On Addition Formulae of KP, mKP and BKP hierarchies

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1 The addition formula for the \(\tau\)-function of the KP hierarchy

Let

\[ [\alpha] = (\alpha, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots), \quad \xi(t, \lambda) = \sum_{n=1}^{\infty} t_n \lambda^n, \quad t = (t_1, t_2, t_3, \ldots). \]

The KP hierarchy is a system of equations for a function \(\tau(t)\) given by

\[ \oint e^{\xi(t'-\lambda)} \tau(t'-[\lambda^{-1}]) \tau(t+[\lambda^{-1}]) \frac{d\lambda}{2\pi i} = 0. \] (1)

Here \(\oint\) means a formal algebraic operator extracting the coefficient of \(z^{-1}\) of Laurent series:

\[ \oint \frac{dz}{2\pi i} \sum_{n=-\infty}^{\infty} a_n z^n = a_{-1}. \]

Set \(t = x + y, t' = x - y\). Then (1) becomes

\[ \oint e^{-2\xi(y, \lambda)} \tau(x-y-[\lambda^{-1}]) \tau(x+y+[\lambda^{-1}]) \frac{d\lambda}{2\pi i} = 0. \] (2)

Set

\[ y = \frac{1}{2} (\sum_{i=1}^{m-1} [\beta_i] - \sum_{i=1}^{m+1} [\alpha_i]). \]

By virtue of the identity

\[ \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x), \]

the exponential factor \(e^{-2\xi(y, \lambda)}\) reduces to a rational function of \(\lambda, \alpha_i, \beta_i\) as

\[ e^{-2\xi(y, \lambda)} = \frac{\prod_{i=1}^{m-1} (1-\beta_i \lambda)}{\prod_{i=1}^{m+1} (1-\alpha_i \lambda)}. \]

Finally shifting the variable \(x\) as

\[ x \rightarrow x + \frac{1}{2} (\sum_{i=1}^{m-1} [\beta_i] - \sum_{i=1}^{m+1} [\alpha_i]), \]

we get the following addition formulae of \(\tau\)-function

\[ \sum_{i=1}^{m+1} (-1)^{i-1} \zeta(x; \beta_1, \ldots, \beta_{m-1}, \alpha_i) \zeta(x; \alpha_1, \ldots, \alpha_i, \ldots, \alpha_{m+1}) = 0, \quad m \geq 2. \] (3)
where
\[ \zeta(x; \alpha_1, \ldots, \alpha_n) = \Delta(\alpha_1, \ldots, \alpha_n) \tau(x + [\alpha_1] + \cdots + [\alpha_n]), \]
\[ \Delta(\alpha_1, \ldots, \alpha_n) = \prod_{i<j} (\alpha_i - \alpha_j), \]
and, \( \hat{\alpha}_i \) denotes to remove \( \alpha_i \).

Example 1 In the case of \( m = 2 \), we have
\[ \alpha_{12} \alpha_{34} \tau(x + [\alpha_1] + [\alpha_2]) \tau(x + [\alpha_3] + [\alpha_4]) - \alpha_{13} \alpha_{24} \tau(x + [\alpha_1] + [\alpha_3]) \tau(x + [\alpha_2] + [\alpha_4]) + \alpha_{14} \alpha_{23} \tau(x + [\alpha_1] + [\alpha_4]) \tau(x + [\alpha_2] + [\alpha_3]) = 0, \tag{4} \]
where \( \alpha_{ij} = \alpha_i - \alpha_j \).

We call (4) ‘the three terms equation’. We have derived (4) from (1). In fact, the converse is true.

Theorem 1 The three terms equation (4) is equivalent to the KP hierarchy (1).

This theorem has been proved by Takasaki and Takebe [25]. They proved the theorem by constructing the wave function of the KP-hierarchy. To do it they used the differential Fay identity which is a certain limit of (4). Here we give an alternative and direct proof of the theorem. Theorem 1 is proved by using the following propositions.

