数理解析研究所講究録
第 1947 巻 2015 年 32-44

Space-time asymptotics of the 2D Navier-Stokes flow in the whole plane

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Key word: asymptotic expansion, temporal decay, large time behavior

1 Introduction

We consider the Navier-Stokes equations in $\mathbb{R}^2$:

$$\text{(N-S)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^2, \end{cases}$$

where $u = u(x,t) = \big(u^1(x,t), u^2(x,t)\big)$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, respectively, while $u_0 = u_0(x) = \big(u_0^1(x), u_0^2(x)\big)$ denotes the given initial velocity.

In two dimensional case, it is well-known that the unique global solution u in the class $L^\infty\big(0,\infty\,;\, L^2(\mathbb{R}^2)\big)$ of (N-S) which satisfy the following integral equation:

$$\text{(IE)} \qquad u(t) = e^{-tA}u_0 - \int_0^t \nabla \cdot e^{-(t-s)A} P(u \otimes u)(s)\, ds, \qquad t > 0,$$

where $A = -\Delta$ is the Laplacian on $\mathbb{R}^2$, $\{e^{-tA}\}_{t\geq 0}$ denotes the heat semigroup, $P = (P_{jk})_{j,k=1}^2$ is the Fujita-Kato bounded projection onto the solenoidal vector fields and $u \otimes u = (u_j u_k)_{j,k=1}^2$.

The decay problem has been one of main interests in mathematical fluid mechanics. Especially, the algebraic time decay is investigated by, for instance, Schonbek [20, 21, 22, 23], Kajikiya and Miyakawa [11], Wiegner [27]. Indeed, under the moment condition on the initial data u_0:

$$\int_{\mathbb{R}^n} (1+|x|)|u_0(x)|\,dx < \infty, \tag{1.1}$$

there exists a weak solution $u(t)$ to the Navier-Stokes equations with the upper bound:

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n+2}{4}}, \qquad t \ge 0. \tag{1.2}$$

See also [4, 5, 6, 8, 15, 16, 2]. Here we note that $(1+t)^{-(n+2)/4}$ of the energy decay is known as the critical rate for general initial data. More precisely, Carpio [3], Fujigaki and Miyakawa [7], Miyakawa and Schonbek [19] showed (1.2) by the asymptotic expansion with the heat kernel function $(4\pi t)^{-n/2}\exp(-|x|^2/4t)$, under (1.1), where the leading terms were definitely described. Furthermore, assuming not only (1.1) but also

$$|u_0(x)| \le C(1+|x|)^{-n-1} \quad \text{and} \quad \int |x|^m |u_0(x)|\,dx < \infty, \tag{1.3}$$

more specific space-time behaviors, especially, higher order asymptotic expansion were proved. See also [25, 24, 17, 18, 10, 1, 9, 13, 12]. Here rises a natural question that whether or not, the moment condition (1.1) on the initial data is essential to control the space-time behavior of the Navier-Stokes flow, more precisely, is necessary to determine the leading order term of the solution of (IE).

In this article, our aim is to derive the space-time asymptotics using the heat kernel function without any moment condition on the initial data like (1.1), i.e., no restriction of the decay at spatial infinity on u_0. Alternatively, we introduce the following profile of initial data:

$$u_0(x_1,x_2) = \big(a^1(x_1)\varphi^1(x_2),\, a^2(x_1)\varphi^2(x_2)\big). \tag{1.4}$$

In two dimensional case, since the solenoidal condition div $u_0 = 0$ is much stringent, we see that a^1 and φ^2 are necessarily differentiable and that u_0 has a representation with the stream function $a^1(x_1)\varphi^2(x_2)$. Hence this structure enables us to determine the leading order terms of asymptotic expansion without any moment condition on initial data, not just to derive a rapid time decay. Of course, it is natural that under restriction on initial data at spatial infinity one can obtain rapid energy decay and also pointwise estimates at spatial infinity of the flow. Furthermore, we discuss the necessary and sufficient condition on the initial data which causes the critical rate $(1+t)^{-1}$ of the energy decay, like Miyakawa and Schonbek [19].

Finally, we discuss the condition on the initial data for the second order asymptotic expansion of the solution of (IE) with the aid of weighted Hardy spaces.

