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HÖLDER REGULARITY OF LIMIT STATE FUNCTIONS IN RANDOM COMPLEX DYNAMICAL SYSTEMS

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ABSTRACT. We study the Hölder regularity of limit state functions of random complex dynamical systems on the Riemann sphere. We employ the multifractal formalism in ergodic theory to investigate the spectrum of Hölder exponents of these functions, which gives rise to a gradation between chaos and order in random complex dynamical systems.

1. INTRODUCTION AND STATEMENT OF RESULTS

Random complex dynamical systems were first studied by J. E. Fornaess and N. Sibony ([FS91]). For the recent studies on random complex dynamical systems we refer to the second author's works [Sum11, Sum13, Sum14]. The study of random complex dynamical systems is deeply related to the dynamics of semigroups of rational maps. We denote by Rat the set of all non-constant rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. A subsemigroup of Rat with semigroup operation being functional composition is called a *rational semigroup*. The first study of the dynamics of rational semigroups was conducted by A. Hinkkanen and G. J. Martin ([HM96]), who were interested in the role of polynomial semigroups (i.e., semigroups of non-constant polynomial maps) while studying various one-complex-dimensional module spaces for discrete groups, and by F. Ren's group ([GR96]), who studied such semigroups from the perspective of random dynamical systems. We refer to Section 2 for a brief introduction.

In this paper, we consider a Markov process on the Riemann sphere $\widehat{\mathbb{C}}$ given by choosing independently and identically distributed from a set of rational maps. To define the process, let I be a finite index set with at least two elements and let $(f_i)_{i\in I}\in(\mathrm{Rat})^I$ be a family of rational maps with degree at least two. For a probability vector $(p_i)_{i\in I}\in(0,1)^I$ with $\sum_{i\in I}p_i=1$ we define the Markov process on $\widehat{\mathbb{C}}$ given by

$$\mathbb{P}(z,A):=\sum_{i\in I}p_i 1_A(f_i(z)),\quad \text{for each } z\in\widehat{\mathbb{C}} \text{ and every Borel set } A\subset\widehat{\mathbb{C}}, \tag{1.1}$$

where 1_A denotes the characteristic function of A. The associated transition operator M of the process acting on the Banach space $C(\widehat{\mathbb{C}})$ of continuous complex-valued functions endowed with the sup-norm is given by

$$M:C(\widehat{\mathbb{C}})\to C(\widehat{\mathbb{C}}),\quad (M\varphi)(z):=\sum_{i\in I}p_i h(f_i(z)),\quad \text{for each } h\in C(\widehat{\mathbb{C}}) \text{ and } z\in\widehat{\mathbb{C}}.$$

A non-zero element $\rho\in C(\widehat{\mathbb{C}})$ is called a *unitary eigenfunction* of M if there exists $a\in\mathbb{C}$ with $|a|=1$ such that $M\rho=a\rho$. Denote by $U\subset C(\widehat{\mathbb{C}})$ the $\mathbb{C}$-vector space of finite linear combinations of unitary eigenfunctions of M. The elements of U are called *limit state functions*.

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Question 1.1. *What can we say about the Hölder regularity of limit state functions?*

1.1. **Motivation.** Before we state our main results on Question 1.1 let us outline our motivation.

1.1.1. *Gradation between chaos and order.* To study all the possible paths of the process defined in (1.1) we consider the dynamics of the semigroup G generated by the family $(f_i)_{i\in I}$. We use

$$G := \langle f_i : i \in I \rangle := \left\{ f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} : n \in \mathbb{N}, (\omega_1, \ldots, \omega_n) \in I^n \right\}$$

to denote the rational semigroup generated by $(f_i)_{i\in I}$. Since G contains elements of degree at least two, there exist points in $\widehat{\mathbb{C}}$ which exhibit a chaotic behavior under the dynamics of G. Namely, it is well known that G has a non-empty Julia set $J(G)$ which is given by

$$J(G) := \left\{ z \in \widehat{\mathbb{C}} : \text{ there exists no non-empty neighborhood } U \text{ of } z \text{ such that } (g_{|U})_{g\in G} \text{ is normal} \right\}.$$

