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A SURVEY ON SOME RESULTS OF VALUED FIELDS IN RECENT MODEL THEORY

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ABSTRACT. We begin with basic theory on valued fields based on the book "Valued fields" written by A.J.Engeler, A.Prestel, published in 2005, Springer Monographs in Mathematics. And then we introduce two results on quantifier elimination of henselian valued fields having nice languages. Finally we present some results on NTP_2 related to henselian valued fields.

1. INTRODUCTION

This survey is organized as follows. In section 2 we recall the definitions of valued fields and valuation rings. In section 3 we review completions of valued fields and a rank of orderd abelian groups giving by the number of proper convex subgroups. In section 4 we discuss extensions of valued fields and a characterization of henselian valued fields. In section 5 we introduce henselizations of valued fields, inertia fields and ramification fields in the separable closure. In section 6, we present a characterization of non-trivial henselian valued fields by galois groups. In section 7, we offer a generalization of Hasse-Minkowski Principle by using henselizations instead of completions. The above sections are completely based on the book "Valued fields" [EP], we only prove easy facts and try to introduce important theorems avoiding technical lemmas in the book. In section 8, we discuss quantifier elimination. For p-adically closed fields we use Macintyre language and for henselian fields we use Denef-Pas language. In final section, we give some definitions in recent model theory and present recent results that $\mathbb{Q}_p$ is dp-minimal in a proper language and the depth of inp-pattern of henselian valued fields is bounded by the depth of inp-patterns of their value groups and residue class fields in Denef-pas language.

2. DEFINITIONS OF VALUED FIELDS, VALUATION RING

Definition 2.1. Let K be a field, Γ be an ordered abelian group. We say that (K, v, Γ) is a valued field, if $v : K \to \Gamma \cup \{\infty\}$ satisfies

(1) $v(x) = \infty \Leftrightarrow x = 0$
(2) $v(K^\times) = \Gamma$
$v(xy) = v(x) + v(y)$ for all $x, y \in K$
i.e. $v : (K^\times, \cdot) \twoheadrightarrow (\Gamma, +)$ is an epimorphism.
(3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$

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The following are easy facts.

Fact 2.2. (1) $v(\pm 1) = 0$
(2) $v(x^{-1}) = -v(x)$
(3) $v(-x) = v(x)$
(4) $v(x) < v(y) \Rightarrow v(x+y) = v(x)$

Proof. (1) : $v(1) = v(1 \cdot 1) = v(1) + v(1)$ and $0 = v(1) = v((-1) \cdot (-1)) = v(-1) + v(-1)$. (2) : $0 = v(1) = v(x \cdot x^{-1}) = v(x) + v(x^{-1})$. (3) : $v(-x) = v(-1) + v(x) = 0 + v(x) = v(x)$. (4) : $v(x+y) \geq \min\{v(x), v(y)\} = v(x)$
If $v(x+y) > v(x)$, then $v(x) = v((x+y)-y) = \min\{v(x+y), v(-y) = v(y)\} > v(x)$, a contradiction. □

Example 2.3. (1) *p-adic valuation :* $v_p : \mathbb{Q} \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$

$$v_p(p^\nu \frac{m}{n}) = \nu,$$

where p is a prime number and $p \not| m, n \in \mathbb{Z}$
(2) $p(X)$*-adic valuation :* $v_{p(X)} : k(X) \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$

$$v_p\left(p(X)^\nu \frac{f(X)}{g(X)}\right) = \nu,$$

where $p(X) \in k[X]$ *is irreducible and* $p(X) \not| f(X), g(X) \in k[X]$

Let (K, v, Γ) be a valued field. Then $\mathcal{O}_v := \{x \in K : v(x) \geq 0\}$ is a subring of K, $\mathcal{M}_v := \{x \in K : v(x) > 0\} \subset \mathcal{O}_v$ is a maximal ideal and the unit of $\mathcal{O}_v$ is $\mathcal{O}_v^\times = \mathcal{O}_v \setminus \mathcal{M}_v$. We also have $x \in \mathcal{O}_v$ or $x^{-1} \in \mathcal{O}_v$ for any $x \in K^\times$.

Definition 2.4. We say that a subring $\mathcal{O}$ of a field K is called a valuation ring of K, if $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$ for any $x \in K^\times$.

Fact 2.5. *If $\mathcal{O}$ is a valuation ring of a field K, then there exists a valuation v on K such that $\mathcal{O} = \mathcal{O}_v$.*

Proof. $\Gamma := (K^\times/\mathcal{O}^\times, +, \leq)$: an ordered abelian group as follows:

$$x\mathcal{O}^\times + y\mathcal{O}^\times := xy\mathcal{O}^\times$$
$$x\mathcal{O}^\times \leq y\mathcal{O}^\times \Leftrightarrow \frac{y}{x} \in \mathcal{O}$$

Put $v(x) := x\mathcal{O}^\times \in \Gamma$. If $v(x) \leq v(y)$, then $\frac{y}{x} \in \mathcal{O}$. As $\frac{x+y}{x} = 1 + \frac{y}{x} \in \mathcal{O}$, we have $v(x+y) \geq v(x) = \min\{v(x), v(y)\}$

