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On the Thin Film Approximation for the Flow of a Viscous Incompressible Fluid down an Inclined Plane

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1 Introduction

In this paper, we consider a two-dimensional motion of liquid film of a viscous and incompressible fluid flowing down an inclined plane under the influence of the gravity and the surface tension on the interface. The motion is mathematically formulated as a free boundary problem for the incompressible Navier–Stokes equations. We assume that the domain $\Omega(t)$ occupied by the liquid at time $t\geq 0$, the liquid surface $\Gamma(t)$, and the rigid plane Σ are of the forms

$$\begin{cases} \Omega(t)=\{(x,y)\in\mathbb{R}^2 \mid 0<y<h_0+\eta(x,t)\}, \\ \Gamma(t)=\{(x,y)\in\mathbb{R}^2 \mid y=h_0+\eta(x,t)\}, \\ \Sigma=\{(x,y)\in\mathbb{R}^2 \mid y=0\}, \end{cases}$$

where h_0 is the mean thickness of the liquid film and $\eta(x,t)$ is the amplitude of the liquid surface. Here we choose a coordinate system (x,y) so that x axis is down and y axis is normal to the plane. The motion of the liquid is described by the velocity $\boldsymbol{u}=(u,v)^T$

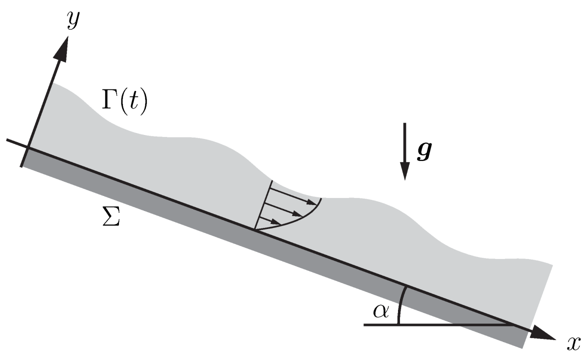

Figure 1: Sketch of a thin liquid film flowing down an inclined plane

and the pressure p satisfying the Navier–Stokes equations

$$\begin{cases} \rho\big(\boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big)=\nabla\cdot\mathbf{P}+\rho g(\sin\alpha,-\cos\alpha)^T & \text{in} \quad \Omega(t),\ t>0, \\ \nabla\cdot\boldsymbol{u}=0 & \text{in} \quad \Omega(t),\ t>0, \end{cases} \tag{1.1}$$

where $\mathbf{P} = -p\mathbf{I} + 2\mu\mathbf{D}$ is the stress tensor, $\mathbf{D} = \frac{1}{2}\big(D\boldsymbol{u} + (D\boldsymbol{u})^T\big)$ is the deformation tensor, $\mathbf{I}$ is the unit matrix, ρ is a constant density of the liquid, g is the acceleration of the gravity, α is the angle of inclination, and μ is the shear viscosity coefficient. The dynamical and kinematic conditions on the liquid surface are

$$\begin{cases} \mathbf{P}\boldsymbol{n} = -p_0\boldsymbol{n} + \sigma H\boldsymbol{n} & \text{on } \Gamma(t),\ t > 0, \\ \eta_t + u\eta_x - v = 0 & \text{on } \Gamma(t),\ t > 0, \end{cases} \tag{1.2}$$

where $\boldsymbol{n}$ is the unit outward normal vector to the liquid surface, that is, $\boldsymbol{n} = \frac{1}{\sqrt{1+\eta_x}}(-\eta_x, 1)^T$, p_0 is a constant atmospheric pressure, σ is the surface tension coefficient, and H is the twice mean curvature of the liquid surface, that is, $H = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x$. The boundary condition on the rigid plane is the non-slip condition

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \Sigma,\ t > 0. \tag{1.3}$$

These equations have a laminar steady solution of the form

$$\eta = 0, \quad u = (\rho g \sin\alpha/2\mu)(2h_0 y - y^2), \quad v = 0, \quad p = p_0 - \rho g \cos\alpha(y - h_0), \tag{1.4}$$

which is called the Nusselt flat film solution. Throughout this paper, we assume that the flow is downward l_0-periodic or approaches asymptotically this flat film solution at spacially infinity.