2 Result

Before stating our results, we introduce the following notations. Let $C_{0,\sigma}^{\infty}(\mathbb{R}^2)$ denotes the set of all C^{∞}-solenoidal vectors ϕ with compact support in $\mathbb{R}^2$, i.e., div $\phi = 0$ in $\mathbb{R}^2$. $L_{\sigma}^{r}(\mathbb{R}^2)$ is the closure of $C_{0,\sigma}^{\infty}(\mathbb{R}^2)$ with respect to the L^r-norm $\|\cdot\|_r$, $1 < r < \infty$. $(\cdot,\cdot)$ is the duality pairing between $L^r(\mathbb{R}^2)$ and $L^{r'}(\mathbb{R}^2)$, where $1/r + 1/r' = 1$, $1 \leq r \leq \infty$. $L^r(\mathbb{R}^2)$ and $W^{m,r}(\mathbb{R}^2)$ denote the usual (vector-valued) L^r-Lebesgue space and L^r-Sobolev space over $\mathbb{R}^2$, respectively. Moreover $H^m(\mathbb{R}^2)$ stands for $W^{m,2}(\mathbb{R}^2)$. $\mathscr{S}(\mathbb{R}^2)$ denotes set of all of the Schwartz functions. $\mathscr{S}'(\mathbb{R}^2)$ denotes the set of all tempered distributions.

To state our theorem, we introduce the explicit representation of the projection operator $P : L^r(\mathbb{R}^2) \to L_{\sigma}^{r}(\mathbb{R}^2)$. By the Fourier transform, we have

$$P_{jk}(\xi) = \delta_{jk} + \frac{i\xi_j i\xi_k}{|\xi|^2} \quad \text{for } j,k = 1,2, \quad (i = \sqrt{-1}).$$

Therefore, putting $F_\ell = (F_{\ell,j,k})_{j,k=1}^{2} = \partial_\ell e^{-tA}P$, we have

$$\widehat{F}_{\ell,j,k}(\xi,t) = i\xi_\ell e^{-t|\xi|^2}\delta_{jk} + \frac{i\xi_\ell i\xi_j i\xi_k}{|\xi|^2}e^{-t|\xi|^2} = i\xi_\ell e^{-t|\xi|^2}\delta_{jk} + i\xi_\ell i\xi_j i\xi_k \int_t^{\infty} e^{-s|\xi|^2}\,ds,$$

for $\ell = 1,2$ since $|\xi|^{-2} = \int_0^{\infty} e^{-s|\xi|^2}\,ds$. Hence we obtain

$$F_{\ell,j,k}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_t^{\infty} \partial_\ell\partial_j\partial_k E_s(x)\,ds \quad \text{for } \ell,j,k = 1,2,$$

where $E_t(x)$ is the heat kernel:

$$E_t(x) = (4\pi t)^{-1}\exp\Big(-\frac{|x|^2}{4t}\Big).$$

Furthermore, we have the following estimate:

$$\|\partial_t^m \partial_x^\alpha F_{\ell,j,k}(\cdot,t)\|_q \le C_q t^{-(3+|\alpha|+2m)/2+1/q}, \qquad t>0, \quad 1 \le q \le \infty, \tag{2.1}$$

for all $m = 0, 1, \ldots$ and all multi-indeces α.

We note that the integral equation (IE) is rewritten with F_ℓ as follows:

$$u(t) = e^{-tA}u_0 - \left(\sum_{\ell,k=1}^{2} \int_0^t F_{\ell,j,k}(t-s) * (u_\ell u_k)(s)\, ds \right)_{j=1,2} . \tag{IE*}$$

Now our theorem read:

Theorem 2.1. *(i) Let $u_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with the following form:*

$$u_0(x_1, x_2) = \big(a^1(x_1)\varphi^1(x_2), a^2(x_1)\varphi^2(x_2)\big).$$

Then the strong solution $u(t)$ of (N-S) satisfies

$$\lim_{t\to\infty} t^{3/2-1/q} \Big\| u_1(t) + \frac{1}{\lambda_*}\partial_2 E_t(\cdot) \int a^1(y_1)\varphi^2(y_2)\, dy + \sum_{\ell,k=1}^{2} F_{\ell,1,k}(\cdot,t) \int_0^\infty \int (u_\ell u_k)(y,s)\, dyds \Big\|_q = 0, \tag{2.2}$$

