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Flow-acoustic interaction in an expansion chamber-pipe system: solution by the method of matched asymptotic expansions

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Abstract

The paper is concerned with the generation of sound by the flow through a closed, cylindrical expansion chamber, follower by a long tailpipe. The sound generation is due to self-sustained flow oscillations in the expansion chamber which, in turn, may generate standing acoustic waves in the tailpipe. The main interest is in the interaction between these two sound sources. Here an analytical, approximate solution of the the acoustic part of the problem is obtained via the method of matched asymptotic expansions.

1 Introduction

Expansion chambers (mufflers) are used in connection with silencers in engine exhaust systems, with the aim of attenuating the sound waves through destructive interference. But the gas flow through the chamber may generate self-excited oscillations, thus becoming a sound generator rather than a sound attenuator [3, 6, 23]. Similar geometries and thus similar problems may be found in, for example, solid propellant rocket motors [8], valves [24], and in corrugated pipes [7].

This paper considers a simple axisymmetric 'silencer model' consisting of an expansion chamber followed by a tailpipe, as shown in Fig. 1.

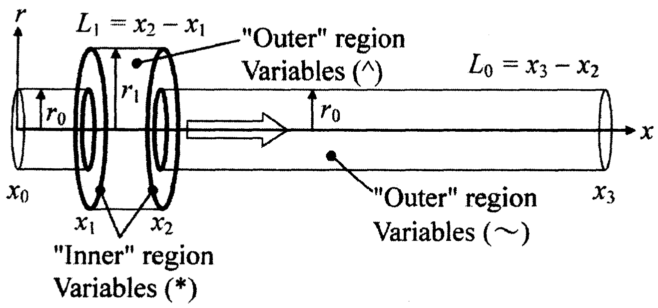

Figure 1: The expansion chamber-tailpipe system. Sketch of the configuration of the problem, and indication of coordinates.

The aim is to contribute to the understanding of the interaction between oscillations of the flow field and the acoustic field. By oscillations of the flow field we mean the self-sustained oscillations of the jet shear layer. The shear layer is unstable and rolls up into a large, coherent vortex (a 'smoke-ring') which is convected downstream with the flow. It cannot pass through the hole in the downstream plate but hits the plate, where it creates a pressure disturbance. The disturbance is thrown back (with the speed of sound) to the upstream plate, where it disturbs the shear layer. This initiates the roll-up of a new coherent vortex. In this way an acoustic feedback loop is formed, making up one type of flow-acoustic interaction.

These so-called hole-tone feedback oscillations [2, 14, 15, 22] may interact with the acoustic axial and (to a much lesser extend) radial eigen-oscillations in the cavity and in the tailpipe [3, 6]. In the present paper we seek to understand the interaction with the axial waves in the tailpipe.

As indicated in Fig. 1, perfect axisymmetry is assumed, and a mathematical model is formulated in terms of the cylindrical axisymmetric coordinates (x, r). The sound-generating flow is represented by a discrete vortex method approach, based on (axisymmetric) vortex rings, as applied also in earlier papers [14, 15]. The acoustic part of the problem can be solved analytically, and completely, in terms of eigenfunction expansions. A travelling wave formulation for a single change in cross-sectional area was considered already in 1944 by Miles [19]. The results of Miles were employed by El-Sharkawy & Nayfeh [5] in an analysis of sound propagation through an expansion chamber. The problem of Miles was reconsidered by Dupère & Dowling [4] in terms of Howe's theory of vortex sound [8, 9].

A 'brute-force' eigenfunction expansion solution, as mentioned above, will actually become quite complicated. A much more manageable approach is possible by taking advantage of characteristic length-scales in the different regions of the problem: (i) tailpipe region, (ii) step (cross section change) regions, and (iii) cavity region. The simplified solutions for these three regions can then be coupled by employing the method of matched asymptotic expansions. Such an approach was used by Lesser & Lewis [16, 17] for a plane (two-dimensional) duct. It is also the approach employed in the present paper.

The paper is divided into eight sections. The (time-domain) governing equations are given and discussed in Section 2. The brief Section 3 is concerned with Fourier transform, used in order to go from a time-domain to a frequency-domain formulation. Section 4 is concerned with non-dimensionalization in terms of parameters with appropriate length- and time-scales. A perturbation expansion of the dependent variables is discussed in Section 5. This is followed by solutions of the simplified governing equations. Asymptotic matching of these solutions is discussed in Section 6. Transformation back to the time-domain is discussed in Section 7. Finally, concluding remarks are made in Section 8.

2 Governing equations

The starting point is taken in the Euler equation [8]

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla p = -\rho \boldsymbol{\mathfrak{L}} H(x - x_1) H(x_2 - x), \tag{1}$$

where ρ is the mean density of the fluid, t is the time, $\mathbf{u} = (u, v)$ is the acoustic particle velocity, p is the acoustic pressure, and

$$\boldsymbol{\mathfrak{L}} = \boldsymbol{\omega} \times \mathbf{v}, \tag{2}$$

which often is called the vortex force, or the Lamb vector [8, 9]. In this expression, $\mathbf{v} = (\mathfrak{u}, \mathfrak{v})$ is flow velocity of the incompressible, sound-generating 'background flow' and $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the vorticity. Finally, $H(s)$ is the Heaviside step function, which equals 0 for $s < 0$ and 1 for $s > 0$. The continuity equation is

$$\begin{aligned} \kappa \frac{\partial p}{\partial t} &= -\nabla \cdot \mathbf{u} \\ &= -\left[\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(rv) \right], \quad \kappa = \frac{1}{\rho c_0^2}, \end{aligned} \tag{3}$$

where c_0 is the speed of sound.