Concerning the instability of this laminar flow, there are vast research literatures in the physical and the engineering point of view. The first investigation of the wave motion of thin film including the effect of the surface tension was provided by Kapitza [10]. Particularly, he considered the case where liquid film flows down a vertical wall, that is, the case $\alpha = \frac{\pi}{2}$. Yih [21] first formulated the linear stability problem of the laminar flow of liquid film flowing down an inclined plane as an eigenvalue problem for the complex phase velocity, more specifically, the Orr-Sommerfeld problem although he neglected the effect of the surface tension. Benjamin [3] took into account the effect of the surface tension and showed that the critical Reynolds number R_c is given by $\mathrm{R}_c = \frac{5}{4}\cot\alpha$ by expanding the normal mode solution in powers of y. (In his original paper, the critical Reynolds number was given by $\mathrm{R}_c = \frac{5}{6}\cot\alpha$. This difference comes from the definition of the Reynolds number, that is, Benjamin used the average speed of the Nusselt flat film solution whereas we use the speed of the solution on the liquid surface as in Benney [4].) Later, Yih [22] showed the same condition by expanding normal mode solution in powers of the aspect ratio of the film which will be denoted by δ in this article. An approach taking into account the nonlinearity was first given by Mei [12] and Benney [4]. While Mei considered the gravity waves, Benney considered the capillary-gravity waves and he recovered Benjamin's and Yih's linear stability theories. Using the mean thickness of the liquid h_0, the characteristic scale of the streamwise direction l_0, and the typical amplitude of the liquid surface a_0, Benney introduced two non-dimensional parameters δ and ε defined by

$$\delta = \frac{h_0}{l_0}, \quad \varepsilon = \frac{a_0}{h_0},$$

respectively, and derived the following single nonlinear evolution equation

$$\begin{aligned}\eta_t =& A(1+\varepsilon\eta)\eta_x + \delta\big(B(1+\varepsilon\eta)\eta_{xx} + \varepsilon C(1+\varepsilon\eta)\eta_x^2\big) \\ &+ \delta^2\big(D(1+\varepsilon\eta)\eta_{xxx} + \varepsilon E(1+\varepsilon\eta)\eta_x\eta_{xx} + \varepsilon^2 F(1+\varepsilon\eta)\eta_x^3\big) \\ &+ \delta^3\big(G(1+\varepsilon\eta)\eta_{xxxx} + \varepsilon H(1+\varepsilon\eta)\eta_x\eta_{xxx} + \varepsilon I(1+\varepsilon\eta)\eta_{xx}^2 \\ &\quad + \varepsilon^2 J(1+\varepsilon\eta)\eta_x^2\eta_{xx} + \varepsilon^3 K(1+\varepsilon\eta)\eta_x^4\big) + O(\delta^4)\end{aligned} \tag{1.5}$$

by the method of perturbation expansion of the solution (u, v, p) with respect to δ under the thin film regime $\delta \ll 1$. Here, $A, B, \dots, K$ are polynomials in $1+\varepsilon\eta$. Thereafter, several authors have followed the Benney's approach. Here, we note that if the Weber number W satisfies the condition $\mathrm{W} = O(1)$, the effect of the surface tension does not appear until the term of $O(\delta^3)$ in the above equation. Since Benney considered the case $\mathrm{W} = O(1)$ and calculated the terms up to $O(\delta^2)$, the effect of the surface tension was omitted in his stability analysis. Consequently, his results showed that linearly unstable waves grow more rapidly in the nonlinear range. Nakaya [13] computed the terms up to $O(\delta^3)$ and showed that the surface tension has a stabilization effect in the development of the monochromatic waves. On the other hand, Gjevik [7] incorporated the effect of the surface tension into the equation by assuming the condition $\mathrm{W} = O(\delta^{-2})$ and investigated the growth of an initially unstable periodic surface perturbation and its nonlinear interaction with the higher harmonics. Their results imply that the surface tension plays an important role in investigating the stability of surface waves, which have already been pointed out by Kapitza [10]. We remark that the condition $\mathrm{W} = O(\delta^{-2})$ holds for many kinds of fluid such as water and alcohol at normal temperature. Moreover, several authors extended the Benney's results to the three-dimensional case. Roskes [16] calculated the terms up to $O(\delta^2)$ and investigated the interactions between two-dimensional and three-dimensional weakly nonlinear waves on liquid film under the condition $\mathrm{W} = O(1)$, which implies that he did not consider the effect of the surface tension. Atherton and Homsy [1] and Lin and Krishna [11] calculated the terms up to $O(\delta)$ and $O(\delta^2)$, respectively, under the condition $\mathrm{W} = O(\delta^{-2})$, namely, they took the effect of surface tension in the equation in three-dimensional case. Furthermore, while they considered the case where $\mathrm{R} = O(1)$, Topper and Kawahara [19] derived approximate equations under the conditions $\mathrm{W} = O(\delta^{-2})$ and $\mathrm{R} = O(\delta)$. More details or a list of useful references about the thin film approximation can be found in [5, 6, 9, 15].