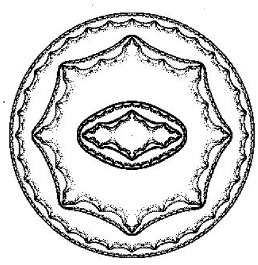
On the other hand, by a recent result of Sumi ([Sum11, Theorem 3.15]), under the assumption that the kernel Julia set $\bigcap_{g\in G} g^{-1}(J(G))$ is empty, we have that the iterates of the transition operator M stabilize. More precisely, we have that

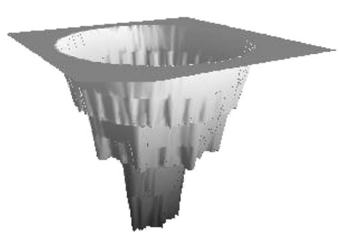
$$C(\widehat{\mathbb{C}}) = U \oplus \left\{ h \in C(\widehat{\mathbb{C}}) : \lim_{n\to\infty} \|M^n(h)\|_\infty \to 0 \right\}.$$

This means that, although the Julia set $J(G)$ is non-empty, the averaging procedure obtained from the iteration of M has a stable behavior. From this point of view, it is natural to investigate the regularity of the limit state functions, which appear in the limit stage of the averaging procedure. The Hölder regularity of limit state functions gives rise to a gradation between chaos and order (see [Sum11, Sum13, Sum14]).

1.1.2. *Singular functions on the Riemann sphere.* Limit state functions can provide examples of devil's staircase-like functions on the Riemann sphere ([Sum11, Sum13]). This type of functions is called a devil's coliseum ([Sum11]). A devil's coliseum is a continuous function which varies only on a thin fractal set. We give the following example from the recent work of Sumi ([Sum11]):

Example 1.2. Let $\varphi_1(z) := z^2 - 1$, $\varphi_2(z) := z^2/4$, $f_i := \varphi_i \circ \varphi_i$ for $i \in \{1,2\}$. We consider the process introduced in (1.1) with $p_1 = p_2 = 1/2$. The space of limit state functions U is the 2-dimensional space given by $U = \mathbb{C}1 \oplus \mathbb{C}T_\infty$, where T_∞ denotes the function of probability of tending to infinity. On the left hand side of the following figure we see the Julia set $J(G)$ of $G = \langle f_1, f_2 \rangle$, on the right hand side we see the limit state function T_∞. Note that T_∞ varies precisely on $J(G)$. We refer to [Sum11] for the details.





1.2. **Main result.** To state our main result, we need further definitions. For a function $\rho : \widehat{\mathbb{C}} \to \mathbb{C}$ we denote by $\text{Höl}(\rho, \cdot)$ the pointwise Hölder exponent of ρ which is for $z \in \widehat{\mathbb{C}}$ given by

$$\text{Höl}(\rho, z) := \sup \left\{ \beta \in \mathbb{R} : \limsup_{y\to z, y\neq z} \frac{|\rho(y) - \rho(z)|}{d(y,z)^\beta} < \infty \right\} \in [0, \infty],$$

where d refers to the spherical distance on $\widehat{\mathbb{C}}$. For $\alpha \in \mathbb{R}$ we define the level sets

$$H(\rho,\alpha) := \left\{ z \in \widehat{\mathbb{C}} : \mathrm{H\ddot{o}l}(\rho,z) = \alpha \right\}.$$

Moreover, we set

$$\alpha_{\min} := \inf\{\alpha \in \mathbb{R} : H(\rho,\alpha) \neq \varnothing\} \quad \text{and} \quad \alpha_{\max} := \sup\{\alpha \in \mathbb{R} : H(\rho,\alpha) \neq \varnothing\}.$$

We say that $G = \langle f_i : i \in I \rangle$ is *hyperbolic* if $\overline{\bigcup_{g \in G \cup \{\mathrm{id}\}} g(\bigcup_{i \in I} \mathrm{CV}(f_i))} \subset \widehat{\mathbb{C}} \setminus J(G)$, where $\mathrm{CV}(f_i)$ refers to the set of critical values of f_i. We say that $(f_i)_{i \in I}$ satisfies the *separation condition* if $f_i^{-1}(J(G)) \cap f_j^{-1}(J(G)) = \varnothing$ for all $i, j \in I$ with $i \neq j$.