Equations (1) and (3) can be combined through elimination of $\mathbf{u}$ to give the non-homogeneous wave equation

$$\frac{1}{c_0^2}\frac{\partial p}{\partial t^2} - \nabla^2 p = \rho\nabla\cdot\boldsymbol{\mathfrak{L}}\,H(x-x_1)H(x_2-x). \tag{4}$$

Equation (1) should be understood in the same way as the Powell-Howe equation (4) normally is understood [8, 9], that is, it is assumed that the observation point (x, r) is well away from the sound source domain, such that the fluid dynamical flow velocity $\mathbf{v} \approx \mathbf{0}$ there (giving also that $\boldsymbol{\mathfrak{L}} \approx \mathbf{0}$).

In terms of a flow (shear layer) representation by a 'necklace' of M discrete axisymmetric vortex rings, located at (x_m, r_m), $m = 1, 2, \cdots, M$, the Lamb vector $\boldsymbol{\mathfrak{L}} = (\mathfrak{L}_x, \mathfrak{L}_r)$ is given by

$$\boldsymbol{\mathfrak{L}} = (\mathfrak{L}_x, \mathfrak{L}_r) = \sum_{m=1}^{M}\Gamma_m\delta(x-x_m)\frac{\delta(r-r_m)}{\pi r}(\mathfrak{v}_m, -\mathfrak{u}_m)\,. \tag{5}$$

Here $\delta(x - x_m)$ is the one-dimensional delta function ([10], p. 55), while $\delta(r - r_m)/\pi r$ is the axisymmetric delta function ([10], p. 306).

3 Fourier transform

Solution of the equations, and asymptotic matching of these solutions, is easier to carry out in the frequency domain, rather than in the time domain. We thus employ the Fourier transform

$$P(\omega) = \int_{-\infty}^{\infty} p(t)e^{\mathrm{i}\omega t}dt, \quad p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\omega)e^{-\mathrm{i}\omega t}d\omega, \tag{6}$$

to obtain the frequency domain Euler equation

$$\mathrm{i}\rho\omega\mathbf{U} = \nabla P + \rho\boldsymbol{L}\,H(x-x_1)H(x_2-x), \tag{7}$$

and the continuity equation

$$\mathrm{i}\omega\kappa P = \nabla\cdot\mathbf{U}. \tag{8}$$

The frequency domain version of (4) takes the form

$$\nabla^2 P + k^2 P = -\rho\nabla\cdot\boldsymbol{L}\,H(x-x_1)H(x_2-x), \tag{9}$$

where $k = \omega/c_0$ is the acoustic wave number. The transformed Lamb vector $\boldsymbol{L} = \boldsymbol{\Omega}\times\mathbf{V}$, where $\boldsymbol{\Omega} = \nabla\times\mathbf{V}$.

4 Scaling

The governing equations (7), (8), and (9) are made non-dimensional by the use of appropriate length scales for each of the three types of domain, (i) pipe domain, (ii) step domain, and (iii) cavity domain. In the following, let $\mathbf{U} = (U, V)$ and let $\mathbf{V} = (\mathfrak{U}, \mathfrak{V})$.

(i) *Pipe domain.* Here the pipe length L_0 is the appropriate length scale in the x direction, while the pipe radius r_0 is appropriate in the r direction. We thus get

$$\tilde{x} = \frac{x}{L_0}, \quad \tilde{r} = \frac{r}{r_0}, \quad \tilde{u} = \frac{U}{\omega L_0}, \quad \tilde{v} = \frac{V}{\omega r_0}, \quad \tilde{p} = \frac{P}{\rho c_0\omega L_0}, \quad \tilde{k} = kL_0. \tag{10}$$

(ii) *Step domain.* Here the geometry is rapidly varying, and we thus make both x and r non-dimensional with r_0. On the other hand, the acoustic velocity components are only varying slowly across the step, and ωL_0 is the appropriate velocity scale in both x and r directions. We thus get

$$x^* = \frac{x}{r_0} = \frac{\tilde{x}}{\epsilon}, \quad r^* = \frac{r}{r_0}, \quad u^* = \frac{U}{\omega L_0}, \quad v^* = \frac{V}{\omega L_0} = \tilde{v}\epsilon, \tag{11}$$
$$p^* = \frac{P}{\rho c_0 \omega L_0}, \quad k^* = kL_0.$$