Many approximate equations are obtained from (1.5). For example, by neglecting the terms of $O(\delta^2+\varepsilon^2)$, we obtain the Burgers equation

$$\eta_t = -2\eta_x - 4\varepsilon\eta\eta_x + \delta B(1)\eta_{xx}$$

with $B(1) = \frac{8}{15}\big(\frac{5}{4}\cot\alpha - \mathrm{R}\big)$, from which we can recover the Benjamin's critical Reynolds number $\mathrm{R}_c = \frac{5}{4}\cot\alpha$. By neglecting the terms of $O(\delta^3+\varepsilon\delta+\varepsilon^2)$, we obtain the KdV–Burgers equation

$$\eta_t = -2\eta_x - 4\varepsilon\eta\eta_x + \delta B(1)\eta_{xx} + \delta^2 D(1)\eta_{xxx},$$

which was named by Johnson [8]. Here, $D(1) = -2 - \frac{22}{63}R^2 + \frac{40}{63}\mathrm{R}\cot\alpha$. Moreover, by neglecting the terms of $O(\delta^4+\varepsilon\delta+\varepsilon^2)$, we obtain the so-called generalized Kuramoto–Sivashinsky equation or Kawahara equation (or more simply KdV–KS equation)

$$\eta_t = -2\eta_x - 4\varepsilon\eta\eta_x + \delta B(1)\eta_{xx} + \delta^2 D(1)\eta_{xxx} + \delta^3 G(1)\eta_{xxxx}$$

with $G(1) = -\frac{2}{3}\mathrm{W}\csc\alpha - \frac{157}{56}\mathrm{R} - \frac{8}{45}\mathrm{R}\cot^2\alpha + \frac{138904}{155925}\mathrm{R}^2\cot\alpha - \frac{1213952}{2027025}\mathrm{R}^3$. Therefore, the effect of the surface tension, namely, the Weber number W first appear in the coefficient of the fourth order derivative term in the case $\mathrm{W} = O(1)$. Now, our purpose is to give a mathematically rigorous justification of these thin film approximations by establishing the error estimate between the solution of Navier–Stokes equations (1.1)–(1.3) and those of the above approximate equations. In order to carry out the justification, the most difficult task is to derive a uniform estimate for the solution of the Navier–Stokes equations with respect to δ in the thin film regime $\delta \ll 1$. In this paper, we will focus on deriving a uniform estimate of the solution with respect to δ when the Reynolds number, the angle of inclination, and the initial date are sufficiently small under the condition $\mathrm{R} = O(1)$ and $O(1) \leq \mathrm{W} \leq O(\delta^{-2})$. In the future research, we will give a mathematically rigorous justification of the thin film approximations.

Concerning a mathematical analysis of the problem, Teramoto [17] showed that the initial value problem to the Navier–Stokes equations (1.1)–(1.3) has a unique solution globally in time under the assumption that the Reynolds number and the initial data are sufficiently small. Furthermore, Nishida, Teramoto, and Win [14] showed the exponential stability of the laminar flow under the assumption that the angle of inclination is sufficiently small in addition to the assumption in [17]. We follow basically the techniques used in the paper [14] and introduce a new energy function to obtain the uniform estimate.

The plan of this paper is as follows. In Section 2, we rewrite the problem in a non-dimensional form and transform the problem in a time dependent domain to a problem in a time independent domain by using an appropriate diffeomorphism. Then, we give our main theorem in this paper. In Section 3, we carry out energy estimates to the transformed equations, which are key estimates to derive a uniform boundedness of the solution in δ. Finally, we derive a uniform estimate of the solution in Section 4.