We also have $x \in \mathcal{O}_v \Leftrightarrow v(x) \geq 0$ in $\Gamma \Leftrightarrow 1\mathcal{O}^\times \leq x\mathcal{O}^\times \Leftrightarrow \frac{x}{1} \in \mathcal{O}$. □

3. Completions of valued fields

Definition 3.1. Let (K, v, Γ) be a valued field and $(a_n)_{n<\omega}$ be a sequence in K.
(1) $\lim_{n\to\infty} a_n = a \Leftrightarrow$ for any $\gamma \in \Gamma$ there exists $n_0 < \omega$ such that for all $n \geq n_0$

$$v(a_n - a) > \gamma$$

(2) $(a_n)_{n<\omega}$ is a Cauchy sequence $\Leftrightarrow$ for any $\gamma \in \Gamma$ there exists $n_0 < \omega$ for all $n, m \geq n_0$

$$v(a_n - a_m) > \gamma$$

(3) (K, v, Γ) is complete, if any Cauchy sequence in K converges in K

Fact 3.2. *(Completion) Any valued field (K, v, Γ) can be embedded into a complete valued field $(\hat{K}, \hat{v}, \hat{\Gamma})$ such that*

(1) *K is dense in $\hat{K}$*
(2) $\Gamma \simeq \hat{\Gamma}$
(3) $\mathcal{O}_v/\mathcal{M}_v \simeq \mathcal{O}_{\hat{v}}/\mathcal{M}_{\hat{v}}$

Definition 3.3. Let Γ be an ordered abelian group. A subgroup $\Delta \leq \Gamma$ is convex if $\gamma \in \Gamma$ with $0 \leq \gamma \leq \delta \in \Delta$, then $\gamma \in \Delta$.

Remark 3.4. (1) Convex subgroups are linearly ordered by inclusion: If $\Delta_1, \Delta_2 \leq \Gamma$ are convex, then $\Delta_1 \leq \Delta_2$ or $\Delta_2 \leq \Delta_1$.
(2) We define the rank of Γ, $\mathrm{rk}(\Gamma) = n$ if there are exactly n-many proper convex subgroups of Γ, i.e. $\{0\} = \Delta_1 < \Delta_2 < \cdots < \Delta_n < \Gamma$ and if $\Delta < \Gamma$ is convex, then $\Delta = \Delta_i$ for some $1 \leq i \leq n$.
(3) If Γ is archimedian, then $\mathrm{rk}(\Gamma) = 1$, in particular $\mathrm{rk}(\mathbb{Z}) = 1$.

Proof. (1) : Otherwise there exist $\delta_1 \in \Delta_1 \setminus \Delta_2, \delta_2 \in \Delta_2 \setminus \Delta_1$. As $-\delta_1 \in \Delta_1 \setminus \Delta_2, -\delta_2 \in \Delta_2 \setminus \Delta_1$, we may assume $\delta_1, \delta_2 \geq 0$. Then $0 \leq \delta_i < \delta_j$ implies $\delta_i \in \Delta_j$, a contradiction. (3) : Let $\{0\} < \Delta \leq \Gamma$ be convex. Fix a $0 < \delta \in \Delta$. As Γ is archimedian, for any $0 < \gamma \in \Gamma$, there exists $n \in \mathbb{N}$ such that $0 < \gamma < n\delta \in \Delta$. So $\Delta = \Gamma$. □

Fact 3.5. (1) $\mathrm{rk}(\Gamma) = 1$ *iff Γ is order-isomorphic to a non-trivial subgroup of* $(\mathbb{R}, +, \leq)$.
(2) *The lexicographic product* $\mathrm{rk}(\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}}) = n$.

For (K, v, Γ), $\overline{K_v} := \mathcal{O}_v/\mathcal{M}_v$ is called the residue class field. For $a \in \mathcal{O}_v$, $\overline{a}$ denotes $a + \mathcal{M}_v \in \overline{K_v}$.
For $f(X) = \sum_{i=0}^{n} a_i X^i \in \mathcal{O}_v[X]$, $\overline{f}(X)$ denotes $\sum_{i=0}^{n} \overline{a_i} X^i \in \overline{K_v}[X]$

Fact 3.6. (1) *If* $\mathrm{rk}(\Gamma) = 1$ *and (K, v, Γ) is complete, then Henselian Lemma holds in (K, v, Γ). i.e. If $f(X) \in \mathcal{O}_v[X]$ and $\overline{f}(\overline{a_0}) = 0, \overline{f}'(\overline{a_0}) \neq 0$ for some $a_0 \in \mathcal{O}_v$, then there exists $a \in \mathcal{O}_v$ such that $f(a) = 0$ and $\overline{a} = \overline{a_0}$.*
(2) *It is known that the above fact does not holds in case of* $\mathrm{rk}(\Gamma) = 2$. *See Remark 2.4.6. on pp.53 in* [EP].

4. Henselian fields

Let $K_1 \subseteq K_2$ be fields, $\mathcal{O}_i \subseteq K_i (i = 1, 2)$ be valuation rings. We say $\mathcal{O}_2$ is an extension of $\mathcal{O}_1$ if $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$. We write $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ if $\mathcal{O}_2$ is an extension of $\mathcal{O}_1$.