(iii) *Cavity domain.* Here we will assume that the long length-scale L_0 is the appropriate one in both x and r directions. We thus get

$$\hat{x} = \frac{x}{L_0}, \quad \hat{r} = \frac{r}{L_0} = \tilde{r}\epsilon, \quad \hat{u} = \frac{U}{\omega L_0} = \frac{\tilde{u}}{\epsilon}, \quad \hat{v} = \frac{V}{\omega L_0}, \quad \hat{p} = \frac{P}{\rho c_0 \omega L_0}, \tag{12}$$
$$\hat{k} = kL_0, \quad \hat{\boldsymbol{L}} = \frac{\boldsymbol{L}}{\rho c_0 \omega}, \quad \hat{\Gamma}_m = \frac{\Gamma_m}{\rho c_0 L_0^2}, \quad \hat{\mathfrak{u}} = \frac{\mathfrak{U}}{\omega L_0}, \quad \hat{\mathfrak{v}} = \frac{\mathfrak{V}}{\omega L_0}.$$

4.1 Scaled governing equations

Using the non-dimensional parameters introduced in the previous Section, we obtain the following scaled, non-dimensional equations.
(i) *Pipe domain*

$$\mathrm{i}k\tilde{u} = \frac{\partial \tilde{p}}{\partial \tilde{x}}, \quad \epsilon^2 \mathrm{i}\tilde{k} = \frac{\partial \tilde{p}}{\partial \tilde{r}}, \quad i\tilde{k}\tilde{p} = \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\tilde{v}}{\tilde{r}} + \frac{\partial \tilde{v}}{\partial \tilde{r}}. \tag{13}$$

(ii) *Step domain*

$$\epsilon \mathrm{i}k^* u^* = \frac{\partial p^*}{\partial x^*}, \quad \epsilon \mathrm{i}k^* v^* = \frac{\partial p^*}{\partial r^*}, \quad \epsilon \mathrm{i}p^* = \frac{\partial u^*}{\partial x^*} + \frac{v^*}{r^*} + \frac{\partial v^*}{\partial r^*}. \tag{14}$$

(iii) *Cavity domain*

$$\mathrm{i}\hat{k}\hat{u} = \frac{\partial \hat{p}}{\partial \hat{x}} + \hat{L}_x, \quad \mathrm{i}\hat{k}\hat{v} = \frac{\partial \hat{p}}{\partial \hat{r}} + \hat{L}_r, \quad \mathrm{i}\hat{k}\hat{p} = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\hat{v}}{\hat{r}} + \frac{\partial \hat{v}}{\partial \hat{r}}, \tag{15}$$

$$\begin{aligned} \frac{\partial^2 \hat{p}}{\partial \hat{x}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{p}}{\partial \hat{r}} + \frac{\partial^2 \hat{p}}{\partial \hat{r}^2} + \hat{k}^2 \hat{p} &= -\hat{\nabla} \cdot \hat{\boldsymbol{L}}, \\ &= -\frac{\partial \hat{L}_x}{\partial \hat{x}} - \frac{1}{\hat{r}}\frac{\partial}{\partial \hat{r}}\left(\hat{r}\hat{L}_r\right). \end{aligned} \tag{16}$$

In all of these equations, $\epsilon = r_0/L_0$ play the role of a small parameter.

5 Perturbation expansion, simplified equations, and solutions

Next the dependent variables p, u, and v (with a tilde, a hat, or an asterisk) are expanded in terms of asymptotic sequences of functions of ϵ (defined just above),

$$\begin{aligned} p &= \alpha_0(\epsilon)p_0 + \alpha_1(\epsilon)p_1 + \alpha_2(\epsilon)p_2 + \cdots, \\ u &= \beta_0(\epsilon)u_0 + \beta_1(\epsilon)u_1 + \beta_2(\epsilon)u_2 + \cdots, \\ v &= \gamma_0(\epsilon)v_0 + \gamma_1(\epsilon)v_1 + \gamma_2(\epsilon)v_2 + \cdots. \end{aligned} \tag{17}$$

In the most general approach, $\alpha_n(\epsilon)$, $\beta_n(\epsilon)$, and $\gamma_n(\epsilon)$ are asymptotic sequences of *unknown* functions which are determined such that matching can be carried out [12]. In the present case it is, however, sufficient to let these functions be simple powers of ϵ. Thus we apply the expansions

$$\begin{aligned} p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots , \\ u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots , \\ v &= v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots . \end{aligned} \tag{18}$$

To the lowest (ϵ^0) order, the governing equations in the pipe section are

$$\begin{aligned} \mathrm{i}\tilde{k}\tilde{u}_0 &= \frac{\partial \tilde{p}_0}{\partial \tilde{x}}, \qquad \frac{\partial \tilde{p}_0}{\partial \tilde{r}} = 0, \\ \mathrm{i}\tilde{k}\tilde{p}_0 &= \frac{\partial \tilde{u}_0}{\partial \tilde{x}} + \frac{\tilde{v}_0}{\tilde{r}} + \frac{\partial \tilde{v}_0}{\partial \tilde{r}}. \end{aligned} \tag{19}$$

The second equation of (19) gives that $\tilde{p}_0$ is a function of x only, i.e. $\tilde{p}_0(x)$. Using this, one finds that the acoustic particle velocity components are governed by the equations

$$\tilde{u}_0 = \frac{1}{\mathrm{i}\tilde{k}}\frac{\partial \tilde{p}_0}{\partial \tilde{x}}, \qquad \tilde{v}_0 = -\frac{\tilde{r}}{\mathrm{i}2\tilde{k}}\left[\frac{\partial^2 \tilde{p}_0}{\partial \tilde{x}^2} + \tilde{k}^2 \tilde{p}_0\right]. \tag{20}$$