and

$$\lim_{t\to\infty} t^{3/2-1/q} \Big\| u_2(t) - \frac{1}{\lambda_*}\partial_1 E_t(\cdot) \int a^1(y_1)\varphi^2(y_2)\, dy + \sum_{\ell,k=1}^{2} F_{\ell,2,k}(\cdot,t) \int_0^\infty \int (u_\ell u_k)(y,s)\, dyds \Big\|_q = 0, \tag{2.3}$$

for $1 \le q \le 2$. Here $\lambda_* \in \mathbb{R}$ is characterized as follows:

$$\lambda_* = \frac{-(a^1, \partial_1 \phi)}{(a^2, \phi)}, \quad \text{or} \quad \lambda_* = \frac{(\varphi^2, \partial_2 \psi)}{(\varphi^1, \psi)}, \tag{2.4}$$

with some $\phi, \psi \in C_0^\infty(\mathbb{R})$ so that $(a^2, \phi) \ne 0$ and $(\varphi^1, \psi) \ne 0$.

(ii) Furthermore, it holds that

$$\lim_{t\to\infty} t^{3/2-1/q} \|u(t)\|_q = 0,$$

if and only if

(2.5)
$$\int_{-\infty}^{\infty} a^1(y_1)dy_1 \int_{-\infty}^{\infty} \varphi^2(y_2)dy_2 = 0 \quad \text{and} \quad \int_0^\infty \int (u_\ell u_k)(y,s)\, dyds = c\delta_{\ell k}$$

for some constant $c > 0$.

(iii) On the other hand, we have

$$\liminf_{t\to\infty} t^{3/2-1/q} \|u(t)\|_q > 0,$$

if and only if (2.5) *does not hold.*

Remark 2.1. (i) The characterization of $\lambda_* \neq 0$ is well-defined. Indeed, λ_* is independent of the choice of $\phi, \psi \in C_0^\infty(\mathbb{R})$. See also Lemma 3.1 below.

(ii) Under our assumption on u_0, the solenoidal condition in weak sense yields the differentiability of a^1 and φ^2. See, Lemma 3.1 below.

(iii) If u_0 satisfies (1.1), then by Fujigaki-Miyakawa [7] we also have the expression:

$$(2.6) \quad \lim_{t\to\infty} t^{3/2-1/q} \Big\| u_j(t) + \sum_k \partial_k E_t(\cdot) \int y_k u_{0,j}(y)\, dy \\ + \sum_{\ell,k=1}^{2} F_{\ell,j,k}(\cdot,t) \int_0^\infty \int (u_\ell u_k)(y,s)\, dyds \Big\|_q = 0.$$

By virtue of the profile of initial data as in Theorem 2.1 and by Lemma 3.1 below, it is easy to see that

$$\int y_1 a^1(y_1)\varphi^1(y_2)\, dy = 0, \qquad \int y_2 a^1(y_1)\varphi^1(y_2)\, dy = \frac{1}{\lambda_*}\int a^1(y_1)\varphi^2(y_2)\, dy,$$
$$\int y_1 a^2(y_1)\varphi^2(y_2)\, dy = -\frac{1}{\lambda_*}\int a^1(y_1)\varphi^2(y_2)\, dy, \qquad \int y_2 a^2(y_1)\varphi^2(y_2)\, dy = 0.$$

Hence the leading terms from the linear part as in (2.6) may correspond to our leading terms if u_0 satisfy both our assumption as in Theorem 2.1 and (1.1).

(iv) For the optimal decay, the necessary and sufficient condition (2.5) is a generalization of that of Miyakawa and Schonbek [19]. Indeed, as is mentioned above, if u_0 satisfies the moment condition (1.1), (2.5) must coincide with that of [19]. Especially, if

$$\int_0^\infty \int (u_\ell u_k) dyds = c\delta_{\ell k} \quad \text{and} \quad \int a^1\, dy_1 \int \varphi^2\, dy_2 \neq 0$$

with some $c \in \mathbb{R}$, then by cancellation of the contribution from the nonlinear term, we may not expect that the flow is asymptotically symmetric in $\mathbb{R}^2$.