Fact 4.1. *Let $K_1 \subseteq K_2$ be fields and $\mathcal{O}_1 \subseteq K_1$ be a valuation ring.*

(1) *There exists an extension $\mathcal{O}_2 \subseteq K_2$ of $\mathcal{O}_1$. See Theorem 3.1.1 on pp.57 in* [EP].
(2) *If $(K_1, \mathcal{O}_1) \subset (K_2, \mathcal{O}_2)$ (i.e. $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$), we have*
   (a) $\mathcal{M}_2 \cap \mathcal{O}_1 = \mathcal{M}_1$
   (b) $\mathcal{O}_2^\times \cap K_1 = \mathcal{O}_1^\times$

Proof. We only prove (2). For (a): As $\mathcal{M}_2 \cap \mathcal{O}_1 \subseteq \mathcal{O}_1$ is a ideal and $\mathcal{M}_1$ is a maximal ideal of $\mathcal{O}_1$, $\mathcal{M}_2 \cap \mathcal{O}_1 \subseteq \mathcal{M}_1$. If $x \in \mathcal{M}_1 \setminus (\mathcal{M}_2 \cap \mathcal{O}_1)$. then $x^{-1} \notin \mathcal{O}_1$, so $x \in \mathcal{O}_1$, a contradiction.
For (b): $\mathcal{O}_2^\times \cap K_1 = \mathcal{O}_2 \cap \mathcal{M}_2^c \cap K_1 = \mathcal{O}_2 \cap K_1 \cap \mathcal{M}_2^c = \mathcal{O}_1 \cap \mathcal{M}_2^c$ (as $\mathcal{O}_2 \cap K_1 = \mathcal{O}_1$) $= \mathcal{O}_1 \setminus \mathcal{O}_1 \cap \mathcal{M}_2 = \mathcal{O}_1 \setminus \mathcal{M}_1$ (by (a)) $= \mathcal{O}_1^\times$. □

By (a) $\mathcal{M}_2 \cap \mathcal{O}_1 = \mathcal{M}_1$, we have

$$\overline{K_1} = \mathcal{O}_1/\mathcal{M}_1 \hookrightarrow \mathcal{O}_2/\mathcal{M}_2 = \overline{K_2}$$

By (b) $\mathcal{O}_2^\times \cap K_1 = \mathcal{O}_1^\times$, we have

$$\Gamma_1 \simeq K_1^\times/\mathcal{O}_1^\times \hookrightarrow K_2^\times/\mathcal{O}_2^\times \simeq \Gamma_2$$

We call $e(\mathcal{O}_2/\mathcal{O}_1) := [\Gamma_2 : \Gamma_1]$ the ramification index, and $f(\mathcal{O}_2/\mathcal{O}_1) := [\overline{K_2} : \overline{K_1}]$ the residue degree.

Fact 4.2. *If* $[K_2 : K_1] = n < \omega$ *then*

$$e(\mathcal{O}_2/\mathcal{O}_1)f(\mathcal{O}_2/\mathcal{O}_1) \leq n.$$

Let K^s be separable closure of K.

Fact 4.3. *(Finite multiplicity)*
Let L be an algebraic extension of K and suppose that $[L \cap K^s : K] < \omega$. Let $\mathcal{O}$ be a valuation ring of K. THEN $|\{\mathcal{O}' : (K, \mathcal{O}) \subseteq (L, \mathcal{O}')\}| \leq [L \cap K^s : K] < \omega$. In particular if L/K is purely inseparable, the extension of $\mathcal{O}$ to L is unique. (As $[L \cap K^s : K] = 1$) Since $L/L \cap K^s$ is purely inseparable, the extension of a valuation ring of $L \cap K^s$ to L is unique. Recall that $\mathrm{dcl}(K) = K_{ins}/K$ *is purely inseparable in the field language.*

Theorem 4.4. *(Conjugation Theorem)*
Suppose that N/K is NORMAL. ($\sigma(N) = N$ for any $\sigma \in$ $\mathrm{Aut}(\widetilde{K}/K)$, *where $\widetilde{K}$ is an algebraic closure of K) If $(K, \mathcal{O}) \subseteq (N, \mathcal{O}'), (N, \mathcal{O}'')$, then there exists $\sigma \in$* $\mathrm{Aut}(N/K)$ *such that $\sigma(\mathcal{O}') = \mathcal{O}''$. Moreover⋯*

(1) *Let v', v'' be valuations on N such that $\mathcal{O}' = \mathcal{O}_{v'}, \sigma(\mathcal{O}') = \mathcal{O}'' = \mathcal{O}_{v''}$. Then $v'' = v' \circ \sigma^{-1}$*
(2) $e(\mathcal{O}'/\mathcal{O}) = e(\mathcal{O}''/\mathcal{O}), f(\mathcal{O}'/\mathcal{O}) = f(\mathcal{O}''/\mathcal{O})$
(3) $\overline{N_{v'}}/\overline{K_v}$ *is also NORMAL, where v is a valuation on K such that $\mathcal{O} = \mathcal{O}_v$.*

Fact 4.5. *Let $(K, \mathcal{O})$ be a valued field. Then the following are equivalent, and such a valued field is called henselian.*

(1) *Henselian Lemma holds in $(K, \mathcal{O})$: If $f(X) \in \mathcal{O}[X]$ and $\overline{f}(\overline{a_0}) = 0, \overline{f'}(\overline{a_0}) \neq 0$ for some $a_0 \in \mathcal{O}$, then there exists $a \in \mathcal{O}$ such that $f(a) = 0$ and $\overline{a} = \overline{a_0}$.*
(2) *If L/K is algebraic, then $\mathcal{O}$ has a unique extension to L. (cf. It is known that $\mathcal{O}$ has many extensions to $K(X)$ if $K(X)/K$ is transcendental.)*

Remark 4.6. $(K, \mathcal{O})$ is henselian $\Leftrightarrow$ $\mathcal{O}$ has a unique extension to K^s.