Applying the second of these equations on the pipe wall $\tilde{r} = \tilde{r}_0$ (where $\tilde{v}_0 = 0$), one finds that the pressure $\tilde{p}_0$ is governed by the one-dimensional wave equation contained within the square brackets [ ] in (20). The solution to this equation, which satisfies the boundary condition $\tilde{p}_0 = 0$ at $\tilde{x} = \tilde{x}_3$, is given by

$$\tilde{p}_0 = \tilde{p}_0(\tilde{x}) = A_0 \left[\cos \tilde{k}\tilde{x} - \cot \tilde{k}\tilde{x}_3 \sin \tilde{k}\tilde{x}\right]. \tag{21}$$

The step sections are, to the lowest order, governed by

$$\begin{aligned} &\frac{\partial p_0^*}{\partial x^*} = 0, \quad \frac{\partial p_0^*}{\partial r^*} = 0, \\ &\mathrm{i}k^* u_0^* = \frac{\partial p_1^*}{\partial x^*}, \quad \mathrm{i}k^* v_0^* = \frac{\partial p_1^*}{\partial r^*}. \end{aligned} \tag{22}$$

The first two equations give that

$$p_0^* = C_0^* = \text{constant}. \tag{23}$$

As the next two equations show, the lowest order velocity components are governed by the next-order (ϵ^1) pressure term, p_1^*. This term is governed by the Laplace equation

$$\frac{\partial^2 p_1^*}{\partial x^{*2}} + \frac{1}{r^*}\frac{\partial p_1^*}{\partial r^*} + \frac{\partial^2 p_1^*}{\partial r^{*2}} = 0. \tag{24}$$

Concentrating here on the step at $x^* = x_2^*$ (refer to Fig. 1), the solution to (24) there can be written as

$$p_1^* = \begin{cases} a_0^{*-} + \sum_n a_n^{*-} \, \mathrm{e}^{-\zeta_n \epsilon_1 x^*} J_0\left(\zeta_n \epsilon_1 r^*\right), \; x^* < x_2^*, \\ a_0^{*+} + \sum_n a_n^{*+} \, \mathrm{e}^{-\zeta_n x^*} J_0\left(\zeta_n r^*\right), \; x^* > x_2^*, \end{cases} \tag{25}$$

where J_0 is the Bessel function of first kind and order zero.

On the step at $x^* = x_2^*$, the following boundary conditions must be satisfied:

$$\begin{aligned} p_1^*(x_2^{*-}, r^*) &= p_1^*(x_2^{*+}, r^*), \quad 0 < r^* < 1, \\ \frac{\partial p_1^*}{\partial x^*}(x_2^{*-}, r^*) &= \frac{\partial p_1^*}{\partial x^*}(x_2^{*+}, r^*), \quad 0 < r^* < 1, \\ \frac{\partial p_1^*}{\partial x^*}(x_2^{*-}, r^*) &= 0, \quad 1 < r^* < 1/\epsilon_1, \end{aligned} \tag{26}$$

where $\epsilon_1 = r_0/r_1$ (see again Fig. 1). As it is not possible to impose these 'strong conditions' on a solution of the form (25), will will instead employ the following equivalent weak conditions,

$$\begin{aligned}\int_0^1 p_1^*(x_2^{*-}, r^*) J_0(\zeta_m r^*) r^* dr^* &= \int_0^1 p_1^*(x_2^{*+}, r^*) J_0(\zeta_m r^*) r^* dr^*, \\ \int_0^1 \frac{\partial p_1^*}{\partial x^*}(x_2^{*-}, r^*) r^* dr^* &= \int_0^1 \frac{\partial p_1^*}{\partial x^*}(x_2^{*+}, r^*) r^* dr^*, \\ \int_1^{1/\epsilon_1} \frac{\partial p_1^*}{\partial x^*}(x_2^{*-}, r^*) r^* dr^* &= 0.\end{aligned} \tag{27}$$

It is noted that the last two conditions can be combined into one, on the form

$$\int_0^{1/\epsilon_1} \frac{\partial p_1^*}{\partial x^*}(x_2^{*-}, r^*) r^* dr^* = \int_0^1 \frac{\partial p_1^*}{\partial x^*}(x_2^{*+}, r^*) r^* dr^*. \tag{28}$$

The application of these boundary conditions will be discussed in connection with the matching of solutions in the next Section.