(v) In higher dimensional case $n \geq 3$, consider

$$u_0(x) = \big(a^1(x_1)\varphi^1(x_2)\eta(x_3,\dots,x_n), a^2(x_1)\varphi^2(x_2)\eta(x_3,\dots,x_n), 0,\dots,0\big).$$

Then we obtain that all properties of Theorem 2.1 still hold for strong solutions with replacing $t^{3/2-1/q}$ by $t^{1/2+n(1-1/q)/2}$ and replacing $\int a^1\,dy_1 \int \varphi^2\,dy_2$ by $\int a^1\,dy_1 \int \varphi^2\,dy_2 \int \eta\,dy_3\dots dy_n$.

As is mentioned in Remark 2.1, if we assume (1.1) in addition to the assumption as in Theorem 2.1 then we obtain the first order asymptotic expansion which corresponds with (2.6). Moreover, under such a situation, we expect the second order expansion for the Stokes flow. On the other hand, for the asymptotic expansion of the nonlinear term we need rapid decay of weighted estimates for the solution of (IE). Recently, Tsutsui [26] investigated the specific weighted estimate of the solution in the weighted Hardy spaces $\mathcal{H}^p(w)$. So introducing the estimate obtained in [26], by slight modification of (1.1) in terms of weighted Hardy space $\mathcal{H}^1(w)$ with a weight w we obtain the second order asymptotic expansion of the solution of (IE). Here,

$$\mathcal{H}^1(w) = \left\{ f \in \mathscr{S}'(\mathbb{R}^2)\,;\ \int_{\mathbb{R}^2} \sup_{\lambda>0} |(\Phi_\lambda * f)(x)| w(x)\,dx < \infty \right\},$$

with some $\Phi \in \mathscr{S}(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \Phi(x)\,dx = 1$, where $\Phi_\lambda(x) = \lambda^{-2}\Phi(x/\lambda)$.

Then we have the following theorem:

Theorem 2.2. *Let $u_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with the form:*

$$u_0(x_1,x_2) = \big(a^1(x_1)\varphi^1(x_2), a^2(x_2)\varphi^2(x_2)\big)$$

and satisfy (1.1). *(i) Then we have* (2.2) *and* (2.3) *for all $1 \leq q \leq \infty$.*

(ii) Furthermore, for every $0 < \varepsilon < 1/2$ let $u_0 \in \mathcal{H}^1(w)$ with $w(x) = |x|^{1-\varepsilon}$ and let $\|u_0\|_2 + \|u_0\|_{\mathcal{H}^1(w)}$ be sufficiently small. Then the strong solution $u(t)$ satisfies

$$\text{(2.7)}\quad \lim_{t\to\infty} t^{2-1/q}\Big\| u_1(t) + \frac{1}{\lambda_*}\sum_{0\leq|\alpha|\leq 1}(-1)^{|\alpha|}(\partial_x^\alpha \partial_2 E_t)(\cdot)\int y^\alpha a^1(y_1)\varphi^2(y_2)\,dy$$
$$+ \sum_{\ell,k=1}^{2}\sum_{|\beta|\leq 1}(-1)^{|\beta|}(\partial_x^\beta F_{\ell,1,k})(\cdot,t)\int_0^\infty \int y^\beta (u_\ell u_k)(y,s)\,dyds\Big\|_q = 0$$

and

$$\text{(2.8)} \quad \lim_{t\to\infty} t^{2-1/q}\Big\| u_2(t) - \frac{1}{\lambda_*}\sum_{0\le|\alpha|\le 1}(-1)^{|\alpha|}(\partial_x^\alpha \partial_1 E_t)(\cdot)\int y^\alpha a^1(y_1)\varphi^2(y_2)\,dy$$

$$+\sum_{\ell,k=1}^{2}\sum_{|\beta|\le 1}(-1)^{|\beta|}(\partial_x^\beta F_{\ell,2,k})(\cdot,t)\int_0^\infty\int y^\beta(u_\ell u_k)(y,s)\,dyds\Big\|_q = 0$$

for all $1 \le q \le \infty$, *where* $\lambda_* \in \mathbb{R}$ *is determined by* (2.4).

Remark 2.2. (i) Under (1.1) we have the rapid decay estimate for the strong solution obtained by Miyakawa [15]. Then we can extend the range of q up to ∞.