Proof. ($\Rightarrow$) is clear.
($\Leftarrow$) : Let L/K be algebraic and suppose that $\mathcal{O}$ has a unique extension $\mathcal{O}^s$ to K^s, then $\mathcal{O}$ has a unique extension $\mathcal{O}^s \cap L \cap K^s$ to $L \cap K^s$. As $L/L \cap K^s$ is purely inseparable, $\mathcal{O}^s \cap L \cap K^s$ has a unique extension $\mathcal{O}_L$ to L, and $\mathcal{O}_L$ is a unique extension of $\mathcal{O}$ to L.

5. Henselization of valued field

Let $G(K^s/K)$ denotes the galois group of K^s over K. $G(K^s/K)$ is a profinite group, a compact Hausdorff totally disconnected topological group.
For a valued field $(K, \mathcal{O})$ and an extension $\mathcal{O}^s$ of $\mathcal{O}$ to K^s, we have the following.

Fact 5.1. *(Henselization $K^h(\mathcal{O}^s)$ of $(K, \mathcal{O}^s)$)*

(1) $G^h(\mathcal{O}^s) := \{\sigma \in G(K^s/K) : \sigma(\mathcal{O}^s) = \mathcal{O}^s\} \le G(K^s/K)$ *is closed.*

(2) $K^h(\mathcal{O}^s) := \mathrm{Fix}(G^h(\mathcal{O}^s))$ *is henselian and the residue fields of K and $K^h(\mathcal{O}^s)$ are same and so are value groups of K and $K^h(\mathcal{O}^s)$.*

(3) *If $(K_1, \mathcal{O}_1)$ is a henselian extension of $(K, \mathcal{O})$, then there exists a K-embedding*

$$\iota : (K^h(\mathcal{O}^s), \mathcal{O}^s \cap K^h(\mathcal{O}^s) \hookrightarrow (K_1, \mathcal{O}_1)$$

i.e. $\iota(\mathcal{O}^s \cap K^h(\mathcal{O}^s)) = \mathcal{O}_1 \cap \iota(K^h(\mathcal{O}^s))$.

Proof. Here we only check that $K^h(\mathcal{O}^s) := \mathrm{Fix}(G^h(\mathcal{O}^s))$ is henselian.
Recall that $(K, \mathcal{O})$ is henselian iff $\mathcal{O}$ has a unique extension to K^s. And we have $K^s/K^h(\mathcal{O}^s)/K$ as $\{\mathrm{id}\} \le G^h(\mathcal{O}^s) \le G(K^s/K)$. So, if $(K^s, \mathcal{O}^s), (K^s, \mathcal{O}') \supseteq (K^h(\mathcal{O}^s), \mathcal{O}^s \cap K^h(\mathcal{O}^s))$,then we need to show that

$$\mathcal{O}^s = \mathcal{O}'$$

By conjugation theorem on normal extensions, there exists $\sigma \in G(K^s/K^h(\mathcal{O}^s))$ such that

$$\sigma(\mathcal{O}^s) = \mathcal{O}'.$$

As $G(K^s/K^h(\mathcal{O}^s))$=$G^h(\mathcal{O}^s)$ we have $\mathcal{O}' = \sigma(\mathcal{O}^s)$=$\mathcal{O}^s$ as desired. □

Theorem 5.2. *(More on conjugation theorem)*
Let N/K be normal and $(N, \mathcal{O}') \supseteq (K, \mathcal{O})$. If $\sigma \in \mathrm{Aut}(N/K)$ be such that $\sigma(\mathcal{O}') = \mathcal{O}'$, then put $\overline{\sigma}(x + \mathcal{M}') := \overline{\sigma(x)} = \sigma(x) + \mathcal{M}'$ for each $x \in \mathcal{O}'$. Then we have $\overline{\sigma} \in \mathrm{Aut}(\overline{N}/\overline{K})$.

As $\sigma \in G^h(\mathcal{O}^s) \le G(K^s/K)$ satisfies $\sigma(\mathcal{O}^s) = \mathcal{O}^s$, we have $\overline{\sigma} \in G(\overline{K^s}/\overline{K})$. Then we have the following fact.

Fact 5.3. (1) $\overline{*} : G^h(\mathcal{O}^s) \twoheadrightarrow G(\overline{K^s}/\overline{K})$ *is continuous epimorphism, where* $\sigma \mapsto \overline{\sigma}$

(2) $G^t(\mathcal{O}^s) := \ker(\overline{*}) \trianglelefteq G^h(\mathcal{O}^s)$ *is closed.*

(3) $G^h(\mathcal{O}^s)/G^t(\mathcal{O}^s) \simeq G(\overline{K^s}/\overline{K})$.

