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Mathematical and Numerical Analyses on a Hamilton-Jacobi-Bellman Equation Governing Ascending Behaviour of Fishes

(邦題：魚類の遡上行動を支配する Hamilton-Jacobi-Bellman 方程式の数理および数値解析)

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Abstract
Ascending behaviour of individual fishes in 1-D open channels is considered as a transport phenomenon governed by a continuous time stochastic process model. A stochastic control problem is formulated that determines the drift of the model based on a minimization principle of physiological energy consumption of the fish during migration. The problem ultimately reduces to solving a Hamilton-Jacobi-Bellman equation governing the optimal ascending velocity, which is a nonlinear and nonconservative parabolic partial differential equation. Mathematical and numerical analyses on the equation are performed for comprehending behaviour of its solutions. Some numerical issues encountered in solving the equation are also discussed.

Key words: Ascending behaviour, stochastic differential equation, stochastic control problem, energy minimization principle, Hamilton-Jacobi-Bellman equation, ascending condition

1. Introduction

Ascending behaviour of fishes in open channels, such as rivers, agricultural drainage canals, and fishways, is a complicated transport phenomenon. Assessment of ascending behaviour of individual fishes is one of the most crucial hydro-environmental research topics because of the urgent need to establish an effective framework for improving and preserving aquatic ecological systems where fishes play central roles. One example is designing a fishway passaging upstream and downstream water bodies of a physical barrier, such as dams and headworks (Katopodis and Williams, 2012). Another example is assessment of ecological functions of surface water systems serving as passages and habitats for fishes, such as stream networks (Cote *et al.*, 2009) and surface agricultural drainage systems (Unami *et al.*, 2010).

Ascending behaviour of fishes is subject to inherent disturbances due to our limited knowledge and environmental and ecological stochasticity. Mathematical models serve as effective means for simulating hydraulic processes in surface water bodies, which provide basic hydraulic information for considering migration of fishes. Although a large number of researches discussed the hydraulic processes, far less number of researches focused on migration of fishes, mainly due to difficulties to find their reasonable mathematical expressions (Liao, 2007; Willis, 2011). It has been suggested that stochastic process models are effective for comprehending migration of fishes, in which the stochasticity embedded in the dynamics is considered (Fujihara and Akimoto, 2010). Although these models are effective for assessing the ascending behaviour of fishes, they assume that the hydraulic processes determine the behaviour without considering biological and ecological feedbacks, such as the physiological energy consumption of fishes during the migration (Brodersen *et al.*, 2008). One possible way to develop a more reasonable model for ascending behaviour of fishes considering hydraulics, biology, and ecology in a feedback manner is to formulate the problem in the context of optimal control based on the stochastic differential equation (SDE) (Øksendal, 2007); however, such a model has not been presented so far.

The purpose of this paper is to present a stochastic process model for ascending behaviour of fishes, in which the hydraulic, ecological, biological effects involved in the dynamics are considered in a feedback manner based on a stochastic control theory. Lagrangian movement of individual fish is considered as a controlled Markov process subject to shallow water flows. A stochastic control problem is then formulated for determining the optimal ascending strategy that harmonizes the two conflicting objectives: minimization of the total physiological energy consumption and maximization of the profit gained when reaching the upstream area.

2. Stochastic process model

This paper focuses exclusively on 1-D problems. The domain of water flows is the 1-D open channel $\Omega=(0,L)$ with its length $L(>0)$ and water depth $h(>0)$. The flow velocity of water in the channel is denoted by V, which is assumed to be unidirectional and its positive direction is same with that of the x abscissa defined along the channel $(V>0)$. The upstream- and downstream- ends of the channel are $x=0$ and $x=L$, respectively. The position of individual fish at the time t is denoted by X_t, which is a continuous time stochastic process. Inspiring from the stochastic process model for Lagrangian movement of solute particles (Yoshioka and Unami, 2013), the SDE governing X_t is proposed as

$$\mathrm{d}X_t=(V-u)\mathrm{d}t+\sqrt{2D}\mathrm{d}B_t \tag{1}$$

where B_t is the 1-D standard Brownian motion (Øksendal, 2007), u is the ascending speed of fish where its positive direction is taken same with that of $-x$, and $D(>0)$ is the dispersivity that modulates the magnitude of the stochasticity involved in the dynamics, which should be related with turbulent intensity of the flow. The ascending velocity u is the control variable of the model, which is assumed to be constrained in the admissible set

$$U=\{u \mid |u|\le u_{\mathrm{M}}\} \tag{2}$$

for a positive constant u_{M} that can be naively taken as the maximum swimming speed of fishes, but would actually vary in both space and time depending on hydraulic and biological conditions. In this paper, u_{M} is assumed to be constant for the sake of brevity and is referred to as the maximum swimming speed. The coefficients V and D are assumed not to involve the control variable u. The generator A of the coupled stochastic process $Y_t=(t,X_t)$ conditioned on $Y_s=(s,x)$ is given by (Øksendal, 2007)

$$AY=\frac{\partial y}{\partial s}+(V-u)\frac{\partial y}{\partial x}+D\frac{\partial^2 y}{\partial x^2} \tag{3}$$

for a sufficiently regular function $y=y(s,x)$.