For the expansion chamber, it is convenient to state the lowest (ϵ^0) order governing equation on the wave equation form

$$\frac{\partial^2 \hat{p}_0}{\partial \hat{x}^2} + \frac{1}{\hat{r}}\frac{\partial \hat{p}_0}{\partial \hat{r}} + \frac{\partial^2 \hat{p}_0}{\partial \hat{r}^2} + \hat{k}^2 \hat{p}_0 = -\hat{\nabla} \cdot \hat{\boldsymbol{L}}. \tag{29}$$

A particular solution can be expressed as

$$\hat{p}_0^{\text{part}} = \iint G \nabla \cdot \hat{\boldsymbol{L}} \hat{r} d\hat{r} d\hat{x} = -\iint \hat{\boldsymbol{L}} \cdot \left(\frac{\partial G}{\partial \hat{x}}, \frac{\partial G}{\partial \hat{r}}\right) \hat{r} d\hat{r} d\hat{x}, \tag{30}$$

where the superscript 'part' refers to 'particular', and

$$G(x, r) = \frac{\mathrm{i}}{2\pi\hat{\delta}^2} \sum_{n=0}^{\infty} \frac{1}{\kappa_n} \frac{J_0(\zeta_n \hat{r}_*/\hat{\delta})}{J_0^2(\zeta_n)} J_0\left(\zeta_n \frac{\hat{r}}{\hat{\delta}}\right) \mathrm{e}^{\mathrm{i}\kappa_n |\hat{x} - \hat{x}_*|} \tag{31}$$

is the Green's function which satisfies the boundary condition $\partial G/\partial \hat{r} = 0$ at $\hat{r} = \hat{r}_1$. Here

$$\kappa_n = \begin{cases} \sqrt{\hat{k}^2 - (\zeta_n/\hat{\delta})^2} & \text{for} \quad |\hat{k}| > \zeta_n/\hat{\delta} \\ \mathrm{i}\sqrt{(\zeta_n/\hat{\delta})^2 - \hat{k}^2} & \text{for} \quad |\hat{k}| < \zeta_n/\hat{\delta} \end{cases},$$

and ζ_n are the the zeros of J_1, the Bessel function of order unity. For convenience, we will write (30) as

$$\hat{p}_0^{\text{part}} = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \hat{f}_n(\hat{r}, \hat{r}_m) \mathrm{e}^{\mathrm{i}\kappa_n |\hat{x} - \hat{x}_m|}, \tag{32}$$

where $(\hat{x}_m, \hat{r}_m)$ are, again, the positions of the free vortex rings present within the cavity, and

$$\hat{f}_n(\hat{r}, \hat{r}_m) = -\frac{\hat{\Gamma}_m}{2\pi\hat{\delta}^2} \frac{J_0\left(\zeta_n \frac{\hat{r}}{\hat{\delta}}\right)}{J_0^2(\zeta_n)} \left[\hat{\mathfrak{v}}_m J_0\left(\zeta_n \frac{\hat{r}_m}{\hat{\delta}}\right) + \mathrm{i}\,\hat{\mathfrak{u}}_m \frac{\zeta_n}{\kappa_n \hat{\delta}} J_1\left(\zeta_n \frac{\hat{r}_m}{\hat{\delta}}\right)\right]. \tag{33}$$

We will also include an 'eigensolution' to the homogeneous version of (29), which likewise satisfies the boundary condition $\partial G/\partial \hat{r} = 0$ at $\hat{r} = \hat{r}_1$. Such a solution can be written as

$$\hat{p}_0^{\text{hom}} = \sum_{n=0}^{\infty} \hat{C}_n J_0\left(\zeta_n \frac{\hat{r}}{\hat{\delta}}\right) \mathrm{e}^{\mathrm{i}\kappa_n \hat{x}}, \tag{34}$$

where the superscript 'hom' refers to 'homogeneous'. The full (complete) solution is thus

$$\hat{p}_0 = \hat{p}_0^{\mathrm{part}} + \hat{p}_0^{\mathrm{hom}}. \tag{35}$$

Evaluation of the acoustic particle velocity components will be based on the 'homogeneous solution' (34) only,

$$\mathrm{i}\hat{k}\hat{u}_0 = \frac{\partial \hat{p}_0^{\mathrm{hom}}}{\partial \hat{x}}, \quad \mathrm{i}\hat{k}\hat{v}_0 = \frac{\partial \hat{p}_0^{\mathrm{hom}}}{\partial \hat{r}}. \tag{36}$$

6 Asymptotic matching of solutions

We first match the solutions (21) and (23), for the pipe and step regions, respectively. Here $\tilde{p}_0(\tilde{x})$ (for the pipe) is considered as the outer expansion and $p_0^*(x^*)$ (for the step) as the inner expansion. The outer variable is $\tilde{x} = x/L_0$, while the inner variable is $x^* = (x - x_2)/r_0 = (\tilde{x} - \tilde{x}_2)/\epsilon$, which gives that $\tilde{x} = \epsilon x^* + \tilde{x}_2$. The matching principle applied here is ([21], p. 266)

Inner expansion of (outer expansion) = Outer expansion of (inner expansion), (37)

which for the present pipe-step matching problem takes the form

$$\lim_{\substack{\epsilon \to 0 \\ \tilde{x} - \tilde{x}_2 \text{ fixed}}} p_0^*\left(\frac{\tilde{x} - \tilde{x}_2}{\epsilon}\right) = \lim_{\substack{\epsilon \to 0 \\ x^* \text{ fixed}}} \tilde{p}_0\left(\epsilon x^* + \tilde{x}_2\right). \tag{38}$$

Evaluation of (38) gives the relation

$$C_0^* = \tilde{A}_0 \left\{\cos \tilde{k}\tilde{x}_2 - \cot \tilde{k}\tilde{x}_3 \sin \tilde{k}\tilde{x}_2\right\}. \tag{39}$$