Proposition 1 The KP hierarchy (1) is equivalent to (3).

Proposition 2 The following formula follows from (4):
\[ \frac{\tau(x + \sum_{i=1}^{m} [\beta_i] - \sum_{i=1}^{m} [\alpha_i])}{\tau(x)} = \prod_{1 \leq i, j \leq m}^{m} \frac{(\beta_i - \alpha_j)}{(\beta_i - \alpha_j) \tau(x)}, \quad m \geq 2. \tag{5} \]

Proposition 3 The Plücker relations for the determinant of the right hand side of (5) give the addition formulae (3).

Proposition 1 is proved using the properties of symmetric functions. Proposition 2 is proved by using the Sylvester’s theorem on determinants.

2 The mKP hierarchy

Let \( \eta(t) \ (l \in \mathbb{Z}) \) be \( \tau \)-functions of the modified KP (mKP) hierarchy. We use the same notation as that for KP hierarchy (\( \{\alpha\}, \xi(t, \lambda), \) etc.).

The mKP hierarchy is given by the bilinear equation of the form
\[ \oint e^{t(t-t') \lambda^{-1} - t' \lambda} \eta(t - [\lambda^{-1}]) \eta(t' + [\lambda^{-1}]) \frac{d\lambda}{2\pi i} = 0, \quad l \geq l'. \tag{6} \]

Set \( t = x - y, t' = x + y \). Then (6) becomes
\[ \oint e^{-2t y \lambda^{-1}} \eta(x - y - [\lambda^{-1}]) \eta(x + y + [\lambda^{-1}]) \frac{d\lambda}{2\pi i} = 0, \quad l \geq l'. \tag{7} \]
Let \( l - l' = k \geq 0 \). Set

\[
y = \frac{1}{2} \left( \sum_{i=1}^{m-2} \beta_i - \sum_{i=1}^{m+k} \alpha_i \right).
\]

The exponential factor in (7) reduces to a rational function of \( \lambda, \alpha_i, \beta_i \) as in the KP case:

\[
\exp \left( -\xi \left( \sum_{i=1}^{m-2} \beta_i - \sum_{i=1}^{m+k} \alpha_i, \lambda \right) \right) = \frac{\prod_{i=1}^{m-2} (1 - \beta_i \lambda)}{\prod_{i=1}^{m+k} (1 - \alpha_i \lambda)}.
\]

Computing the integral as the KP case and shift the variable \( x \) as

\[
x \rightarrow x + \frac{1}{2} \left( \sum_{i=1}^{m-2} \beta_i - \sum_{i=1}^{m+k} \alpha_i \right),
\]

and we get the following addition formulae of the mKP hierarchy:

\[
\sum_{i=1}^{m+k} (-1)^{i-1} \zeta_l(x; \beta_1, \ldots, \beta_{m-2}, \alpha_i) \zeta_{l+k}(\alpha_1, \hat{\alpha}_i, \ldots, \alpha_{m+k}) = 0
\]

\( l \in \mathbb{Z}, \ k \geq 0, \ m \geq 2 \),

where

\[
\zeta_l(x; \alpha_1, \ldots, \alpha_n) = \Delta(\alpha_1, \ldots, \alpha_n) \tau_l(x + \sum_{i=1}^{n} \alpha_i).
\]

**Example 2** The case \( l - l' = 1 \) and \( m = 2 \) of (8) is

\[
\alpha_{23} \tau_l(x + [\alpha_1]) \tau_{l+1}(x + [\alpha_2] + [\alpha_3])
- \alpha_{13} \tau_l(x + [\alpha_2]) \tau_{l+1}(x + [\alpha_1] + [\alpha_3])
+ \alpha_{12} \tau_l(x + [\alpha_3]) \tau_{l+1}(x + [\alpha_1] + [\alpha_2]) = 0.
\]

We call this equation (9) 'the three terms equation of the mKP hierarchy'.