3. Stochastic control problem

3.1 Hamilton-Jacobi-Bellman equation

A stochastic control problem is formulated in order to determine the ascending velocity u. Literatures indicate that fishes minimize physiological energy consumption during migration depending on local hydraulic conditions (Brodersen *et al.*, 2008). Assuming that the fish strategically ascends the open channel Ω toward the upstream boundary $x=0$ based on a physiological energy consumption minimization principle, in which the value function J^u to be maximized is proposed as

$$J^u(s,x)=\mathrm{E}^{s,x}\left[\int_s^{\bar{T}}\left(-\frac{1}{2}u^2\right)\mathrm{d}t+G\left(\bar{T},Y_{\bar{T}}\right)\right] \text{ with } \bar{T}=\min\left(T,\tau^{s,x}\right) \tag{4}$$

where $\mathrm{E}^{s,x}[\cdot]$ represents the expectation conditioned on $Y_s=(s,x)$, T is the terminal time, $\tau^{s,x}$ is the first exit time of the process Y_t from the spatio-temporal domain $\Xi=\Omega\times(-\infty,T)$, and $G(\ge 0)$ is the profit specified on the boundary $\partial\Xi$ of Ξ. The profit G is specified on the boundary $\partial\Xi$ as

$$G(s,0)=P(>0),\ G(s,L)=0,\text{ and } G(x,T)=0, \tag{5}$$

so that the fish gains the profit if and only if it approaches the upstream-end $x=0$.

The optimal control variable maximizing the value function J^u is denoted by u^*. Based on the dynamic programming principle (Øksendal, 2007), the HJBE governing the maximized value function

$$\Phi(s,x)=\max_{u\in U} J^u(s,x)=J^{u^*}(s,x) \tag{6}$$

is derived as

$$\max_{u\in U}\left(L\Phi-\frac{1}{2}u^2\right)=\left(L\Phi-\frac{1}{2}u^2\right)_{u=u_*}=0, \tag{7}$$

which is rewritten as

$$\frac{\partial\Phi}{\partial s}+D\frac{\partial^2\Phi}{\partial x^2}+\max_{u\in U}\left((V-u)\frac{\partial\Phi}{\partial x}-\frac{1}{2}u^2\right)=\frac{\partial\Phi}{\partial s}+D\frac{\partial^2\Phi}{\partial x^2}+\left((V-u)\frac{\partial\Phi}{\partial x}-\frac{1}{2}u^2\right)_{u=u^*}=0. \tag{8}$$

The last term in the middle of Eq.(8) is a quadratic function of u, and its maximization is achieved by the optimal control variable u^*, which is analytically given by

$$u^*=-\chi\frac{\partial\Phi}{\partial x}-(1-\chi)u_{\mathrm{M}}\operatorname{sgn}\left(\frac{\partial\Phi}{\partial x}\right) \tag{9}$$

where $\chi=\chi\left(\left|\frac{\partial\Phi}{\partial x}\right|\le u_{\mathrm{M}}\right)$ is the characteristic function, namely

$$\chi=1\ \text{ for }\ \left|\frac{\partial\Phi}{\partial x}\right|\le u_{\mathrm{M}}\ \text{ and }\ \chi=0\ \text{ for }\ \left|\frac{\partial\Phi}{\partial x}\right|>u_{\mathrm{M}}. \tag{10}$$

Substituting Eq.(9) into Eq.(8) yields

$$\frac{\partial\Phi}{\partial s}+(V+v)\frac{\partial\Phi}{\partial x}+D\frac{\partial^2\Phi}{\partial x^2}-\frac{1-\chi}{2}u_{\mathrm{M}}^2=0 \tag{11}$$

with the auxiliary variable

$$v=\frac{\chi}{2}\frac{\partial\Phi}{\partial x}+(1-\chi)u_{\mathrm{M}}\operatorname{sgn}\left(\frac{\partial\Phi}{\partial x}\right). \tag{12}$$

The HJBE(11) is a nonlinear and nonconservative parabolic partial differential equation with a source term whose analytical solution is not available except for some simplified cases. Hereafter, all the know functions involved in the problem are assumed to be constant unless otherwise specified for the sake of simplicity. Then, it is reasonable to assume stationarity of the dynamics ($\Phi=\Phi(x)$ and $T\to+\infty$) where the HJBE(11) reduces to

$$(V+v)\frac{\mathrm{d}\Phi}{\mathrm{d}x}+D\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2}-\frac{1-\chi}{2}u_{\mathrm{M}}^2=0, \tag{13}$$

which is a non-linear ordinary differential equation. The problem thus reduces to finding the solution to Eq.(13) with the known functions determined from a shallow water flow field.

3.2 Ascending condition

The ascending condition of the fish is defined so that passage efficiency of the channel Ω can be analytically assessed with the present model. The ascending condition in this paper is given by

$$V_{\mathrm{g}}=V-u^*<0\ \text{ in }\ \Omega, \tag{14}$$

which means that the ground velocity V_{g} of the fish with the optimal ascending velocity u^* is negative (is directed toward the upstream) everywhere in the channel Ω. Eq.(14) is rewritten with Eq.(9) as

$$V_{\mathrm{g}}=V-u^*=V+\chi\frac{\partial\Phi}{\partial x}+(1-\chi)u_{\mathrm{M}}\operatorname{sgn}\left(\frac{\partial\Phi}{\partial x}\right)\ \text{ in }\ \Omega. \tag{15}$$

According to Eq.(14), fishes do not ascend the channel if $V-u_{\mathrm{M}}>0$ in Ω. Such a trivial condition is out of the interest of this paper and the condition $V-u_{\mathrm{M}}\le 0$ is assumed to be satisfied in Ω.