(4) *We call $K^t(\mathcal{O}^s) := \mathrm{Fix}(G^t(\mathcal{O}^s))$ the inertia field of $\mathcal{O}^s$ over K.*
We also have $K^s/K^t(\mathcal{O}^s)/K^h(\mathcal{O}^s)/K$ as $\{\mathrm{id}\} \le G^t(\mathcal{O}^s) \le G^h(\mathcal{O}^s) \le G(K^s/K)$.
Moreover if $K^t(\mathcal{O}^s)/N/L/K^h(\mathcal{O}^s)$ with $[N : L] < \omega$, then the ramification index

$$e(\mathcal{O}^s \cap N/\mathcal{O}^s \cap L) = 1$$

and $f(\mathcal{O}^s \cap N/\mathcal{O}^s \cap L) = [N : L]$, so N/L is an UNRAMIFIED extension.

Theorem 5.4. *(A part of conjugation theorem)*
Let N/K be normal. Let v', v'' be valuations on N such that $\mathcal{O}' = \mathcal{O}_{v'}, \sigma(\mathcal{O}') = \mathcal{O}'' = \mathcal{O}_{v''}$. Then $v'' = v' \circ \sigma^{-1}$.

So if $\sigma \in G(K^s/K)$ and $\sigma(\mathcal{O}^s) = \mathcal{O}^s$ i.e. $\sigma \in G^h$, then $v^s \circ \sigma = v^s$, where $v^s : K^s \twoheadrightarrow \Gamma^s$ be a valuation corresponding to $\mathcal{O}^s$. So $v^s(x) = v^s(\sigma(x))$ for all $x \in (K^s)^\times$.
In particular, For any $\sigma \in G^h, x \in (K^s)^\times$,

$$\frac{\sigma(x)}{x} \in (\mathcal{O}^s)^\times, \overline{\frac{\sigma(x)}{x}} \in (\overline{K^s})^\times$$

Fact 5.5. *Let $v^t : K^t \twoheadrightarrow \Gamma^t$ be a valuation corresponding to $\mathcal{O}^t$.
Then there exists a well-defined epimorphism $\psi : G^t \twoheadrightarrow \mathrm{Hom}(\Delta^s/\Delta^t, (\overline{K^s})^\times)$,*

$$\psi(\sigma)(\delta + \Delta^t) = \overline{\frac{\sigma(x)}{x}} \in (\overline{K^s})^\times$$

*, where $v^s(x) = \delta \in \Delta^s$ and $x \in (K^s)^\times$.
And $G^v = \ker(\psi) \trianglelefteq G^t$ is closed.
We have*

$$G^t/G^v \simeq \mathrm{Hom}(\Delta^s/\Delta^t, (\overline{K^s})^\times)$$

*We call $K^v := \mathrm{Fix}(G^v)$ the ramification field of $\mathcal{O}^s$ over K.
As we $G^v \trianglelefteq G^t \trianglelefteq G^h \leq G(K^s/K)$, we have*

$$K^s/K^v/K^t/K^h/K.$$

It is known that K^v/K^h is galois and if $K^v/N/L/K^t$ and $[N/L] < \omega$ then the ramificatin index

$$e(\mathcal{O}^s \cap N/\mathcal{O}^s \cap L) = [N : L]$$

and $f(\mathcal{O}^s \cap N/\mathcal{O}^s \cap L) = 1$, so N/L is an RAMIFIED extension.

Compare that if $K^t/N/L/K^h$ and $[N/L] < \omega$ then the ramificatin index $e(\mathcal{O}^s \cap N/\mathcal{O}^s \cap L) = 1$ and $f(\mathcal{O}^s \cap N/\mathcal{O}^s \cap L) = [N : L]$, N/L is unramified.

6. Galois characterization of henselian fields

There exist a non-trivial henselian valued field K, a field L without any non-trivial henselian valuation such that

$$G(K^s/K) \simeq G(L^s/L)$$

in the following each case. (See pp.136-137 in [EP])

(1) Γ_K is divisible.
(2) Γ is p-divisible for any prime $p \neq ch(\overline{K})$.
(3) $G(\overline{K}^s/\overline{K}) \neq \{\mathrm{id}\}$ and $(\Gamma_K : p\Gamma_K) = p \neq ch(\overline{K})$

By informations on galois group of K, it is hard to see whether a non-trivial henselian valuation on K exists or not, but for the following well-extracted valued fields, so-called "tamely branching valued fields at p", we have a characterization on the existence of a non-trivial henselian valuation.

Excluding the above bad cases (1), (2), (3), we define the following.

Definition 6.1. We say that (K, v, Γ) is tamely branching at p, if $p \neq ch(\overline{K})$ and Γ is not p-divisible, and if $(\Gamma, p\Gamma) = p$ then $p^\infty | G(\overline{K}^s/\overline{K})$.

