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$ then $P_2 \subseteq U$.*

Let J be a jellyfish such that $J_F = V(A_1)$ and put $G = B \rtimes_{J_F} J$. Then the following hold:

(1) *G is a 0-extension of A_1.*
(2) *If $U \subseteq G$, $U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$ then $P_2 \subseteq U$.*

(3) $A < G$.

(4) *If G is normal to f then $G \in \mathbf{K}_f$.*

Proof. Proof for (1) and (2) are similar to that for Lemma 2.8.

(3) Suppose G is normal to f and $U \subseteq G$. We show that U is normal to f.

Let $H = A_1 \rtimes_{V(A_1)} J$. $H \in \mathbf{K}_f$ by Lemma 2.8. We have $U = (U \cap B) \otimes_{U \cap A_1} (U \cap H)$.

If $U \subseteq B$ or $U \subseteq H$ then U is normal to f since $B, H \in \mathbf{K}_f$.

We assume that $U \cap B$ and $U \cap H$ are proper extensions of $U \cap A_1$.

Case $\delta(U \cap A_1) < \delta(U \cap B)$ and $\delta(U \cap A_1) < \delta(U \cap H)$.

Since $B \in \mathbf{K}_f$ and $H \in \mathbf{K}_f$, $U \cap B$ and $U \cap H$ are normal to f. U is normal to f by Lemma 1.15.

Case $\delta(U \cap A_1) = \delta(U \cap B)$ and $\delta(U \cap A_1) < \delta(U \cap H)$.

Let $c = |V(U \cap B) - V(U \cap A_1)|$. Since $U \cap B$ is normal, $U \cap A_1$ is c-normal. Since $\delta(U \cap A_1) < \delta(U \cap H)$, $U \cap H = (U \cap A_1) \rtimes_{D_F} D$ for some proper induced subgraph D of U. Since $c \geq 1$, $U \cap H$ is also c-normal by Lemma 2.6 (4). Therefore U is normal because $\delta(U) = \delta(U \cap H)$ and $|V(U) - V(U \cap H)| = c$.

Case $\delta(U \cap A_1) = \delta(U \cap H)$. In this case, $U \cap A_1 = A_1$, and $U \cap H = H$. Since $A_1 \leq B$, $\delta(U \cap B) \geq \delta(A_1)$. U is a 0-extension of $U \cap B$. Hence, $\delta(U) = \delta(U \cap B) \geq \delta(A_1) = \delta(B) = \delta(G)$. Since G is normal, $\delta(G) \geq f(|V(G)|) \geq f(|V(U)|)$. Therefore, U is normal to f. □

3. Model Completeness

Proposition 3.1. *Suppose $A \in \mathbf{K}_f$. There is $B \in \mathbf{K}_f$ such that $A < B$ and B is critical to f.*

Proof. Suppose $A \in \mathbf{K}_f$. By adding an isolated point to make a strong extension, we can assume that $|V(A)| \geq 1$. Let $A_1 = A \otimes P_2$ where P_2 is a graph with 2 vertices and no edge. We can assume that $P_2 \subseteq A_1$. Note that $|A_1| \geq 3$.

Let N be a largest integer x such that $\delta(A_1) \geq f(x)$. Since $A_1 \in \mathbf{K}_f$, and A_1 is not critical, $N > |A_1|$. Let $N = m|A_1| + r$ with $0 \leq r < |A_1|$.

If $r = 0$, put $B_0 = A_1$. If $r = 1$, put $B_0 = A_1 \rtimes_{V(P_2)} W$ where W is a wedge. If $r = 2$, put $B_0 = (A_1 \rtimes_{V(P_2)} W) \rtimes_{V(P_2)} W$. If $r \geq 3$, put $B_0 = A_1 \rtimes_{V(A_0)} J'$ where $P_2 \subseteq A_0 \subseteq A_1$ with $|V(A_0)| = r$, and J' is a jellyfish with $J'_F = V(A_0)$.

In any of these cases, we have the following:

- B_0 is a 0-extension of A_1;
- if $U \subseteq B_0$, $U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$ then $P_2 \subseteq U$;
- $A < B_0$; and
- $B_0 \in \mathbf{K}_f$.

Let J be a jellyfish with $J_F = V(A_1)$. For $i = 1, \dots, m-1$, put $B_i = B_{i-1} \rtimes_{V(A_1)} J$.

Then by Lemma 2.10, we have the following: For each $i = 1, \dots, m-1$,

- B_i is a 0-extension of A_1;
- if $U \subseteq B_i$, $U \not\subseteq A_1$ and $\delta(U/U \cap A_1) = 0$ then $P_2 \subseteq U$;
- $A < B_i$; and
- $B_i \in \mathbf{K}_f$.

By the construction, $|V(B_{m-1})| = N$ and $\delta(B_{m-1}) = \delta(A_1)$. Therefore, $A < B_{m-1}$ and B_{m-1} is critical to f, and $B_{m-1} < \mathbf{K}_f$. □

Now, we prove that the generic graph of $(\mathbf{K}_f, <)$ is model complete.

Proof of Theorem 1.9. Let M be a generic graph for $(\mathbf{K}_f, <)$.