4. Mathematical analysis on the HJBE

Mathematical analysis on the HJBE(13) is performed. The HJBE(13) is non-dimensionalized for the sake of brevity of the analysis. Eqs.(12) and (13) are non-dimensionalized as

$$(1+w)\frac{\mathrm{d}\phi}{\mathrm{d}y}+\frac{1}{p}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}-\frac{1-\chi}{2}w_{\mathrm{M}}^2=0 \tag{16}$$

and

$$w=\frac{\chi}{2}\frac{\mathrm{d}\phi}{\mathrm{d}y}+(1-\chi)w_{\mathrm{M}}\,\mathrm{sgn}\left(\frac{\mathrm{d}\phi}{\mathrm{d}y}\right), \tag{17}$$

respectively, using the non-dimensional variables

$$y=\frac{x}{L},\quad \phi=\frac{\Phi}{VL},\quad w_{\mathrm{M}}=\frac{u_{\mathrm{M}}}{V},\quad p=\frac{VL}{D},\ \text{and}\ P_0=\frac{P}{VL}. \tag{18}$$

The following two cases w_{M} are considered in this paper, which are

Case (a): $w_{\mathrm{M}}=+\infty$ ($u_{\mathrm{M}}=+\infty$: unbounded case)

and

Case (b): $0<w_{\mathrm{M}}<+\infty$ ($0<u_{\mathrm{M}}<+\infty$: bounded case).

4.1 Case (a): unbounded case ($w_{\mathrm{M}}=+\infty$)

This is an idealized case where the admissible set U is identified with the 1-D space. $\mathbb{R}$ although it has been indicated that there certainly exists an upper bound of the maximum swimming speed for each fish (Iosilveskii and Weihs, 2008). In this case, Eqs.(16) and (17) reduce to

$$\left(1+\frac{1}{2}\frac{\mathrm{d}\phi}{\mathrm{d}y}\right)\frac{\mathrm{d}\phi}{\mathrm{d}y}+\frac{1}{p}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}=0. \tag{19}$$

Assuming that Eq.(19) has a classical solution, application of the variable transformation

$$\psi=e^{\frac{p\phi}{2}} \tag{20}$$

to it leads to

$$\frac{\mathrm{d}\psi}{\mathrm{d}y}+\frac{1}{p}\frac{\mathrm{d}^2\psi}{\mathrm{d}y^2}=0, \tag{21}$$

which is analytically solvable. The model is therefore tractable in this case. If $P_0\ne 2$, the solution to Eq.(21) is analytically derived with the transformed boundary conditions

$$\psi(0)=e^{\frac{pP_0}{2}}\ \text{and}\ \psi(1)=1 \tag{22}$$

as

$$\psi=\frac{1-e^{p\left(\frac{P_0}{2}-1\right)}+\left(e^{\frac{pP_0}{2}}-1\right)e^{-py}}{1-e^{-p}}. \tag{23}$$

By Eqs.(20) and (23), the solution to Eq.(19) is derived as

$$\phi=\frac{2}{p}\ln\left(\frac{1-e^{p\left(\frac{P_0}{2}-1\right)}+\left(e^{\frac{pP_0}{2}}-1\right)e^{-py}}{1-e^{-p}}\right) \tag{24}$$

with its gradient

$$\frac{\mathrm{d}\phi}{\mathrm{d}y}=\frac{2\left(e^{\frac{pP_0}{2}}-1\right)}{\left(e^{p\left(\frac{P_0}{2}-1\right)}-1\right)e^{py}-\left(e^{\frac{pP_0}{2}}-1\right)}. \tag{25}$$

The steady solution for $P_0=2$ is derived with the application of the L'Hospital's rule to Eq.(24), which is given by

$$\phi=2-2y \tag{26}$$

with its gradient

$$\frac{\mathrm{d}\phi}{\mathrm{d}y}=-2. \tag{27}$$

For $p>>1$ and $P_0>2$, the maximum absolute value of the gradient $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ is evaluated as

$$\left|\frac{\mathrm{d}\phi}{\mathrm{d}y}\right|_{y=1}=-\left.\frac{\mathrm{d}\phi}{\mathrm{d}y}\right|_{y=1}=O\left(e^{p\left(\frac{P_0}{2}-1\right)}\right), \tag{28}$$

indicating that there exists a boundary layer near $y=1$ with the width of $O\left(e^{-p\left(\frac{P_0}{2}-1\right)}\right)$. **Figures 1(a)** and **1(b)** show profiles of the solution (25) for different values of p, showing that there certainly exists one sharp boundary layer in each solution profile. According to Eq.(14), the optimal ground velocity V_g in the present case diverges near $y=1$ as p increases.

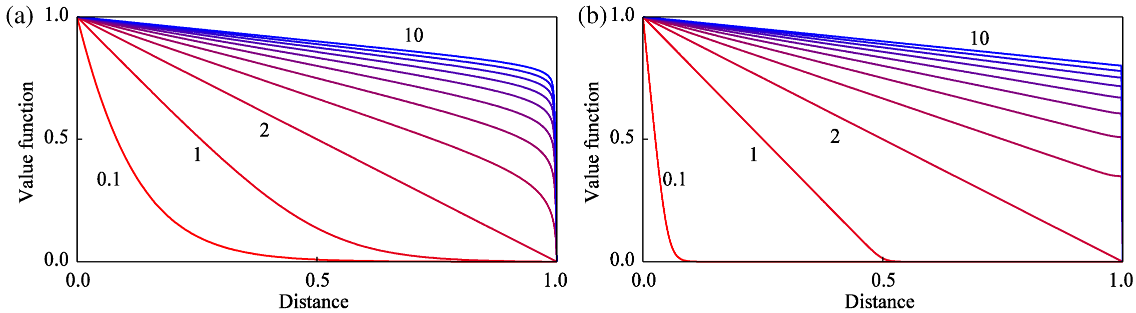

Figure 1: Steady solutions in Eq.(24) with (a) $p=10$ and (b) $p=100$ for the boundary values $P_0=0.1$ and $P_0=1,2\ldots,10$. The solutions are normalized with P_0.