In this case, we have

**Theorem 2** The three terms equation (9) is equivalent to the mKP hierarchy (6).

Theorem 2 has been proved by Takebe. We give another and direct proof of it. Similarly to the case of the KP hierarchy, this theorem is proved by using the following propositions.

**Proposition 4** The mKP hierarchy (6) is equivalent to (8).

**Proposition 5** The following equation follows from (9):

\[
\tau_{l+1}(x + \sum_{i=1}^{n} \alpha_i) = \tau_l(x) \det \begin{bmatrix}
\tau(x + [\alpha_1] - [\beta_1]) & \cdots & \tau(x + [\alpha_1] - [\beta_{n-1}]) & \tau(x + [\alpha_1] + [\alpha_2]) \\
\tau(x + [\alpha_2] - [\beta_1]) & \cdots & \tau(x + [\alpha_2] - [\beta_{n-1}]) & \tau(x + [\alpha_2] + [\alpha_3]) \\
\vdots & \ddots & \vdots & \vdots \\
\tau(x + [\alpha_n] - [\beta_1]) & \cdots & \tau(x + [\alpha_n] - [\beta_{n-1}]) & \tau(x + [\alpha_n] + [\alpha_1])
\end{bmatrix}
= C \det \begin{bmatrix}
\tau(x + [\alpha_1] - [\beta_1]) & \cdots & \tau(x + [\alpha_1] - [\beta_{n-1}]) & \tau(x + [\alpha_1] + [\alpha_2]) \\
\tau(x + [\alpha_2] - [\beta_1]) & \cdots & \tau(x + [\alpha_2] - [\beta_{n-1}]) & \tau(x + [\alpha_2] + [\alpha_3]) \\
\vdots & \ddots & \vdots & \vdots \\
\tau(x + [\alpha_n] - [\beta_1]) & \cdots & \tau(x + [\alpha_n] - [\beta_{n-1}]) & \tau(x + [\alpha_n] + [\alpha_1])
\end{bmatrix}
\]
where

\[
C = \frac{\prod_{i=1}^{n} \prod_{j=1}^{n-1} (\alpha_i - \beta_j)}{(\prod_{i<j}^{n-1} \beta_{ij})(\prod_{i>j}^{n} \alpha_{ij})}
\]

**Proposition 6** The Plücker relations for the determinant of right hand side of (10) gives (8) with \(k = 1\).

**Lemma 1** Equation (8) follows from (9).

Using free fermions, we can derive the equation (10).

Following [1] let \(\psi_n, \psi_n^*\) be free fermionic operators with the following anticommutation relations:

\[
[\psi_n, \psi_m]_+ = [\psi_n^*, \psi_m^*]_+ = 0, \quad [\psi_n, \psi_m^*]_+ = \delta_{mn}.
\]

They generate an infinite dimensional Clifford algebra. We define the generating functions of free fermions as

\[
\psi(\lambda) = \sum_{i=1}^{\infty} \psi_i \lambda^i, \quad \psi^*(\lambda) = \sum_{i=1}^{\infty} \psi_i^* \lambda^{-i}.
\]

For \(n \in \mathbb{Z}\), set

\[
H(x) = \sum_{n=1}^{\infty} x_n H_n, \quad H_n = \sum_{i \in \mathbb{Z}} :\psi_i \psi_{i+n}^*:.
\]

Then we introduce a vacuum \(|0\rangle\) and the dual vacuum \((0|\rangle\). These vacuum have the following properties:

\[
\psi_n |0\rangle = 0, \quad (n < 0), \quad \psi_n^* |0\rangle = 0, \quad (n \geq 0)
\]

\[
(0)\psi_n = 0, \quad (n \geq 0), \quad (0)\psi_n^* = 0, \quad (n < 0)
\]