(ii) Fujigaki and Miyakawa [7] and Miyakawa [15] showed higher order asymptotic expansion assuming pointwise estimates on the initial data as in (1.3). Of course, they simultaneously obtained the precise pointwise estimate for the solution in space and time. However, as for the second order asymptotic expansion we need no pointwise estimate on u_0 as in (1.3).

3 Outline of proof

The following lemmata for the Stokes flow are essential role to prove our theorems.

Lemma 3.1. *Let* $u_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *with the following form:*

$$u_0(x_1,x_2) = \big(a^1(x_1)\varphi^1(x_2), a^2(x_1)\varphi^2(x_2)\big).$$

Then it holds that $a^1, \varphi^2 \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ *and that* u_0 *is expressed as:*

(3.1)

$$u_0(x_1,x_2) = \frac{1}{\lambda_*}\big(-a^1(x_1)\partial_2\varphi^2(x_2), \partial_1 a^1(x_1)\varphi^2(x_2)\big) \quad \textit{for a.e. } (x_1,x_2)\in\mathbb{R}^2$$

where $\lambda_* \in \mathbb{R}$ *is characterized as :*

$$\text{(2.4)} \qquad \lambda_* = \frac{-(a^1,\partial_1\phi)}{(a^2,\phi)}, \quad or \quad \lambda_* = \frac{(\varphi^2,\partial_2\psi)}{(\varphi^1,\psi)},$$

with some $\phi,\psi \in C_0^\infty(\mathbb{R})$ *so that* $(a^2,\phi)\neq 0$ *and* $(\varphi^1,\psi)\neq 0$.

Proof. Since $u_0 \in L^2_\sigma(\mathbb{R}^2)$, i.e., div $u_0 = 0$, we have $i\xi_1\widehat{a}^1(\xi_1)\widehat{\varphi}^1(\xi_2)+i\xi_2\widehat{a}^2(\xi_1)\varphi^2(\xi_2) = 0$ for all $\xi_1, \xi_2 \in \mathbb{R}$. Therefore there exists a constant λ_* such that

$$\frac{i\xi_1\widehat{a}^1(\xi_1)}{\widehat{a}^2(\xi_1)} = \frac{-i\xi_2\widehat{\varphi}^2(\xi_2)}{\widehat{\varphi}^1(\xi_2)} \equiv \lambda_*, \tag{3.2}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$. Hence (3.2) implies

$$\begin{cases} \partial_1 a^1(x_1) = \lambda_* a^2(x_1), & \text{a.e. } x_1, \\ -\partial_2\varphi^2(x_2) = \lambda_*\varphi^1(x_2), & \text{a.e. } x_2. \end{cases} \tag{3.3}$$

Since $a^2, \varphi^1 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, it is easy to obtain $a^1, \varphi^2 \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ and (3.1).

Now it remains to give an expression of λ_* by the given data a^1, a^2, φ^1 and φ^2 with the aid of test functions, and to show $\lambda_* \in \mathbb{R}$ and well-definedness of λ_*. Let $\phi, \psi \in C_0^\infty(\mathbb{R})$. Consider $\phi(x_1)\psi(x_2)$ as a smooth function on $\mathbb{R}^2$. Then we have

$$\begin{aligned} 0 &= \big(u_0, \nabla(\phi\psi)\big) \\ &= \int_{\mathbb{R}^2} \big(a^1(x_1)\varphi^1(x_2)\cdot\partial_1\phi(x_1)\psi(x_2) + a^2(x_1)\varphi^2(x_2)\cdot\phi(x_1)\partial_2\psi(x_2)\big)\,dx_1dx_2 \\ &= \int_{\mathbb{R}} a^1(x_1)\partial_1\phi(x_1)\,dx_1 \int_{\mathbb{R}} \varphi^1(x_2)\psi(x_2)\,dx_2 + \int_{\mathbb{R}} a^2(x_1)\phi(x_1)\,dx_1 \int_{\mathbb{R}} \varphi(x_2)\partial_2\psi(x_2)\,dx_2 \\ &= (a^1, \partial_1\phi)(\varphi^1, \psi) + (a^2, \phi)(\varphi^2, \partial_2\psi). \end{aligned}$$