By Eq.(25), the solution (24) satisfies the ascending condition

$$1-w^*=1+\frac{\mathrm{d}\phi}{\mathrm{d}y}<0 \tag{29}$$

if

$$P_0>\frac{2}{p}\ln\left(\frac{1+e^p}{2}\right). \tag{30}$$

Eq.(30) is satisfied if

$$P_0\geq 2, \tag{31}$$

which does not depend on the parameter p. The right-hand side of Eq.(30) is a non-increasing function of p. Eq.(30) therefore states that the ascending condition is more relaxed as p decreases (or equivalently as D increases), indicating that the increase of turbulence would enhance the ascending behaviour of fishes. Eq.(31) is rewritten in a non-dimensional form as

$$P \geq 2VL, \tag{32}$$

showing that the fishes ascend the channel Ω if the profit gained when reaching the upstream-end $y = 0$ is sufficiently large for fixed V and L.

4.2 Case (b): bounded case ($w_{\mathrm{M}} = +\infty$)

In this case, Eqs.(12) and (13) reduce to

$$\left(1 + \frac{1}{2}\frac{\mathrm{d}\phi}{\mathrm{d}y}\right)\frac{\mathrm{d}\phi}{\mathrm{d}y} + \frac{1}{p}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} = 0 \tag{33}$$

if $\chi = 1$ and to

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} + w_{\mathrm{M}}\left|\frac{\mathrm{d}\phi}{\mathrm{d}y}\right| + \frac{1}{p}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - \frac{1}{2}w_{\mathrm{M}}^2 = 0 \tag{34}$$

if $\chi = 0$. Straightforward calculations show that the condition $\chi = 1$ is satisfied over Ω if

$$P_0 \leq 2 \leq w_{\mathrm{M}}. \tag{35}$$

Similarly, the condition $\chi = 0$ is satisfied over the domain Ω if

$$1 < w_{\mathrm{M}} < 2 \text{ and } 2 < \frac{w_{\mathrm{M}}^2}{2(w_{\mathrm{M}} - 1)} < P_0. \tag{36}$$

Application of an elliptic maximum principle to Eq.(34) leads to $\frac{\mathrm{d}\phi}{\mathrm{d}y} < 0$ in Ω, which reduces Eq.(34) to

$$(1 - w_{\mathrm{M}})\frac{\mathrm{d}\phi}{\mathrm{d}y} + \frac{1}{p}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} = \frac{1}{2}w_{\mathrm{M}}^2. \tag{37}$$

Eq.(37) has the analytical solution

$$\phi = P_0 + \frac{w_{\mathrm{M}}^2}{2(w_{\mathrm{M}} - 1)}y - \left(P_0 + \frac{w_{\mathrm{M}}^2}{2(w_{\mathrm{M}} - 1)}\right)\frac{e^{p(w_{\mathrm{M}}-1)y} - 1}{e^{p(w_{\mathrm{M}}-1)} - 1} \tag{38}$$

with its gradient

$$\frac{\mathrm{d}\phi}{\mathrm{d}y} = \frac{w_{\mathrm{M}}^2}{2(w_{\mathrm{M}} - 1)} - p\left((w_{\mathrm{M}} - 1)P_0 + \frac{w_{\mathrm{M}}^2}{2}\right)\frac{e^{p(w_{\mathrm{M}}-1)y}}{e^{p(w_{\mathrm{M}}-1)} - 1} < 0, \tag{39}$$

showing that the solution (38) satisfies the ascending condition if $w_{\mathrm{M}} > 1$, namely if the maximum swimming speed $u_{\mathrm{M}} > 1$ is larger than the flow velocity V. Although not presented in this paper, it has been numerically confirmed that the solution ϕ satisfies the ascending condition if both $P_0 > 2$ and $w_{\mathrm{M}} > 2$ are satisfied. Maximum absolute value of the gradient $\frac{\mathrm{d}\phi}{\mathrm{d}y}$ is evaluated as

$$\left|\frac{\mathrm{d}\phi}{\mathrm{d}y}\right|_{y=1} = -\left.\frac{\mathrm{d}\phi}{\mathrm{d}y}\right|_{y=1} = O(p) \ (p >> 1), \tag{40}$$

indicating that there exists a boundary layer near $y = 1$ for sufficiently large p, but which is less sharp compared with that in the unbounded case where the layer has exponential width. **Figures 2(a)** and **2(b)** show the solutions to the HJBE in the bounded case with $w_M = 1.5$ for different values of p. Similarly, **Figures 3(a)** and **3(b)** show the solutions with $w_M = 2.5$. **Figures 2** and **3** show that there exists one sharp boundary layer in each solution profile but is not apparently sharper than that in the corresponding unbounded case subject to the same values of p and P_0 in **Figure 1**. **Figures 2** and **3** show that the boundary layer becomes sharper as w_{M} increases.