Theorem 1.3 (JS13b). *Suppose that $G = \langle f_i : i \in I \rangle$ is hyperbolic and satisfies the separation condition. For the Markov process introduced in (1.1) above, suppose that U contains a non-constant limit state function and let $\rho \in U$ be non-constant. The following two statements hold:*

(1) *The numbers $\alpha_{\min}$ and $\alpha_{\max}$ do not depend on the choice of non-constant elements $\rho \in U$. If $\alpha_{\min} < \alpha_{\max}$ then the dimension function given by $\alpha \mapsto \dim_H(H(\rho,\alpha))$, $\alpha \in (\alpha_{\min}, \alpha_{\max})$, is a positive, real-analytic and strictly concave function with maximum $\dim_H(J(G))$.*

(2) *We have $\alpha_{\min} = \alpha_{\max}$ if and only if there exist $\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})$, $(a_i) \in \mathbb{C}^I$ and $\lambda \in \mathbb{R}$ such that, for all $i \in I$,*

$$\varphi \circ f_i \circ \varphi^{-1}(z) = a_i z^{\pm \deg(f_i)} \quad \text{and} \quad \log(\deg(f_i)) = \lambda \log p_i.$$

In this case we have $H(\rho, \alpha_{\min}) = J(G)$.

Note that Theorem 1.3 in particular applies to Example 1.2 with $\alpha_{\min} < \alpha_{\max}$.

2. Preliminaries on Rational Semigroups

Throughout, let I be a finite index set with at least two elements and let $(f_i)_{i \in I} \in (\mathrm{Rat})^I$ be a family of rational maps with degree at least two.

Definition 2.1 ([Sum00]). The *skew product map* associated with $f = (f_i)_{i \in I}$ is given by

$$\tilde{f} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}} \times \widehat{\mathbb{C}}, \quad \tilde{f}(\omega, z) := (\sigma(\omega), f_{\omega_1}(z)),$$

where $\sigma : I^{\mathbb{N}} \to I^{\mathbb{N}}$ denotes the left-shift given by $\sigma(\omega_1, \omega_2, \dots) := (\omega_2, \omega_3, \dots)$, for $\omega = (\omega_1, \omega_2, \dots) \in I^{\mathbb{N}}$. For $\omega \in I^{\mathbb{N}}$ we define

$$F_\omega := \left\{ z \in \widehat{\mathbb{C}} : \left(f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1} \right)_{n \in \mathbb{N}} \text{ is normal in a neighbourhood of } z \right\} \text{ and } J_\omega := \widehat{\mathbb{C}} \setminus F_\omega.$$

For each $\omega \in I^{\mathbb{N}}$, we set $J^\omega := \{\omega\} \times J_\omega$ and we set

$$J(\tilde{f}) := \overline{\bigcup_{\omega \in I^{\mathbb{N}}} J^\omega}, \quad F(\tilde{f}) := \left(I^{\mathbb{N}} \times \widehat{\mathbb{C}} \right) \setminus J(\tilde{f}),$$

where the closure is taken with respect to the product topology on $I^{\mathbb{N}} \times \widehat{\mathbb{C}}$. Let $\pi_1 : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to I^{\mathbb{N}}$ and $\pi_{\widehat{\mathbb{C}}} : I^{\mathbb{N}} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ denote the canonical projections.

We refer to [Sum00, Proposition 3.2] for the proof of the following proposition.

Proposition 2.2. *The following three statements hold:*

(1) For each $\omega \in I^{\mathbb{N}}$ we have $\tilde{f}(J^\omega) = J^{\sigma\omega}$ and $(\tilde{f}_{|\pi_1^{-1}(\omega)})^{-1}(J^{\sigma\omega}) = J^\omega$.

(2) $\tilde{f}(J(\tilde{f})) = J(\tilde{f})$, $\tilde{f}^{-1}(J(\tilde{f})) = J(\tilde{f})$.
(3) $\pi_{\widehat{\mathbb{C}}}(J(\tilde{f})) = J(G)$.

For a holomorphic map $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and $z \in \widehat{\mathbb{C}}$, the norm of the derivative of h at $z \in \widehat{\mathbb{C}}$ with respect to the spherical metric is denoted by $\|h'(z)\|$.