Let T be the theory of M in the graph language. Since T is countably categorical, M is saturated. So, every finite type (over empty set) is realised in M. Our aim is to show that T is model compete.

Claim 1. *Every finite type realised in M is generated by a single existential formula of the graph language.*

Let A be a finite subgraph of M. We show that $\mathrm{tp}(A)$ is generated by an existential formula. Consider the closure $\mathrm{cl}(A)$ of A in M. $\mathrm{cl}(A)$ is also finite becuase M is a generic graph. By Proposition 3.1, there is $B \in \mathbf{K}_f$ such that $\mathrm{cl}(A) < B$ and B is critical to f. Since $\mathrm{cl}(A) < B$ and $\mathrm{cl}(A) < M$, we can embed B in M over $\mathrm{cl}(A)$ as a closed subset of M.

We can assume that $B \subseteq M$ and $\mathrm{cl}(A) < B < M$.

Suppose $V(A) = \{a_1, \dots, a_n\}$ and $V(B) = \{b_1, \dots, b_m\}$. Let

$$\psi(x_1, \dots, x_n, y_1, \dots, y_m) = \mathrm{qftp}(a_1, \dots, a_n, b_1, \dots, b_m)$$

be a formula representing the quantifier-free type of (A, B). Then $(a_1, \dots, a_n)$ realises an existential formula

$$\exists y_1, \dots, \exists y_m \psi(x_1, \dots, x_n, y_1, \dots, y_m).$$

Let $\varphi(x_1, \dots, x_n)$ denote this formula. We show that $\varphi(x_1, \dots, x_n)$ determines $\mathrm{tp}(a_1, \dots, a_n)$.

Let $\{c_1, \dots, c_n\} \subseteq V(M)$ be arbitrary. Assume that $(c_1, \dots, c_n)$ satisfies $\varphi(x_1, \dots, x_n)$ in M. We show that $(c_1, \dots, c_n)$ realises $\mathrm{tp}(a_1, \dots, a_n)$.

There is $d_1, \dots, d_m \in V(M)$ such that $M \models \psi(c_1, \dots, c_n, d_1, \dots, d_m)$. Then

$$\mathrm{qftp}(c_1, \dots, c_n, d_1, \dots, d_m) = \mathrm{qftp}(a_1, \dots, a_n, b_1, \dots, b_m).$$

Hence, there is a graph isomorphism σ_0 such that $\sigma_0(d_i) = b_i$ for $i = 1, \dots, m$ and $\sigma_0(c_i) = a_i$ for $i = 1, \dots, n$. Put

$$C = M|\{c_1, \dots, c_n\} \text{ and } D = M|\{d_1, \dots, d_m\}.$$

Then $\sigma_0 : D \to B$ is a graph isomorphism such that $\sigma_0|C$ is a graph isomorphism from C to A.

D is also critical to f. Then $D \subseteq U \subseteq M$ with U finite implies that $U \in \mathbf{K}_f$ and thus $\delta(D) < \delta(U)$ by Lemma 1.7. Hence D is also closed in M. Therefore, σ_0 can be extended to an graph automorphism σ of M by Fact 1.8. Hence, $\mathrm{tp}(c_1,\dots,c_n) = \mathrm{tp}(a_1,\dots,a_n)$. The claim is proved.

By the claim, every formula is equivalent to an existential formula modulo T. Therefore, T is model complete. □

ACKNOWLEDGEMENTS

The author appreciates valuable discussions with Koichiro Ikeda, Akito Tsuboi, Masanori Sawa, and Genki Tatsumi.

REFERENCES

[1] J.T. Baldwin and K. Holland, Constructing ω-stable structures: model completeness, Ann. Pure Appl. Log. **125**, 159–172 (2004)
[2] J.T. Baldwin and S. Shelah, Randomness and semigenericity, Trans. Am. Math. Soc. **349**, 1359–1376 (1997)
[3] J.T. Baldwin and N. Shi, Stable generic structures, Ann. Pure Appl. Log. **79**, 1–35 (1996)
[4] K. Holland, Model completeness of the new strongly minimal sets, J. Symb. Log. **64**, 946–962 (1999)
[5] E. Hrushovski, A stable $\aleph_0$-categorical pseudoplane, preprint, 1988.
[6] E. Hrushovski, A new strongly minimal set, Ann. Pure Appl. Log. **62**, 147–166 (1993)
[7] K. Ikeda, Simplicity of generic structures (in Japanese), Kokyuroku of RIMS **1390** (Kyoto University), 9-17 (2004).
[8] K. Ikeda, H. Kikyo, Model complete generic structures, to appear in the Proceedings of the 13th Asian Logic Conference, World Scientific.
[9] B. Kim, *Simplicity Theory*, Oxford, 2014.
[10] F.O. Wagner, Relational structures and dimensions, In *Automorphisms of first-order structures*, Clarendon Press, Oxford, 153–181 (1994).
[11] F.O. Wagner, *Simple Theories*, Kluwer, 2000.

神戸大学大学院システム情報学研究科　桔梗　宏孝

GRADUATE SCHOOL OF SYSTEM INFORMATICS, KOBE UNIVERSITY, 1-1 ROKKODAI, NADA, KOBE 657-8501, JAPAN

E-mail address: kikyo@kobe-u.ac.jp