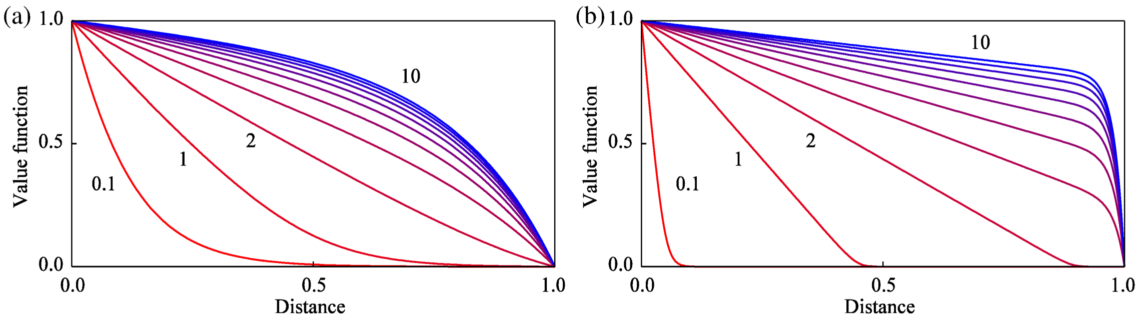

Figure 2: Steady solutions in the bounded case for $w_M = 1.5$ with (a) $p = 10$ and (b) $p = 100$ for the boundary values $P_0 = 0.1$ and $P_0 = 1, 2 \ldots, 10$. The solutions are normalized with P_0.

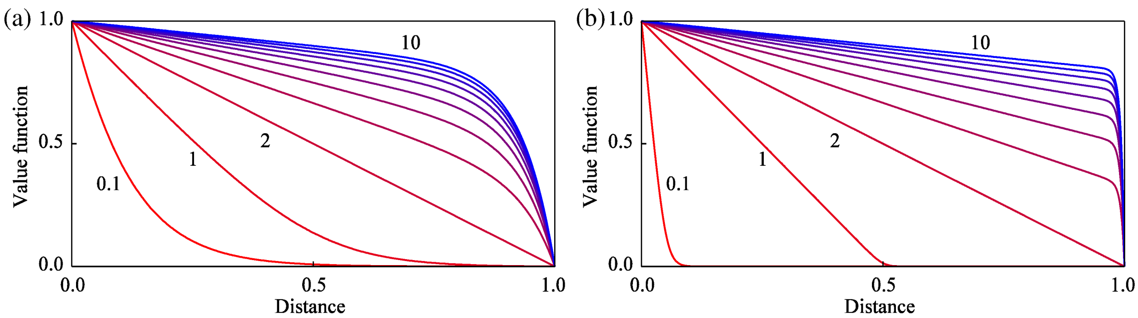

Figure 3: Steady solutions in the bounded case for $w_M = 2.5$ with (a) $p = 10$ and (b) $p = 100$ for the boundary values $P_0 = 0.1$ and $P_0 = 1, 2 \ldots, 10$. The solutions are normalized with P_0.

If the two parameters P_0 and w_M satisfy neither Eqs.(35) nor (36), then there may exist at least one point $\varsigma \in \Omega$ serving as the interface of the sub-domain with $\chi = 0$ and that with $\chi = 1$. Numerical simulations for a variety of the parameter values (P_0, p) suggest that there exists at most one ς for each solution profile with fixed (P_0, p). It has also been numerically checked that the solution is such that $\chi = 1$ in $(0, \varsigma)$ and $\chi = 0$ in $(\varsigma, 1)$ if $w_M > 2$, and $\chi = 0$ in $(0, \varsigma)$ and $\chi = 1$ in $(\varsigma, 1)$ if $w_M < 2$. No analytical expression of ς has been derived so far.

Under the other conditions in the bounded case, analytical solution to the HJBE is not available. In addition, regularity of the solution to the HJBE in such cases is not a trivial issue and comprehending its mathematical properties requires the use of an appropriate mathematics. In this paper, a regularization method for the HJBE is presented for smoothness of its solution, which is later implemented into a numerical method. Firstly, the function $h_K = h_K(a)$ is defined as

$$h_K = \begin{cases} a & (|a| \le K) \\ K\,\mathrm{sgn}(a) & (|a| > K) \end{cases} \tag{41}$$

with its gradient

$$\frac{\partial h_K}{\partial a} = \begin{cases} 1 & (|a| \le K) \\ 0 & (|a| > K) \end{cases} \tag{42}$$

where $K(>0)$ is a positive, bounded constant. Eqs.(41) and (42) show that h_K and $\frac{\partial h_K}{\partial a}$ are bounded. Secondly, the characteristics function χ is regularized as

$$\chi_\varepsilon = H_\varepsilon\left(\frac{\mathrm{d}\phi}{\mathrm{d}y} + w_\mathrm{M}\right) - H_\varepsilon\left(\frac{\mathrm{d}\phi}{\mathrm{d}y} - w_\mathrm{M}\right), \tag{43}$$

where ε is a sufficiently small positive constant and H_ε represents the regularized Heaviside function

$$H_\varepsilon(a) = \frac{1}{2}\left(1 + \tanh\left(\frac{a}{\varepsilon}\right)\right) \tag{44}$$

whose partial derivative $\frac{\partial H_\varepsilon}{\partial a}$ is bounded as

$$(0<)\frac{\partial H_\varepsilon}{\partial a} = \frac{1}{\varepsilon \cosh^2\left(\frac{a}{\varepsilon}\right)}. \tag{45}$$