Recall that a profinite group $G := \varprojlim G_i$ is said to be divided by p^∞ (we write as $p^\infty | G$), if for any $n \in \mathbb{N}$, p^n divides $|G_i|$ for some i

Theorem 6.2. *The following are equivalent.*

(1) *K has a non-trivial henselian valuation, tamely branching at p.*
(2) *$G(K^s/K)$ has a non-procyclic p-Sylow subgroup $P \not\cong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ having a non-trivial abelian normal closed group $A \trianglelefteq P$*

Recall some definitions for the above theorem : A profinite group $G := \varprojlim G_i$ is said to be procyclic, if each G_i is cyclic. A subgroup P is said to be p-Sylow in a profinite group G if P is a maximal closed subgroup of G such that if p^n divides G, so does P.

7. Local-Global Principle for weak isotropy

$(K, \leq)$ is semiordered if $(K, +)$ is an ordered abelian group and if $0 \leq a$ then $0 \leq ab^2$ for each $a, b \in K$.

$(K, \leq)$ is ordered if $(K, +)$ is an ordered abelian group and if $0 \leq a, b$ then $0 \leq ab$ for each $a, b \in K$. Then

$$\sum K^2 := \left\{ \sum_{i=1}^{n} x_i^2 : x_i \in K, n < \omega \right\} \subseteq \{x \in K : x \geq 0\}.$$

Let $\rho = (a_1, \cdots, a_n)$, where $a_i \in K \setminus \{0\}$ for each $1 \leq i \leq n$.
We say that ρ is weakly isotropic in K, if there exist $\sigma_1, \cdots, \sigma_n \in \sum K^2$ such that $\sum_{i=1}^n a_i\sigma_i$ and $(\sigma_1, \cdots, \sigma_n) \neq (0, \cdots, 0)$. If $\sigma_i \in K^2$ for each i, then ρ is said to be isotropic in K.

The following is a classical well-known result, Hasse-Minkowski Principle: ρ is isotropic in $\mathbb{Q}$ if and only if ρ is isotropic in $\mathbb{R}$ and in $\mathbb{Q}_p$ for all prime p, where $\mathbb{R}$ and $\mathbb{Q}_p$ are completions of $(\mathbb{Q}, |*|), (\mathbb{Q}, v_p(*))$ respectively.

A generalization of H-M Principle by using HENSELIZATIONS instead of completions is the following theorem.

Theorem 7.1. *(Bröcker-Prestel Local-Global Principle for weak isotropy)*
Let $(K, \leq)$ be an ordered field and $\rho = (a_1, \cdots, a_n)$, where $a_i \in K \setminus \{0\}$ for each $1 \leq i \leq n$. Then the following are equivalent.

(1) *ρ is weakly isotropic in K*
(2) *ρ is weakly isotropic in $\mathbb{R}$ for every embedding of K into $\mathbb{R}$ and ρ is weakly isotropic in every henselization (K^h, v^h) of (K, v), where v is a non-trivial valuation on K such that its residue class field $\overline{K_v}$ is semiordered.*

8. Quantifier elimination and languages of valued fields

Definition 8.1. We say that a valued field (K, v) is p-adically closed, if

(1) (K, v) is henselian.
(2) $\overline{K} = \mathbb{F}_p$.
(3) $v(K)$ is discrete with $v(p)$ as minimal positive element.
(4) $v(K)/v(p)\mathbb{Z}$ is divisible.

For each valued field (K, v), the following language $\mathcal{L}_{\text{Mac}}$ is given by A.Macintyre: $\mathcal{L}_{\text{Mac}}$:=the field language $\cup\{V(x)\} \cup \{P_n(x) : 1 < n \in \omega\}$, where $V(K) = \mathcal{O}_v$ and $P_n(K) = \{x \in K : \exists y \in K(x = y^n)\}$.

We say that K is p-adic if K is a $\mathcal{L}_{\text{Mac}}$-substructure of a p-adically closed field.
We have the following.

Fact 8.2. (1) *If K is p-adically closed, then* $\mathrm{Th}(K)_{\mathcal{L}_{\mathrm{Mac}}}$ *admits quantifier elimination.* [M]

(2) *If K is a p-adic field and* $\mathrm{Th}(K)_{\mathcal{L}_{\mathrm{Mac}}}$ *admits quantifier elimination, then K is p-adically closed.* [MMvdD]

The following language is called Denef-Pas language : there are tree sorts, the field sort K, the residue class field sort $\overline{K}$ and the value group sort Γ. The field sort and the residue sort use the ring language and Γ uses the order abelian group language and one constant symbol ∞. Moreover there are two cross sort function symbol $v : K \to \Gamma \cup \{\infty\}$ which stands for the valuation and $ac : K \to \overline{K}$ which stands for an angular component map which satisfies the following conditions.

(1) $ac(0) = 0$
(2) $ac|K^{\times} : (K^{\times}, \cdot) \to (\overline{K}^{\times}, \cdot)$ is a homomorphism.
(3) $ac(x) = x + \mathcal{M}_v$ where $x \in \mathcal{O}_v \setminus \mathcal{M}_v$.

$\mathcal{L}_{\mathrm{RRPr}}$ denotes the expanded language of Denef-Pas language whose the value group sort uses the Presburger language $\{+, <, 0, 1\} \cup \{D_n(x) : 1 < n \in \omega\}$.