Next we will match (23) for the (downstream) step with (32) for the cavity. That is to say, in the cavity, only the particular solution $\hat{p}_0^{\mathrm{part}}$ will be considered. (The homogeneous part of the cavity-solution, $\hat{p}_0^{\mathrm{hom}}$, will be determined in connection with matching of axial velocity components; see a little later.) The outer variable is now $\hat{x} = \tilde{x} = x/L_0$. The inner variable is $x^* = (x - x_2)/r_0 = (\hat{x} - \hat{x}_2)/\epsilon$, giving $\hat{x} = \epsilon x^* + \hat{x}_2$. A limiting process similar to (38) now gives

$$C_0^* = \sum_m \sum_n \hat{f}_n(\hat{r}, \hat{r}_m) \mathrm{e}^{\mathrm{i}\kappa_n |\hat{x}_2 - \hat{x}_m|}. \tag{40}$$

As (23) prescribes, C_0 is to be a constant. We thus take the mean value over $\hat{r}$,

$$\begin{aligned} \int_0^{\hat{\delta}} C_0^* d\hat{r} &= \sum_m \sum_n \int_0^{\hat{\delta}} \hat{f}_n(\hat{r}, \hat{r}_m) \mathrm{e}^{\mathrm{i}\kappa_n |\hat{x}_2 - \hat{x}_m|} d\hat{r} \quad \Rightarrow \\ C_0^* &= \sum_m \sum_n \frac{1}{\hat{\delta}} \int_0^{\hat{\delta}} \hat{f}_n(\hat{r}, \hat{r}_m) \mathrm{e}^{\mathrm{i}\kappa_n |\hat{x}_2 - \hat{x}_m|} d\hat{r}. \end{aligned} \tag{41}$$

In this way we obtain the pressure within the pipe on the form

$$\tilde{p}_0 = \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{1}{\hat{\delta}} \int_0^{\hat{\delta}} \hat{f}_n(\hat{r}, \hat{r}_m) d\hat{r} \mathrm{e}^{\mathrm{i}\kappa_n |\hat{x}_2 - \hat{x}_m|} \frac{\sin \tilde{k}(\tilde{x}_3 - \tilde{x})}{\sin \tilde{k}(\tilde{x}_3 - \tilde{x}_2)}. \tag{42}$$

As to the averaging over $\hat{r}$, it is noted that averaging over the cross-sectional area, on the form $\int_0^{\hat{\delta}} \cdots \hat{r} d\hat{r}$, probably is more natural; this integral is however equal to zero. Evaluation of the averaging integral in (41), (42) gives

$$\frac{1}{\hat{\delta}} \int_0^{\hat{\delta}} \hat{f}_n(\hat{r}, \hat{r}_m) d\hat{r} = -\frac{\hat{\Gamma}_m}{2\pi\hat{\delta}^2} \frac{1 - \frac{\pi}{2}\mathbf{H}_1(\zeta_n)}{J_0(\zeta_n)} \left[\hat{\mathfrak{v}}_m J_0\left(\zeta_n \frac{\hat{r}_m}{\hat{\delta}}\right) + \mathrm{i}\,\hat{\mathfrak{u}}_m \frac{\zeta_n}{\kappa_n \hat{\delta}} J_1\left(\zeta_n \frac{\hat{r}_m}{\hat{\delta}}\right)\right], \tag{43}$$

where $\mathbf{H}_1$ is the Struve function of order unity ([1], p. 496).

It is interesting to note that (42) has a form similar to the case where the pressure pulsations in the pipe are driven by an oscillating piston at $\tilde{x} = \tilde{x}_2$ ([13], p. 176). The pressure amplitude will go to infinity at the pipe resonance frequencies $\tilde{k}\tilde{\ell} = j\pi$, $j = 1, 2, \cdots$, where $\tilde{\ell} = \tilde{x}_3 - \tilde{x}_2$ is the length of the pipe. Contrary to resonance in (solid) mechanical oscillators with viscous damping, this (case of infinite amplitude) remains true even when viscosity is included (see again [13], p. 176).

Next we will consider matching of the axial velocity components. For the pipe section we have $\partial\tilde{p}_0/\partial\tilde{x} = \mathrm{i}\tilde{k}\tilde{u}_0$, giving

$$\tilde{u}_0 = \mathrm{i}\tilde{A}_0 \left\{ \sin\tilde{k}\tilde{x} + \cot\tilde{k}\tilde{x}_3 \cos\tilde{k}\tilde{x} \right\}. \tag{44}$$

For the step we have $\partial p_1^*/\partial x^* = \mathrm{i}k^* u_0^*$, giving (just downstream of the step)

$$u_0^* = Y_0^{*+} a_0^{*+} + \sum_{n=1}^{\infty} Y_n^{*+} a_n^{*+} e^{-\lambda_n x^{*+}} J_0(\lambda_n r^*), \tag{45}$$

where the coefficients Y_n^{*+}, and likewise Y_n^{*-} just upstream of the step, are acoustic admittances ([11], [18], p. 104), defined by

$$Y_n^{*\pm} = \frac{1}{\mathrm{i}k^* p_{1n}^{*\pm}} \frac{\partial p_{1n}^{*\pm}}{\partial x^{*\pm}}. \tag{46}$$

Here $p_{1n}^{*\pm}$ is the n'th term in the expansion (25).