Definition 2.3 ([Sum98]). For each $n \in \mathbb{N}$ and $(\omega, z) \in J(\tilde{f})$, we set $(\tilde{f}^n)'(\omega, z) := (f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1})'(z)$. We say that $\tilde{f}$ (or the rational semigroup $G = \langle f_i : i \in I \rangle$) is expanding if there exist constants $C > 0$ and $\lambda > 1$ such that for all $n \in \mathbb{N}$,

$$\inf_{(\omega,z) \in J(\tilde{f})} \| (\tilde{f}^n)'(\omega, z) \| \geq C\lambda^n,$$

where $\| (\tilde{f}^n)'(\omega, z) \|$ denotes the norm of the derivative of $f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}$ at z with respect to the spherical metric.

Remark 2.4. It follows from Proposition 2.6 below that, for a rational semigroup $G = \langle f_i : i \in I \rangle$, the notion of expandingness is independent of the choice of the generator system.

Definition 2.5 ([Sum98]). A rational semigroup G is hyperbolic if $P(G) \subset \widehat{\mathbb{C}} \setminus J(G)$, where $P(G)$ denotes the postcritical set of G given by

$$P(G) := \overline{\bigcup_{g \in G} \mathrm{CV}(g)}.$$

The next proposition characterises when G is expanding.

Proposition 2.6 ([Sum98]). *$G = \langle f_i : i \in I \rangle$ is expanding if and only if G is hyperbolic.*

3. On the Proof of the Main Result

The proof consists mainly of two parts. In the first part we give a dynamical description of the level sets $H(\rho, \alpha)$. It turns out that these sets can be described in terms of the limiting behaviour of quotients of Birkhoff sums with respect to the skew product map $\tilde{f}$. In the second part, we derive the main result by employing the multifractal formalism in ergodic theory.

3.1. **Dynamical description of the level sets.** It is not difficult to see that

$$\mathrm{H\ddot{o}l}(\rho, z) = \liminf_{r \to 0} \frac{\log \sup_{y \in B(z,r)} |\rho(y) - \rho(z)|}{\log r}, \quad \text{for } z \in \widehat{\mathbb{C}}.$$

We aim to give a dynamical description with respect to the dynamical system $(J(\tilde{f}), \tilde{f})$. Define potentials

$$\tilde{\zeta} : J(\tilde{f}) \to \mathbb{R} \quad \tilde{\zeta}(\tau, z) := -\log \|f'_{\tau_1}(z)\|, \quad \text{for } (\tau, z) \in J(\tilde{f}),$$

and

$$\tilde{\psi} : J(\tilde{f}) \to \mathbb{R}, \quad \tilde{\psi}(\tau, z) := \log p_{\tau_1},, \quad \text{for } (\tau, z) \in J(\tilde{f}),$$

where $(p_i)_{i \in I}$ refers to the probability vector of the process introduced in (1.1). We denote by $S_n \tilde{\zeta}$ resp. $S_n \tilde{\psi}$ the Birkhoff sums of $\tilde{\zeta}$ resp. $\tilde{\psi}$ with respect to $(J(\tilde{f}), \tilde{f})$. We can now state the main lemma. Note that for each $z \in J(G)$ there exists a unique $\omega \in I^{\mathbb{N}}$ such that $(\omega, z) \in J(\tilde{f})$.

Lemma 3.1. *For each $(\omega, z) \in J(\tilde{f})$ we have*

$$\liminf_{n \to \infty} \frac{S_n \tilde{\psi}((\omega, z))}{S_n \tilde{\zeta}((\omega, z))} = \mathrm{H\ddot{o}l}(\rho, z).$$

Proof. We may suppose that $M\rho = \rho$. We give a sketch of the proof, the details can be found in [JS13]. Since $z \in J(G)$ and G is hyperbolic, there exists $r_0 > 0$ such that, for all $n \in \mathbb{N}$, there exists a holomorphic map $\phi_n : B((f_{\omega_n} \circ \cdots \circ f_{\omega_1})(z), r_0) \to \widehat{\mathbb{C}}$ with $(f_{\omega_n} \circ \cdots \circ f_{\omega_1}) \circ \phi_n = \mathrm{id}$ and $\phi_n((f_{\omega_n} \circ \cdots \circ f_{\omega_1})(z)) = z$. Put $B_n := \phi_n(B((f_{\omega_n} \circ \cdots \circ f_{\omega_1})(z), r_0))$. For $a, b \in B_n$ we have

$$\begin{aligned}\rho(a) - \rho(b) &= (M^n\rho)(a) - (M^n\rho)(b) \\ &= \sum_{(\tau_1,\ldots,\tau_n)\in I^n} p_{\tau_1} \cdot \cdots \cdot p_{\tau_n} (\rho((f_{\tau_n} \circ \cdots \circ f_{\tau_1})(a)) - \rho((f_{\tau_n} \circ \cdots \circ f_{\tau_1})(b)))\end{aligned}$$