We need the shifted vacua \(|l\rangle\) and the dual vacua \((l|\rangle\) defined by

\[
|l\rangle = \begin{cases} \psi_{l-1} \cdots \psi_0 |0\rangle, & n > 0 \\ \psi_l^* \cdots \psi_n^* |0\rangle, & n < 0 \end{cases}
\]

\[
(l|\rangle = \begin{cases} \langle 0|\psi_l^* \cdots \psi_n^* - 1 \rangle, & n > 0 \\ \langle 0|\psi_{-1} \cdots \psi_n, & n < 0 \end{cases}
\]

It is easy to check the following properties:

\[
\psi_n |l\rangle = 0, \quad n < l, \quad \psi_n^* |l\rangle = 0, \quad n \geq l
\]

\[
(l|\psi_n = 0, \quad n \geq l, \quad (l|\psi_n^* = 0, \quad n < l.
\]

**Proposition 7** We get the equation (10) by the following equation:

\[
\frac{\langle l|\psi^*(\alpha_1^{-1}) \cdots \psi^*(\alpha_n^{-1}) \psi(\beta_{n-1}^{-1}) \cdots \psi(\beta_1^{-1}) e^{H(x)} g |l + 1\rangle}{\langle l|e^{H(x)} g |l \rangle} = (-1)^{n-1} \det \begin{pmatrix} a_{11} & \ldots & a_{1,n-1} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \ldots & a_{n,n-1} & b_n \end{pmatrix}.
\]
where
\[
  a_{ij} = \frac{\langle l|\psi^{*}(\alpha_{i}^{-1})\psi(\beta_{j}^{-1})e^{H(x)}g|l\rangle}{\langle l|e^{H(x)}g|l\rangle},
\]
\[
  b_{i} = \frac{\langle l|\psi^{*}(\alpha_{i}^{-1})e^{H(x)}g|l+1\rangle}{\langle l|e^{H(x)}g|l\rangle},
\]
and
\[
  G = \{ g \in A | \exists g^{-1}, gVg^{-1} = V, gV^{*}g^{-1} = V^{*} \}, \quad V = \oplus_{i \in \mathbb{Z}} \mathbb{C}\psi_{i}, \quad V^{*} = \oplus_{i \in \mathbb{Z}} \mathbb{C}\psi_{i}^{*},
\]
and \( A \) is the Clifford algebra.

Equation (11) can be derived by using the generalized Wick’s theorem. For \( l \in \mathbb{Z} \), we can get the (10) by considering
\[
  \tau_{l}(x) = \langle l|e^{H(x)}g|l\rangle, \quad g \in G.
\]

3 The BKP hierarchy

Let \( \tau(t) \) be the \( \tau \)-function of the BKP hierarchy. In this case, the time variable is \( t = (t_{1}, t_{3}, t_{5}, \cdots) \). We set
\[
  [\alpha]_{o} = (\alpha, \frac{\alpha^{3}}{3}, \frac{\alpha^{5}}{5}, \cdots), \quad \tilde{\xi}(t, \lambda) = \sum_{n=1}^{\infty} t_{2n-1}\lambda^{2n-1}.
\]
The BKP hierarchy is defined by
\[
  \oint e^{\overline{\xi}(t-t', \lambda)}\tau(t-2[\lambda^{-1}]_{o})\tau(t'+2[\lambda^{-1}]_{o}) \frac{d\lambda}{2\pi i \lambda} = \tau(t)\tau(t'). \quad (12)
\]
Set \( t = x + y, t' = x - y \). We get
\[
  \oint e^{-2\tilde{\xi}(y)\lambda}\tau(x - y - 2[\lambda^{-1}]_{o})\tau(x + y + 2[\lambda^{-1}]_{o}) \frac{d\lambda}{2\pi i \lambda} = \tau(x+y)\tau(x-y). \quad (13)
\]
Set
\[
  y = \sum_{i=1}^{n}[\alpha_{i}]_{o}.
\]
By separating \(-2 \sum_{n=1}^{\infty} t_{2n-1}\lambda^{2n-1}\) as
\[
  -2 \sum_{n=1}^{\infty} t_{2n-1}\lambda^{2n-1} = -\sum_{n=1}^{\infty} t_{n}\lambda^{n} + \sum_{n=1}^{\infty} t_{n}(-\lambda)^{n},
\]
we get
\[
  \exp \left( -2\tilde{\xi} \left( \sum_{i=1}^{n}[\alpha_{i}]_{o}, \lambda \right) \right) = \prod_{i=1}^{n} \frac{1 - \alpha_{i}\lambda}{1 + \alpha_{i}\lambda}.
\]
Computing the integral by taking residues as before and shifting $x$ appropriately, we have