Hence we obtain

$$\frac{-(a^1, \partial_1\phi)}{(a^2, \phi)} = \frac{(\varphi^2, \partial_2\psi)}{(\varphi^1, \psi)} \tag{3.4}$$

for all $\phi, \psi \in C_0^\infty(\mathbb{R})$ with $(a^2, \phi) \neq 0$ and $(\varphi^1, \psi) \neq 0$. Therefore it is easy to see that (3.4) implies the characterization of λ_* is independent of the choice of ϕ and ψ. □

Lemma 3.2. *Let $u_0 \in L^2_\sigma(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ with the form:*

$$u_0(x_1, x_2) = \big(a^1(x_1)\varphi^1(x_2), a^2(x_1)\varphi^2(x_2)\big).$$

Then for $1 \le q \le \infty$, it holds that

$$\lim_{t\to\infty} t^{3/2-1/q} \left\| e^{-tA}u_0 - \frac{1}{\lambda_*}\big(-\partial_2 E_t(\cdot), \partial_1 E_t(\cdot)\big) \int a^1(y_1)\varphi^2(y_2)\,dy \right\|_q = 0,$$

where $\lambda_* \in \mathbb{R}$ is determined by (2.4).

Proof. By Lemma 3.1, we note that

$$u_0(x_1,x_2) = \frac{1}{\lambda_*}\big(-a^1(x_1)\partial_2\varphi^2(x_2), \partial_1 a^1(x_1)\varphi^2(x_2)\big) \quad \text{for a.e. } (x_1,x_2) \in \mathbb{R}^2.$$

Noting $\partial_{x_j}E_t(x-y) = -\partial_{y_j}E_t(x-y)$, by integral by parts we have

$$\begin{aligned} e^{-tA}[a^1\varphi^1](x) &= \int_{\mathbb{R}^2} E_t(x-y)a^1(y_1)\varphi^1(y_2)\,dy \\ &= -\frac{1}{\lambda_*}\int_{\mathbb{R}^2} E_t(x-y)a^1(y_1)\partial_{y_2}\varphi^2(y_2)\,dy \\ &= -\frac{1}{\lambda_*}\int_{\mathbb{R}^2} \partial_{x_2}E_t(x-y)a^1(y_1)\varphi^2(y_2)\,dy. \end{aligned} \tag{3.5}$$

Hence we have

$$\begin{aligned} e^{-tA}[a^1\varphi^1](x) &+ \frac{1}{\lambda_*}\partial_2 E_t(x)\int a^1(y_1)\varphi^2(y_2)\,dy \\ &= -\frac{1}{\lambda_*}\int \big[\partial_{x_2}E_t(x-y) - \partial_{x_2}E_t(x)\big]a^1(y_1)\varphi^2(y_2)\,dy \end{aligned}$$

Therefore, by change of variables $x' = x/\sqrt{t}$ and the generalized Minkovski inequality for the integral, we obtain

$$\begin{aligned} &\left\| e^{-tA}[a^1\varphi^1] + \frac{1}{\lambda_*}\partial_2 E_t(\cdot)\int a^1(y_1)\varphi^2(y_2)\,dy \right\|_q \\ &\qquad \le \frac{t^{-3/2+1/q}}{|\lambda_*|}\int \|\partial_2 E_1(\cdot - y/\sqrt{t}) - \partial_2 E_1(\cdot)\|_q\, a^1(y_1)\varphi^2(y_2)\,dy. \end{aligned}$$

Here $\|\partial_2 E_1(\cdot - y/\sqrt{t}) - \partial_2 E_1(\cdot)\|_q$ is bounded in t and y, and we have

$$\lim_{t\to\infty} \|\partial_2 E_1(\cdot - y/\sqrt{t}) - \partial_2 E_1(\cdot)\|_q = 0 \quad \text{for fixed } y.$$

By Lebesgue's convergence theorem, we have

$$\lim_{t\to\infty} t^{3/2-1/q}\left\| e^{-tA}[a^1\varphi^1] + \frac{1}{\lambda_*}\partial_2 E_t(\cdot)\int a^1(y_1)\varphi^2(y_2)\,dy \right\|_q = 0.$$

Furthermore, by the same argument, we obtain

$$\lim_{t\to\infty} t^{3/2-1/q}\left\| e^{-tA}[a^2\varphi^2] - \frac{1}{\lambda_*}\partial_1 E_t(\cdot)\int a^1(y_1)\varphi^2(y_2)\,dy \right\|_q = 0.$$