By Eq.(45), the conditions

$$|a\chi_\varepsilon|, \left|a\frac{\partial \chi_\varepsilon}{\partial a}\right| < +\infty \tag{46}$$

are derived. Thirdly, define the function f as

$$f = (1 + w_\varepsilon)a - \frac{1-\chi_\varepsilon}{2} w_\mathrm{M}^2 \tag{47}$$

with the regularized w given by

$$w_\varepsilon = \frac{1}{2}\chi_\varepsilon h_K - (1-\chi_\varepsilon) w_\mathrm{M}, \tag{48}$$

which is bounded because

$$|w_\varepsilon| = \left|\frac{1}{2}\chi_\varepsilon h_K - (1-\chi_\varepsilon) w_\mathrm{M}\right| \le \frac{1}{2}|h_K| + w_\mathrm{M} < +\infty. \tag{49}$$

The partial derivative $\frac{\partial w_\varepsilon}{\partial a}$ is expressed as

$$\frac{\partial w_\varepsilon}{\partial a} = \frac{\partial}{\partial a}\left(\frac{1}{2}\chi_\varepsilon h_K - (1-\chi_\varepsilon) w_\mathrm{M}\right) = \frac{1}{2}\left(\frac{\partial \chi_\varepsilon}{\partial a} h_K + \chi_\varepsilon \frac{\partial h_K}{\partial a}\right) + \frac{\partial \chi_\varepsilon}{\partial a} w_\mathrm{M}, \tag{50}$$

which leads to

$$\left|a\frac{\partial w_\varepsilon}{\partial a}\right| = \left|\frac{1}{2}\left(a\frac{\partial \chi_\varepsilon}{\partial a} h_K + \chi_\varepsilon a\frac{\partial h_K}{\partial a}\right) + a\frac{\partial \chi_\varepsilon}{\partial a} w_\mathrm{M}\right| \le \frac{1}{2}\left|a\frac{\partial \chi_\varepsilon}{\partial a}\right||h_K| + \frac{1}{2}|a\chi_\varepsilon|\left|\frac{\partial h_K}{\partial a}\right| + w_\mathrm{M}\left|a\frac{\partial \chi_\varepsilon}{\partial a}\right| < +\infty. \tag{51}$$

Eqs.(49), (50), and (51) lead to.

$$\left|\frac{\partial f}{\partial a}\right| = \left|1 + w_\varepsilon + a\frac{\partial w_\varepsilon}{\partial a} + \frac{1}{2} w_\mathrm{M}^2 \frac{\partial \chi_\varepsilon}{\partial a}\right| < 1 + w_\varepsilon \left|a\frac{\partial w_\varepsilon}{\partial a}\right| + \frac{1}{2} w_\mathrm{M}^2 \left|\frac{\partial \chi_\varepsilon}{\partial a}\right| < +\infty. \tag{52}$$

By Eq.(52) and the Schauder's fixed point theorem, the regularized HJBE, which is given by

$$(1 + w_\varepsilon)\frac{\mathrm{d}\phi}{\mathrm{d}y} + \frac{1}{p}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} - \frac{1-\chi_\varepsilon}{2} w_\mathrm{M}^2 = 0 \tag{53}$$

with a non-zero ε has a unique classical solution subject to the boundary conditions because it has the bounded drift and source terms in the sense of Yamomoto and Oishi (2006). A numerical analogue of the present regularization is used in the following section.

5. Numerical analysis on the HJBE

This section focuses on numerical simulation of a HJBE in a periodic open channel, which can be regarded as a fishway having a longitudinally periodic structure. Numerical issues encountered when solving the HJBE are also discussed. The channel Ω is assumed to be infinitely long and have a longitudinally periodic structure with the period $l(>0)$. One such example is a vertical slot fishway having a number of pools and slots (Yoshioka *et al.*, 2014b). The coefficients V and D are assumed to be time-independent and spatially periodic with the period l. Without the loss of generality, let $\Omega_l = (0,l)$ be one of the reaches in Ω. Assume that the solution Φ to the HJBE(13) can be decomposed in Ω_l as

$$\Phi = \overline{\Phi} - gx \tag{54}$$

with

$$\overline{\Phi}(0) = \overline{\Phi}(l) = 0 \text{ and } \frac{\mathrm{d}\overline{\Phi}}{\mathrm{d}x}(0) = \frac{\mathrm{d}\overline{\Phi}}{\mathrm{d}x}(l) \tag{55}$$

where $g(>0)$ is a constant representing the longitudinal trend of the solution Φ and $\overline{\Phi}$ is its periodic element. Formally, larger trend means higher profit that the fishes would obtain. Substituting Eq.(54) into Eq.(13) yields

$$W\frac{\mathrm{d}\overline{\Phi}}{\mathrm{d}x} + D\frac{\mathrm{d}^2\overline{\Phi}}{\mathrm{d}x^2} + S = 0 \tag{56}$$

with the auxiliary variables

$$W = V + \frac{\chi}{2}\left(\frac{\mathrm{d}\overline{\Phi}}{\mathrm{d}x} - 2g\right) - (1-\chi)u_{\mathrm{M}} \text{ and } S = g\left(\frac{\chi}{2}g - V + (1-\chi)u_{\mathrm{M}}\right) \tag{57}$$

where an elliptic maximum principle of the solution Φ has been applied to obtain Eqs.(56) and (57) to determines the signs of some coefficients. The optimal ascending velocity u^* in the present case is

$$u^* = -\chi\left(\frac{\mathrm{d}\overline{\Phi}}{\mathrm{d}x} - g\right) + (1-\chi)u_{\mathrm{M}} \tag{58}$$

and the corresponding optimal ground velocity is

$$V_g = V - u^* = V + \chi\left(\frac{\mathrm{d}\overline{\Phi}}{\mathrm{d}x} - g\right) - (1-\chi)u_{\mathrm{M}}. \tag{59}$$