Fact 8.3. *Let $S = (K, \overline{K}, \Gamma \cup \{\infty\}, v, ac)$ be an $\mathcal{L}_{\mathrm{RRPr}}$-structure.*

(1) *If K is henselian and* $\mathrm{ch}(K) = \mathrm{ch}(\overline{K}) = 0$, *then* $\mathrm{Th}(S)_{\mathcal{L}_{\mathrm{RRPr}}}$ *admits quantifier elimination in the K-sort.* [P]

(2) *If* $\mathrm{ch}(K) = \mathrm{ch}(\overline{K})$, *$\Gamma$ has a minimal positive element γ and $\Gamma/\gamma\mathbb{Z}$ is divisible, and* $\mathrm{Th}(S)_{\mathcal{L}_{\mathrm{RRPr}}}$ *admits quantifier elimination in the K-sort, then (K, v) is henselian.* [Y]

9. Some recent results on valued fields in model theory

We review some definitions in pure model theory.

Definition 9.1. Let $p(x)$ be a partial type over A.

(1) We define the dp-rank of $p(x)$, denoted $\mathrm{dprk}(p(x))$, be the supremum of κ for which there exist $b \models p(x)$ and mutually indiscernible sequence $(a_\alpha)_{\alpha<\kappa}$ over A such that none of them is indiscernible over bA.

(2) We say that there is an ict-pattern of depth κ in $p(x)$, if there exist an array $(a_{i,j})_{i<\kappa, j<\omega}$ and a sequence of formulas $(\varphi_i(x, y_i) : i < \kappa)$ such that $p(x) \cup \{\varphi_i(x, a_{i,s(i)}) : i < \kappa\} \cup \{\neg\varphi_i(x, a_{i,j}) : i < \kappa, j \neq s(i)\}$ is consistent for each $s : \kappa \to \omega$.

(3) We say that there is an inp-pattern of depth κ in $p(x)$, if there exists an array $(a_{i,j})_{i<\kappa, j<\omega}$ a sequence of formulas $(\varphi_i(x, y_i) : i < \kappa)$ and $\{k_i < \omega : i < \kappa\}$ such that
   (a) $\{\varphi_i(x, a_{i,j}) : j < \omega\}$ is k_i-inconsistent for each $i < \kappa$.
   (b) $\{\varphi_i(x, a_{i,s(i)} : i < \kappa\} \cup p(x)$ is consistent for each $s : \kappa \to \omega$.

(4) We define the burden of $p(x)$, denoted $\mathrm{bdn}(p(x))$, be the supremum of the depths of all inp-patterns in $p(x)$.

(5) Let T be a theory. For $n < \omega$, $\kappa^n_{inp}(T)$ denotes the smallest cardinal such that there is no inp-pattern $((a_{i,j})_{j<\omega}, \varphi_i(x, y_i), k_i)_{i<\kappa}$ of depth κ with $lh(x) \leq n$.

Remark 9.2. Let $p(x)$ be a partial type.

(1) $\mathrm{bdn}(p(x)) \leq \mathrm{dprk}(p(x))$. See Proposition 10 in [A].

(2) $\mathrm{dprk}(p(x)) > \kappa$ if and only if there is an ict-pattern of depth κ in $p(x)$. See Proposition 2.6 in [KOU].
(3) If $\kappa^n_{inp}(T)$ is infinite for some $n < \omega$, then $\kappa_{inp}(T) := \sup_{n<\omega} \kappa^n_{inp}(T) = \kappa^n_{inp}(T) = \kappa^1_{inp}(T)$. See Corollary 2.9 in [C].

Now we mention recent results on valued fields in model theory.

Fact 9.3. [DGL] $\mathcal{L}_{\mathrm{vf}}$ *:=the ring language* $\cup\{v(x) \le v(y)\}$. *Then* $\mathrm{Th}(\mathbb{Q}_p)_{\mathcal{L}_{\mathrm{vf}}}$ *is* dp-*minimal i.e.* $\mathrm{dprk}(x = x) = 1$ *in* $\mathrm{Th}(\mathbb{Q}_p)_{\mathcal{L}_{\mathrm{vf}}}$. *See section 6 in* [DGL].

Definition 9.4. Let T be a theory.

(1) T is independent if there exists $\varphi(x, y)$, $\{a_i : i < \omega\}$ and $\{b_s : s \subseteq \omega\}$ such that $\models \varphi(a_i, b_s)$ if and only if $i \in s$.
(2) T is dependent if T is not independent.
(3) T has TP_2 (the tree property of the second kind) if there exists $\varphi(x, y), k \in \omega$ and an array $(a_{i,j} : i, j < \omega)$ such that
   (a) $\{\varphi_i(x, a_{i,j}) : j < \omega\}$ is k-inconsistent for each $i < \omega$.
   (b) $\{\varphi_i(x, a_{i,s(i)}) : i < \omega\}$ is consistent for each $s : \omega \to \omega$.
(4) T is NTP_2 if it does not have TP_2.

Remark 9.5. (1) T is dependent $\Leftrightarrow$ $\mathrm{dprk}(p(x)) < |T|^+$ for any partial type $p(x) \Leftrightarrow \mathrm{dprk}(p(x)) < \infty$ for any partial type $p(x)$. See Fact.2.6 in [OU].
(2) T is $\mathrm{NTP}_2 \Leftrightarrow \mathrm{bdn}(p(x)) < |T|^+$ for any partial type $p(x) \Leftrightarrow \mathrm{bdn}(p(x)) < \infty$ for any partial type $p(x) \Leftrightarrow \kappa_{inp}(T) \le |T|^+$. See Lemma 3.2 in [C].
(3) If T is dependent, then $\kappa_{inp}(T) \le |T|^+$, so T is NTP_2. See Proposition 10 in [A].