Now

$$\lim_{\substack{\epsilon \to 0 \\ \tilde{x} - \tilde{x}_2 \text{ fixed}}} u_0^* \left(\frac{\tilde{x} - \tilde{x}_2}{\epsilon} \right) = \lim_{\substack{\epsilon \to 0 \\ x^* \text{ fixed}}} \tilde{u}_0 \left(\epsilon x^* + \tilde{x}_2 \right) \tag{47}$$

gives

$$Y_0^{*+} a_0^{*+} = \mathrm{i}\tilde{A}_0 \left\{ \sin\tilde{k}\tilde{x}_2 + \cot\tilde{k}\tilde{x}_3 \cos\tilde{k}\tilde{x}_2 \right\}. \tag{48}$$

Next, for the cavity we use $\partial\hat{p}_0^{\mathrm{hom}}/\partial\hat{x} = \mathrm{i}\hat{k}\hat{u}_0$. Matching then gives

$$Y_0^{*-} a_0^{*-} = \sum_{n=0}^{\infty} \mathrm{i}\kappa_n J_0 \left(\zeta_n \frac{\hat{r}}{\hat{\delta}} \right) \hat{C}_n e^{\mathrm{i}\kappa_n \hat{x}_2}, \tag{49}$$

where (28) gives that Y_0^{*+} is related to Y_0^{*-} as follows:

$$Y_0^{*-} a_0^{*-} = \epsilon_1^2 Y_0^{*+} a_0^{*+}, \tag{50}$$

where, again, $\epsilon_1 = r_0/r_1$.

In order to determine the coefficients $\hat{C}_n$, we multiply both sides of (49) by $J_0\left(\zeta_m \frac{\hat{r}}{\hat{\delta}}\right) \hat{r}/\hat{\delta}$, and integrate over $\hat{r}$,

$$Y_0^{*-} a_0^{*-} \int_0^1 J_0 \left(\zeta_m \frac{\hat{r}}{\hat{\delta}} \right) \frac{\hat{r}}{\hat{\delta}} d\frac{\hat{r}}{\hat{\delta}} = \sum_{n=0}^{\infty} \mathrm{i}\kappa_n \hat{C}_n e^{\mathrm{i}\kappa_n \hat{x}_2} \int_0^1 J_0 \left(\zeta_m \frac{\hat{r}}{\hat{\delta}} \right) J_0 \left(\zeta_n \frac{\hat{r}}{\hat{\delta}} \right) \frac{\hat{r}}{\hat{\delta}} d\frac{\hat{r}}{\hat{\delta}}. \tag{51}$$

This gives

$$\mathrm{i}\kappa_0 \hat{C}_0 = Y_0^{*-} a_0^{*-}. \tag{52}$$

Thus we obtain the axial feedback velocity component within the cavity on the form

$$\hat{u}_0 = \epsilon_1^2 \frac{e^{\mathrm{i}\kappa_n |\hat{x} - \hat{x}_2|}}{\hat{k}} \cot\tilde{k}(\tilde{x}_3 - \tilde{x}_2) \sum_{m=1}^{M} \sum_{n=0}^{\infty} \frac{1}{\hat{\delta}} \int_0^{\hat{\delta}} \hat{f}_n(\hat{r}, \hat{r}_m) d\hat{r} e^{\mathrm{i}\kappa_n |\hat{x}_2 - \hat{x}_m|}. \tag{53}$$

As by the pressure equation (42), the velocity amplitude will go to infinity at the pipe resonance frequencies $\tilde{k}\tilde{\ell} = j\pi$, $j = 1, 2, \cdots$, $\tilde{\ell} = \tilde{x}_3 - \tilde{x}_2$.

7 Time domain expressions

Finally, the most important frequency-domain expressions (42) and (53) (with (43)) are reverted to the time domain by employing the second of the equations (6). It seems to be most convenient to invert certain blocks one at a time (see the Appendix), and then couple these blocks via the convolution theorem ([20], p. 464). The final expression for the pressure within the tailpipe (42) is

$$\tilde{p}_0(\tilde{x},\tau) = \hat{\delta}^{-2}\sum_{j=1}^{\infty}\sum_{m=1}^{M}\sum_{n=0}^{\infty}\int_0^\tau\int_0^\tau \frac{\Gamma_m(\alpha)}{J_0(\zeta_n)}\left\{1-\frac{\pi}{2}\mathbf{H}_1(\zeta_n)\right\}\times \tag{54}$$
$$\times(-1)^{j+1}\tilde{\ell}^{-1}\sin\frac{j\pi}{\tilde{\ell}}(\tilde{x}_3-\tilde{x})\sin\frac{j\pi}{\tilde{\ell}}(\tau-\alpha-\beta)\times$$
$$\times\left[\hat{\mathfrak{v}}_m(\alpha)J_0\left(\zeta_n\frac{\hat{r}_m}{\hat{\delta}}\right)\frac{\partial}{\partial\hat{x}_2}H_0^{(1)}\left(T_{mn}(\beta)\right)-\hat{\mathfrak{u}}_m(\alpha)J_1\left(\zeta_n\frac{\hat{r}_m}{\hat{\delta}}\right)H_0^{(1)}\left(T_{mn}(\beta)\right)\right]d\alpha d\beta,$$

where $\tilde{\ell}=\tilde{x}_3-\tilde{x}_2$ is the length of the tailpipe, and

$$T_{mn}(\tau)=\begin{cases}\frac{\zeta_n}{\hat{\delta}}\sqrt{\tau^2-|\hat{x}_2-\hat{x}_m|^2} & \text{for}\quad \tau^2>|\hat{x}_2-\hat{x}_m|^2\\ \mathrm{i}\frac{\zeta_n}{\hat{\delta}}\sqrt{|\hat{x}_2-\hat{x}_m|^2-\tau^2} & \text{for}\quad \tau^2<|\hat{x}_2-\hat{x}_m|^2\end{cases}. \tag{55}$$