$$
\sum_{i=1}^{n} (-1)^{i-1} \frac{\tau(x + 2[\alpha_i]_0)}{\tau(x)} A_{1 \ldots i \ldots n}^{-1} \frac{\tau(x + 2 \sum_{t \neq i}^{n} [\alpha_t]_0)}{\tau(x)} - A_{1 \ldots n}^{-1} \frac{\tau(x + 2 \sum_{l=1}^{n} [\alpha_l]_0)}{\tau(x)} = 0,
$$

$n$: odd, (14)

$$
\sum_{i=1}^{n} (-1)^{i-1} \frac{\tau(x + 2[\alpha_i]_0 + 2[\alpha_n]_0)}{\tau(x)} \frac{\alpha_i}{\tilde{\alpha}_{ii}} A_{1 \ldots i \ldots n-1}^{-1} \frac{\tau(x + 2 \sum_{t \neq i}^{n} [\alpha_t]_0)}{\tau(x)} - A_{1 \ldots n}^{-1} \frac{\tau(x + 2 \sum_{l=1}^{n} [\alpha_l]_0)}{\tau(x)} = 0,
$$

$n$: even. (15)

Here $A_{1 \ldots n}$ is defined by

$$
A_{1 \ldots n} = \prod_{1=i<j}^{n} \frac{\tilde{\alpha}_{ij}}{\alpha_{ij}}, \quad \tilde{\alpha}_{ij} = \alpha_i + \alpha_j, \quad \alpha_{ij} = \alpha_i - \alpha_j.
$$

Example 3 The case $n = 3$ of (14) is

$$
\frac{\tau(x + 2 \sum_{i=1}^{3} [\alpha_i]_0)}{\tau(x)} = A_{123} \frac{\tau(x + 2[\alpha_1]_0)}{\tau(x)} \frac{\alpha_{23}}{\tilde{\alpha}_{23}} \frac{\tau(x + 2[\alpha_2]_0 + 2[\alpha_3]_0)}{\tau(x)} - \frac{\tau(x + 2[\alpha_2]_0)}{\tau(x)} \frac{\alpha_{13}}{\tilde{\alpha}_{13}} \frac{\tau(x + 2[\alpha_1]_0 + 2[\alpha_3]_0)}{\tau(x)} + \frac{\tau(x + 2[\alpha_3]_0)}{\tau(x)} \frac{\alpha_{12}}{\tilde{\alpha}_{12}} \frac{\tau(x + 2[\alpha_1]_0 + 2[\alpha_2]_0)}{\tau(x)}.
$$

We call Equation (16) 'the four terms equation of the BKP hierarchy'.

Example 4 The case of $n = 4$ of (15) is

$$
\frac{\tau(x + 2 \sum_{i=1}^{4} [\alpha_i]_0)}{\tau(x)} = A_{1234} \frac{\tau(x + 2[\alpha_1]_0 + 2[\alpha_4]_0)}{\tau(x)} \frac{\alpha_{14}}{\tilde{\alpha}_{14}} \frac{\tau(x + 2[\alpha_2]_0 + 2[\alpha_3]_0)}{\tau(x)} - \frac{\tau(x + 2[\alpha_2]_0 + 2[\alpha_3]_0)}{\tau(x)} \frac{\alpha_{23}}{\tilde{\alpha}_{23}} \frac{\tau(x + 2[\alpha_1]_0 + 2[\alpha_4]_0)}{\tau(x)} + \frac{\tau(x + 2[\alpha_3]_0 + 2[\alpha_4]_0)}{\tau(x)} \frac{\alpha_{12}}{\tilde{\alpha}_{12}} \frac{\tau(x + 2[\alpha_1]_0 + 2[\alpha_2]_0)}{\tau(x)}.
$$

Equation (17) of example 4 can be derived from Equation (16).