For the sake of brevity of the analysis, Eqs.(56) and (57) are non-dimensionalized as

$$\overline{W}\frac{\mathrm{d}\overline{\phi}}{\mathrm{d}y} + D\frac{\mathrm{d}^2\overline{\phi}}{\mathrm{d}y^2} + \overline{S} = 0 \tag{60}$$

with

$$W = \overline{\overline{V}} + \frac{\chi}{2}\left(\frac{\mathrm{d}\overline{\phi}}{\mathrm{d}y} - 2\overline{g}\right) - (1-\chi)w_{\mathrm{M}} \text{ and } \overline{S} = \overline{g}\left(\frac{\chi}{2}\overline{g} - \overline{\overline{V}} + (1-\chi)w_{\mathrm{M}}\right) \tag{61}$$

where

$$y = \frac{x}{l},\ \overline{\phi} = \frac{\overline{\Phi}}{\overline{V}l},\ w_{\mathrm{M}} = \frac{u_{\mathrm{M}}}{\overline{V}},\ p = \frac{\overline{V}l}{D},\ \overline{\overline{V}} = \frac{V}{\overline{V}},\ \overline{g} = \frac{g}{\overline{V}l}, \tag{62}$$

and $\overline{V}$ is the reference velocity scale. The boundary conditions in Eq.(55) are non-dimensionalized as well. The bars "$\bar{\ }$" and "$\bar{\bar{\ }}$" are omitted from the variables in the following.

The computational domain is uniformly divided into 420 elements with 421 nodes where the periodic boundary conditions are applied at the boundaries $y=0$ and $y=1$, so that the problem is solved on the 1-D torus. The flow velocity is specified as

$$V = \begin{cases} r & (0 < y < 1/6l) \\ 1 & (1/6l < y < 5/6l) \\ r & (5/6l < y < 6/6l) \end{cases} \tag{63}$$

where the parameter r is chosen to be 4 in the present numerical simulation. The areas with the larger and smaller flow velocities represent slot and pool areas, respectively. The conforming Petrov-Galerkin finite element method (CPGFEM) is used for numerically solving Eq.(60) (Yoshioka *et al.* 2015, in press). According to the computational results from a priori performed numerical simulations, it has been indicated that a numerical counterpart of the regularization method presented in **Section 4.2** is crucial for solving the HJBE(60). If the CPGFEM without the regularization method is used for solving Eq.(60), its numerical solutions involve spurious oscillations and/or unphysically flatten profiles. It has been confirmed that the choice $K = 0.2(\Delta y)^{-1}$ where Δy represents the mesh size gives reasonable numerical solutions for sufficiently fine meshes ($\Delta y < 0.0025$).

Figures 4(a) through **4(c)** show the numerical solutions in the 2-D phase space (y, g) with $p = 5$ for different values of the non-dimensionalized maximum swimming speed w_M. Similarly, **Figures 5(a)** through **5(c)** show the computed optimal ground velocity V_g in the 2-D phase space (y, g) with $p = 5$ for different values of w_M. The computational results show that the ascending condition is satisfied for sufficiently high values of the trend g. The computational cases with lower and higher values of p have also been performed. The results with the lower values of p ($p = 0.5$ and $p = 1$) are qualitatively same with those in **Figures 4** and **5** except for that the numerical solutions are much smoother. On the other hand, the numerical solutions for the high values of p ($p \geq 10$) exhibit longitudinal oscillations, which are considered to be numerical artifacts.

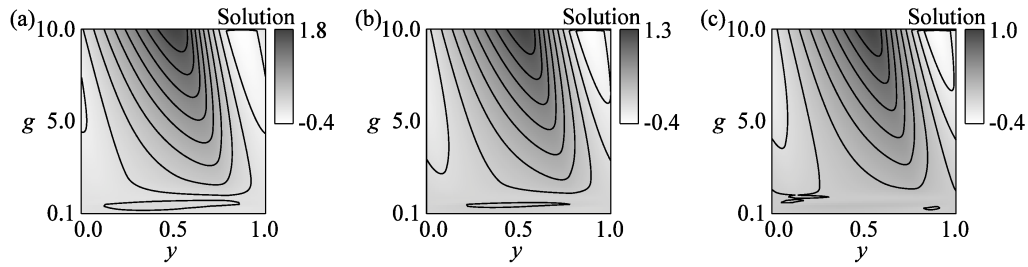

Figure 4: Steady solutions in the bounded case for $p = 5$ and the trend $0.1 \leq g \leq 10$ with (a) $w_M = 5$, (b) $w_M = 6$, and (b) $w_M = 7$. The counters correspond to the ten-section lines of the maximum and minimum values.