Definition 9.6. [C] For a finite set of formulas, $R(\kappa, \Delta)$ denotes the minimal length of a sequence of singletons sufficient for the existence of a Δ-indiscernible subsequences of length κ. Then we have $R(n, \Delta) < \omega$ by finite Ramsey theorem, $R(\omega, \Delta) = \omega$ by infinite Ramsey theorem, and $R(\kappa^+, \Delta) \le \beth_\omega(\kappa)$ by Erdös-Rado theorem.

Fact 9.7. [C] *Let* $S = (K, \Gamma \cup \{\infty\}, \overline{K}, v, ac)$ *be a henselian valued field with* $\mathrm{ch}(K) = \mathrm{ch}(\overline{K}) = 0$ *in the Denef-Pas language. THEN we have*

$$\kappa^1_{inp}(S) \le R(\kappa^1_{inp}(\overline{K}) \times \kappa^1_{inp}(\Gamma) + 2, \Delta)$$

for some finite set Δ *of formulas. As any ordered abelian group is NIP, so we always have* $\kappa^1_{inp}(\Gamma) \le |T|^+$.

(1) *If* $\overline{K}$ *is* NTP_2, *then* S *is* NTP_2, *because* $\kappa^1_{inp}(\overline{K}) \times \kappa^1_{inp}(\Gamma) \le |T|^+$, *so we have* $\kappa^1_{inp}(S) \le R(|T|^+ + 2, \Delta) < \beth_\omega(|T|^+) < \infty$.
(2) *If* $\overline{K}$ *and* Γ *are strong (i.e.* $\kappa^1_{inp}(\overline{K}), \kappa^1_{inp}(\Gamma) \le \omega$*), then* S *is strong.*
(3) *If* $\overline{K}$ *and* Γ *have finite burden (i.e.* $\kappa^1_{inp}(\overline{K}), \kappa^1_{inp}(\Gamma) < \omega$*), then* S *has finite burden.*
(4) *If* $\overline{K}$ *and* Γ *are strongly dependent (i.e.* $\kappa^1_{inp}(\overline{K}), \kappa^1_{inp}(\Gamma) \le \omega$ *and* $\overline{K}$ *and* Γ *are dependent), then* S *is stronly dependent, because it is known that if* $\overline{K}$ *is dependent, then* S *is dependent by Delon's theorem.*

Example 9.8. (1) *Let* $S = (K = \prod_{p:\mathrm{prime}} \mathbb{Q}_p/\mathcal{U}, \Gamma \cup \{\infty\} = \mathbb{Z} \cup \{\infty\}, \overline{K} = \prod_{p:\mathrm{prime}} \mathbb{F}_p/\mathcal{U}, v, ac)$. *As* $\overline{K} = \prod_{p:\mathrm{prime}} \mathbb{F}_p/\mathcal{U}$ *is pseudofinite and any pseudofinite field is pseudo-algebraically closed and not separably closed, so* $\overline{K}$

has independence property by Duret's theorem. As Γ *has strict order property and* $\Gamma \simeq K^{\times}/\mathcal{O}_v^{\times}$ *and it is known that* $\mathcal{O}_p$ *is definable in* $\mathbb{Q}_p$ *in the field language, uniformly in p, it follows that* $\mathcal{O}_v$ *is definable in* K *in the field language, so* S *has independence property and strict order property in the field language. On the other hand, as* $\overline{K} = \prod_{p:\text{prime}} \mathbb{F}_p/\mathcal{U}$ *and* $\Gamma = \mathbb{Z}$ *have finite burden, so* S *has finite burden in the Denef-Pas language.*

(2) *Let* K *be a field and* Γ *an ordered group.* $K((\Gamma))$ *denotes the set of formal power series* $f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$, *where* $a_\gamma \in K$ *for each* $\gamma \in \Gamma$ *and the support of* $f : \mathrm{supp}(f) = \{\gamma \in \Gamma : a_\gamma \neq 0\}$ *is well-ordered. For* $f = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma, g = \sum_{\gamma \in \Gamma} b_\gamma t^\gamma$, *addition* $f + g = \sum_{\gamma \in \Gamma} (a_\gamma + b_\gamma) t^\gamma$ *and muliplication* $f \cdot g = \sum_\gamma \left(\sum_{\delta + \epsilon = \gamma} a_\delta b_\epsilon \right) t^\gamma$ *are well-defined and* $K((\Gamma))$ *is a field. Put* $v(0) := \infty$ *and* $v(f) := \min(\mathrm{supp}(f))$, *then* $(K((\Gamma)), v, \Gamma)$ *is a henselian valued field with the residue class field* K *(see pp.82,83,92 in* [EP]*). So if* K *is an* NTP$_2$ *field, then* $S = (K((\Gamma)), \Gamma \cup \{\infty\}, K, v, ac)$ *is* NTP$_2$.

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