The final expression for the axial feedback velocity component within the cavity (53) is

$$\hat{u}_0(\hat{x},\tau) = -\frac{\epsilon_1^2}{\hat{\delta}^2}\sum_{j=1}^{\infty}\sum_{m=1}^{M}\sum_{n=0}^{\infty}\int_0^\tau\int_0^\tau\int_0^\tau \frac{\Gamma_m(\alpha)}{J_0(\zeta_n)}\left\{1-\frac{\pi}{2}\mathbf{H}_1(\zeta_n)\right\}\times \tag{56}$$
$$\times\left[\hat{\mathfrak{v}}_m(\alpha)J_0\left(\zeta_n\frac{\hat{r}_m}{\hat{\delta}}\right)\frac{\partial}{\partial\hat{x}_2}H_0^{(1)}\left(T_{mn}(\beta)\right)-\hat{\mathfrak{u}}_m(\alpha)J_1\left(\zeta_n\frac{\hat{r}_m}{\hat{\delta}}\right)H_0^{(1)}\left(T_{mn}(\beta)\right)\right]\times$$
$$\times\tilde{\ell}^{-1}\sin\frac{j\pi}{\tilde{\ell}}(\tau-\alpha-\beta-\gamma)H(\gamma-|\hat{x}_2-\hat{x}|)d\alpha d\beta d\gamma.$$

8 Concluding remarks

1. Analytical (approximate) expressions have been obtained, via matched asymptotic expansions, for the pressure and the axial component of the acoustic feedback velocity in a cavity-pipe system.

2. The radial component of the acoustic feedback velocity does not come into play by the order of analysis considered here. It is intuitively understandable that a radial velocity component borne from the purely axial pipe oscillations necessarily must be very small. Yet its effect might not be negligibly small, and it would be of interest to continue the analysis to higher orders.

3. Just as by the pressure in the pipe, the amplitude of the axial component of the acoustic feedback velocity becomes infinite at the pipe resonance frequencies. This indicates the possibility of lock-in of the self-sustained flow oscillations in the cavity to the resonant acoustic pipe pressure oscillations.

4. Future work will, first and foremost, be concerned with numerical computations based on the present results. As to extensions of the analytical work (besides the higher order terms mentioned just above) inclusion of the free space solution (downstream from the free pipe end), along the lines discussed in the second of the two papers by Lesser & Lewis [17], would be interesting.

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Appendix. Fourier inversions

In the inversion of (42), a useful result is that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\tilde{k}(\tilde{x}_3-\tilde{x})}{\sin\tilde{k}\tilde{\ell}}\mathrm{e}^{-\mathrm{i}\tilde{k}\tau}d\tilde{k}=\frac{1}{\pi\tilde{\ell}}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi}{\tilde{\ell}}(\tilde{x}_3-\tilde{x})\sin\frac{n\pi}{\tilde{\ell}}\tau. \tag{57}$$

This result has been obtained by the method of residues. Similarly, for the inversion of (53), use is made of that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\cot\tilde{k}\tilde{\ell}\mathrm{e}^{-\mathrm{i}\tilde{k}\tau}d\tilde{k}=-\frac{1}{\pi\tilde{\ell}}\sum_{n=1}^{\infty}\sin\frac{n\pi}{\tilde{\ell}}\tau, \tag{58}$$

and that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\mathrm{i}\kappa_n}\mathrm{e}^{\mathrm{i}\kappa_n|\hat{x}_2-\hat{x}_m|}\mathrm{e}^{-\mathrm{i}\tilde{k}\tau}d\tilde{k}=-\frac{1}{2}H_0^{(1)}\left(T_{mn}(\tau)\right), \tag{59}$$

where $T_{mn}(\tau)$ is given by (55). From (59) we can obtain that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\kappa_n|\hat{x}_2-\hat{x}_m|}\mathrm{e}^{-\mathrm{i}\tilde{k}\tau}d\tilde{k}=-\frac{1}{2}\frac{\partial}{\partial\hat{x}_2}H_0^{(1)}\left(T_{mn}(\tau)\right). \tag{60}$$

Finally, in (56) it has been used also that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\hat{k}}\mathrm{e}^{\mathrm{i}\hat{k}|\hat{x}_2-\hat{x}_m|}\mathrm{e}^{-\mathrm{i}\hat{k}\tau}d\hat{k}=H(\tau-|\hat{x}_2-\hat{x}_m|), \tag{61}$$

where H is the Heaviside unit step function, as defined in connection with (1).

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