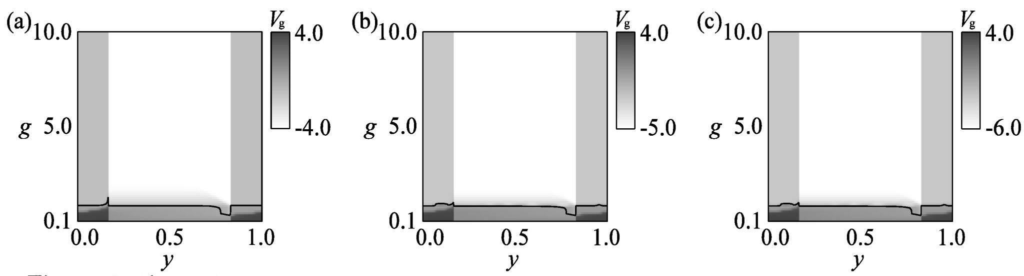

Figure 5: The optimal ground speed in the bounded case for $p = 2$ and the trend $0.1 \leq g \leq 10$ with (a) $w_M = 5$, (b) $w_M = 6$, and (b) $w_M = 7$. The black lines correspond to the contour line for $V_g = 0$.

6. Conclusions

A stochastic process model for analytical assessment of ascending behaviour of individual fishes was proposed and basic properties of its solutions are investigated both from analytical and numerical point of views. A regularized counterpart has also been presented and was applied to numerical simulation of ascending behaviour of fishes in a longitudinally periodic channel. Although the presented analytical solutions are for the dynamics under the simplified cases, they would serve as basics for comprehending mathematical properties of the HJBE.

In this paper, the model was applied to the problems in 1-D open channels. Extension of the model to the problems in locally 1-D open channel networks, which are connected graphs where hydraulic properties are appropriately distributed, is possible if internal boundary conditions are appropriately specified at junctions. Horizontally 2-D counterpart of the model has already been proposed in Yoshioka *et al.* (2014b) where the flow field is computed with the 2-D shallow water equations. Other problems not focused on in this paper involve migration of fish schools, which can be at least partially solved with an alternative SDE based on an appropriate mean field approximation technique. Migration with moving and resting regimes, which is a typical behaviour of fishes, was not addressed as well. This research topic will be addressed with a regime-switching diffusion process model that the authors recently developed, which is a continuous time SDE coupled with a discrete-state Markov process (Yin and Zhu, 2010; Yoshioka *et al.*, 2014a).

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References

[1] Brodersen, J., Nilsson, P.A., Ammitzbøll, J., Hansson, L.A., Skov, C., and Brönmark, C. (2008) Optimal swimming speed in head currents and effects on distance movement of winter-migrating fish, *PLOS ONE*, Vol. 3, No. 5, 7pp. DOI: 10.1371/journal.pone.0002156.

[2] Cote, D., Kehler, D.G., Bourne, C., and Wiersma, Y.F. (2009) A new measure of longitudinal connectivity for stream networks, *Landscape Ecology*, Vol. 24, No. 1, pp. 101-113.

[3] Fujihara, M. and Akimoto, M. (2010) A numerical model of fish movement in a vertical slot fishway, *Fisheries Engineering*, Vol. 47, No. 1, pp. 13-18.

[4] Iosilveskii, G. and Weihs, D. (2008) Speed limits on swimming of fishes and cetaceans, *Journal of the Royal Society, Interface*, Vol. 5, No. 20, pp. 329-338.

[5] Katopodis, C. and Williams, J.G. (2012) The developments of fish passage research in a historical context, *Ecological Engineering*, Vol. 48, pp. 8-18.

[6] Liao, J.C. (2007) A review of fish swimming mechanics and behaviour in altered flows, *Philosophical Transactions of the Royal Society* (*Biological Sciences*), Vol. 362, pp. 1973-1993.

[7] Øksendal, B. (2007) *Stochastic Differential Equations*, Springer-Verlag, Berlin, 1-167.

[8] Unami, K., Ishida, K., Kawachi, T., Maeda, S., and Takeuchi, J. (2010) A stochastic model for behavior of fish ascending an agricultural drainage system, *Paddy and Water Environment*, Vol. 8, No. 2, pp. 105-111.

[9] Willis, J. (2011) Modelling swimming aquatic animals in hydrodynamic models, *Ecological Engineering*, Vol. 222, pp. 3869-3887.

[10] Yamamoto, T. and Oishi, S. (2006) A mathematical theory for numerical treatment of nonlinear two-point boundary value problems, Japan Journal of Industrial and Applied Mathematics, Vol. 23, No. 1, pp. 31-62.

[11] Yin, G.G. and Zhu, C. (2010) *Hybrid Switching Diffusions*, Springer Science+Business Media, LLC, pp. 1-67.

[12] Yoshioka, H. and Unami, K. (2013) A cell-vertex finite volume scheme for solute transport equations in open channel networks, *Probabilistic Engineering Mechanics*, Vol. 31, pp. 30-38.

[13] Yoshioka, H., Unami, K., and Fujihara, M. (2014a) A regime-switching diffusion process model for longitudinal dispersion phenomena in vegetated open channels, *Proceedings of the 63rd National Congress on Theoretical and Applied Mechanics*, OS-15-02-03, pp. 1-2.

[14] Yoshioka, H., Unami, K., and Fujihara, M. (2014b) A stochastic process model for ascending behaviour of fishes in fishways, *Proceedings of the 22nd Japan Rainwater Catchment Systems Association Congress*, pp. 31-36.

[15] Yoshioka, H., Unami, K., and Fujihara, M. (2015) Mathematical analysis on a conforming finite element scheme for advection-dispersion-decay equations on connected graphs, *Journal of Japan Society of Civil Engineers (Special issue on Applied Mechanics)*. 12pp. (in press).