Then,

Theorem 3 The four terms equation (16) is equivalent to the bilinear identity of the BKP hierarchy (12).

Theorem 3 is proved by Takasaki [23]. Here we give an alternative and direct proof of it.

In order to explain the strategy, we introduce the Pfaffian. Set $A = (\alpha_{ij})_{1 \leq i, j \leq 2m}$ is a skew-symmetric matrix with the degree $2m$. Then, the Pfaffian is defined by

$$
\text{det} A = (Pf A)^2, \quad Pf A = a_{12}a_{34} \cdots a_{2m-1, 2m} - \cdots.
$$

Following [8] we denote $Pf A$ by $(1, 2, 3, \ldots, 2m)$:

$$
Pf A = (1, 2, 3, \ldots, 2m).
$$
It is directly defined by

\[(1, 2, 3, \ldots, 2m) = \sum \text{sgn}(i_1, \ldots, i_{2m}) \cdot (i_1, i_2)(i_3, i_4) \cdots (i_{2m-1}, i_{2m}), \quad (i, j) = a_{ij},\]

where the sum is over all permutations of \((1, \ldots, 2m)\) such that

\[i_1 < i_3 < \cdots < i_{2m-1}, \quad i_1 < i_2, \cdots, i_{2m-1} < i_{2m},\]

and \(\text{sgn}(i_1, \ldots, i_{2m})\) is the signature of the permutations \((i_1, \ldots, i_{2m})\).

The Pfaffian can be expanded as

\[(1, 2, 3, \ldots, 2m) = \sum_{j=2}^{2m} (-1)^j (1, j)(2, 3, \ldots, j, \ldots, 2m).\]

For example, in the case of \(m=2\),

\[(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).\]

Let us define the components of Pfaffian by

\[
(0, j) = \frac{\tau(x + 2[\alpha_j]_0)}{\tau(x)}, \quad (i, j) = \frac{\alpha_{ij}}{\tilde{\alpha}_{ij}} \frac{\tau(x + 2[\alpha_\ast \cdot]_0 + 2[\alpha_j]_0)}{\tau(x)}.
\]

Then, we rewrite (16) and (17) as

\[
\frac{\tau(x + 2\sum_{i=1}^{3}[\alpha_i]_0)}{\tau(x)} = A_{123}(0, 1, 2, 3), \quad (18)
\]

\[
\frac{\tau(x + 2\sum_{i=1}^{4}[\alpha_i]_0)}{\tau(x)} = A_{1234}(1, 2, 3, 4). \quad (19)
\]

Theorem 3 can be proved similarly to the KP case using the following propositions.

**Proposition 8** The BKP hierarchy (12) is equivalent to (14) and (15).

**Proposition 9** The following equations follow from (16):

\[
\frac{\tau(x + 2\sum_{i=1}^{n}[\alpha_i]_0)}{\tau(x)} = A_{1\ldots n}(0, 1, 2, \ldots, n), \quad n: \text{odd,} \quad (20)
\]

\[
\frac{\tau(x + 2\sum_{i=1}^{n}[\alpha_i]_0)}{\tau(x)} = A_{1\ldots n}(1, 2, \ldots, n), \quad n: \text{even.} \quad (21)
\]

There exists an analogue of the Plücker relations for Pfaffians [18].

Then we have

**Proposition 10** The Plücker relation for the Pfaffians of the right hand side of (20) and (21) give the addition formulae (14) and (15) respectively.
References