After making r_0 sufficiently small, we can deduce the following: since $(f_i)_{i\in I}$ satisfies separation condition and $\rho_{|F(G)}$ is locally constant on $F(G)$ by [Sum11, Theorem 3.15(1)], we have for all $a, b \in B_n$

$$\rho(a) - \rho(b) = p_{\omega_1} \cdot \cdots \cdot p_{\omega_n} (\rho((f_{\omega_n} \circ \cdots \circ f_{\omega_1})(a)) - \rho((f_{\omega_n} \circ \cdots \circ f_{\omega_1})(b))).$$

Since ρ varies on the $J(G)$ by [Sum11], we deduce that $\sup_{a,b\in B_n} |\rho(a) - \rho(b)| \asymp p_{\tau_1} \cdot \cdots \cdot p_{\tau_n} \asymp e^{S_n \tilde{\psi}(\omega,z)}$. Finally, by Koebe's distortion theorem, we have that B_n is close to a ball of radius $r_n := \|(f_{\omega_n} \circ \cdots \circ f_{\omega_1})'(z)\|^{-1} \asymp e^{S_n \tilde{\zeta}(\omega,z)}$, which finishes the proof. □

3.2. **Application of the Multifractal Formalism.** The multifractal formalism goes back to the work of [Man74, FP85, HJK+86] motivated by statistical physics. We employ the multifractal formalism for level sets given by quotients of Birkhoff sums with respect to the skew product associated with a rational semigroup ([JS13]). For a similar kind of multifractal formalism for conformal repellers we refer to [PW97, Pes97].

The *free energy function* is the unique function $t : \mathbb{R} \to \mathbb{R}$ such that $\mathscr{P}\left(\beta\tilde{\psi} + t(\beta)\tilde{\zeta}, \tilde{f}\right) = 0$ for each $\beta \in \mathbb{R}$, where $\mathscr{P}\left(\cdot, \tilde{f}\right)$ denotes the topological pressure with respect to $\tilde{f}$ ([Wal82]). The convex conjugate of t ([Roc70, Section 12]) is given by

$$t^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}, \quad t^*(c) := \sup_{\beta\in\mathbb{R}} \{\beta c - t(\beta)\}, \quad c \in \mathbb{R}.$$

Since the dynamical system $(J(\tilde{f}), \tilde{f})$ is expanding and the potentials $\tilde{\zeta}$ and $\tilde{\psi}$ are Hölder continuous, it is well known that t is real-analytic (see e.g. [Rue78, Pes97]). Consequently, its convex conjugate function t^* is real-analytic on its domain. The multifractal formalism now relates the Hausdorff dimension of the level-sets $\pi_{\widehat{\mathbb{C}}}\left\{(\omega, z) \in J(\tilde{f}) : \lim_{n\to\infty} S_n\tilde{\psi}/S_n\tilde{\zeta} = \alpha\right\}$ to the function t^*:

Theorem 3.2 ([JS13]). *For each $\alpha \in (\alpha_{\min}, \alpha_{\max})$ we have*

$$\pi_{\widehat{\mathbb{C}}}\left\{(\omega, z) \in J(\tilde{f}) : \lim_{n\to\infty} S_n\tilde{\psi}/S_n\tilde{\zeta} = \alpha\right\} = -t^*(-\alpha).$$

Finally, let us remark that the spectrum degenerates if and only if the potentials $\tilde{\zeta}$ and $\tilde{\psi}$ are linearly dependent in the cohomology class of bounded continuous functions. Employing a result of A. Zdunik [Zdu90] and proceeding as in [SU12], the statemtent in Theorem 1.3 (2) follows.

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