This completes the proof of Lemma 3.2. $\square$

Since we have the asymptotic expansion of the Stokes flow with leading order therms and $\|e^{-tA}u_0\|_2 \leq t^{-\frac{n+2}{4}}$ for large $t > 0$, we may derive the first order asymptotic expansion for the nonlinear terms according to Fujigaki and Miyakawa [7]. Let

$$
\begin{aligned}
w(t) &= \big(w_1(t), w_2(t)\big) = -\int_0^t \nabla \cdot e^{-(t-s)A} P(u \otimes u)(s)\, ds \\
&= \left(-\sum_{\ell,k=1}^{2} \int_0^t F_{\ell,j,k}(t-s) * (u_\ell u_k)(s)\, ds \right)_{j=1,2}
\end{aligned}
\tag{3.6}
$$

Lemma 3.3 (Fujigaki and Miyakawa [7]). *Let u_0 satisfy the assumption of Theorem 2.1. Then we have*

$$
\lim_{t\to\infty} t^{3/2-1/q} \left\| w_j(t) + \sum_{\ell,k=1}^{2} F_{\ell,j,k}(\cdot, t) \int_0^\infty \int (u_\ell u_k)(y,s)\, dyds \right\|_q = 0
$$

for all $1 \leq q \leq 2$ and $j = 1, 2$.

Finally, we consider the second order asymptotic expansion for nonlinear term under the first order moment condition in terms of the weighted Hardy space. For this purpose we introduce the following theorem proved by Tsutsui [26]:

Theorem 3.1 (Tsutsui [26]). *Let $n \geq 2$, $1 \leq p < \infty$, $-n/p < \alpha < n(1 - 1/p) + 1$ and $w(x) = |x|^{\alpha p}$. Then there exists $\delta > 0$ such that for any $u_0 \in L^n \cap \mathcal{H}^p(w)$ with $\|u_0\|_n + \|u_0\|_{\mathcal{H}^p(w)} < \delta$ and* div $u_0 = 0$, *we can construct a solution $u \in L^\infty\big(0,\infty; L^n \cap \mathcal{H}^p(w)\big) \cap C\big([0,\infty); L^n \cap \mathcal{H}^p(w)\big) \cap C^\infty\big((0,\infty)\times\mathbb{R}^n\big)$ of (IE) satisfying*

$$
\lim_{t\to 0} \|u(t) - u_0\|_n = \lim_{t\to 0} \|u(t) - u_0\|_{\mathcal{H}^p(w)} = 0, \qquad \sup_{t>0} t^{1/2} \|\nabla u(t)\|_{\mathcal{H}^q(w)} < \infty.
$$

Moreover, for $q \in [p, \infty)$ and $\beta \in (-n/q, n(1 - 1/q) + 1)$ with $\beta \leq \alpha$, the solution u satisfies the following property:

$$
\|u(t)\|_{\mathcal{H}^q(\sigma)} \leq C t^{-\frac{n}{2}(1/p-1/q)-\frac{\alpha-\beta}{2}} \delta
$$

for all $t > 0$ with $\sigma(x) = |x|^{\beta q}$.

Here we put $p = 1$ $\alpha = 1 - \varepsilon$, $q = 2$ and $\beta = 1/2$ in Theorem 3.1 in case of $n = 2$. Let $u_0 \in L^2_\sigma(\mathbb{R}^2) \cap \mathcal{H}^1(w)$ with $w(x) = |x|^{1-\varepsilon}$ and $\|u_0\|_2 + \|u_0\|_{\mathcal{H}^1(w)}$ is sufficiently small. Then we obtain a unique strong solution $u(t)$ with the following estimate

$$\int_{\mathbb{R}^2} |y||u(y,t)|^2\, dy \le Ct^{-3/2+\varepsilon} \qquad \text{for } t > 0, \tag{3.7}$$

since $L^1_{loc}(\mathbb{R}^2) \cap \mathcal{H}^2(\sigma) \hookrightarrow L^2(\sigma)$ with $\sigma(x) = |x|$, where $L^2(\sigma) = \{u \in L^1_{loc}(\mathbb{R}^2)\,;\, \int |u(x)|^2\sigma(x)\, dx < \infty\}$.

By virtue of (3.7), we obtain the second order expansion of the nonlinear Duhamel term according Fujigaki and Miyakawa [7].

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