Notation. We put $\Omega = \mathbb{G}\times(0,1)$ and $\Gamma = \mathbb{G}\times\{y=1\}$, where $\mathbb{G}$ is the flat torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ or $\mathbb{R}$. For a Banach space X, we denote by $\|\cdot\|_X$ the norms in X. For $1 \le p \le \infty$, we put $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$, $\|u\| = \|u\|_{L^2}$, $|u|_{L^p} = \|u(\cdot,1)\|_{L^p(\mathbb{G})}$, and $|u|_0 = |u|_{L^2}$. We denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. For $s \ge 0$, we denote by $H^s(\Omega)$ and $H^s(\Gamma)$ the L^2 Sobolev spaces of order s on Ω and Γ, respectively. The norms of these spaces are denoted by $\|\cdot\|_s$ and $|\cdot|_s$. For a function $u = u(x,y)$ on Ω, a Fourier multiplier $P(D_x)$ $(D_x = -\mathrm{i}\partial_x)$ is defined by

$$(P(D_x)u)(x,y) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} P(n)\hat{u}_n(y)\mathrm{e}^{2\pi\mathrm{i}nx} & \text{in the case } \mathbb{G} = \mathbb{T}, \\ \displaystyle\int_{\mathbb{R}} P(\xi)\hat{u}(\xi,y)\mathrm{e}^{2\pi\mathrm{i}\xi x}\mathrm{d}\xi & \text{in the case } \mathbb{G} = \mathbb{R}, \end{cases}$$

where $\hat{u}_n(y) = \int_0^1 u(x,y)\mathrm{e}^{-2\pi\mathrm{i}nx}\,\mathrm{d}x$ is the Fourier coefficient and $\hat{u}(\xi,y) = \int_{\mathbb{R}} u(x,y)\mathrm{e}^{-2\pi\mathrm{i}\xi x}\mathrm{d}x$ is the Fourier transform in x. We put $\nabla_\delta = (\delta\partial_x, \partial_y)^T$, $\Delta_\delta = \nabla_\delta\cdot\nabla_\delta$, and $D_\delta^k f = \{(\delta\partial_x)^i\partial_y^j f \,|\, i+j=k\}$. For operators A and B, we denote by $[A,B] = AB - BA$ the commutator. We put

$$\partial_y^{-1}f(x,y) = -\int_y^1 f(x,z)\mathrm{d}z.$$

$f \lesssim g$ means that there exists a non-essential positive constant C such that $f \le Cg$ holds.

2 Reformulation of the problem and main result

We first rewrite (1.1)–(1.3) in a non-dimensional form. We will consider fluctuations on the stationary laminar flow given by (1.4), so that we rescale the independent and dependent variables by

$$\begin{cases} x = l_0 x', \quad y = h_0 y', \quad t = t_0 t', \\ \eta = a_0\eta', \quad u = U_0(\bar{u}' + \varepsilon u'), \quad v = \varepsilon V_0 v', \quad p = p_0 + \varepsilon P_0 p', \end{cases}$$

where $U_0 = \rho g h_0^2 \sin\alpha/2\mu$, $V_0 = (h_0/l_0)U_0$, $t_0 = l_0/U_0$, $\bar{u}' = 2y' - y'^2$, and $P_0 = \rho g h_0 \sin\alpha$. Putting these into (1.1)–(1.3) and dropping the prime sign in the notation, we obtain

(2.1)
$$\begin{cases} \delta\boldsymbol{u}_t^\delta + ((\boldsymbol{U} + \varepsilon\boldsymbol{u}^\delta)\cdot\nabla_\delta)\boldsymbol{u}^\delta + (\boldsymbol{u}^\delta\cdot\nabla_\delta)\boldsymbol{U} + \dfrac{2}{\mathrm{R}}\nabla_\delta p - \dfrac{1}{\mathrm{R}}\Delta_\delta\boldsymbol{u}^\delta = \boldsymbol{0} & \text{in } \Omega_\varepsilon(t),\ t>0, \\ \nabla_\delta\cdot\boldsymbol{u}^\delta = 0 & \text{in } \Omega_\varepsilon(t),\ t>0, \end{cases}$$

$$(2.2)\qquad \begin{cases} \big(\boldsymbol{D}_\delta(\varepsilon\boldsymbol{u}^\delta + \boldsymbol{U}) - \varepsilon p\boldsymbol{I}\big)\boldsymbol{n}^\delta \\ \quad = \Big(-\dfrac{1}{\tan\alpha}\varepsilon\eta + \dfrac{\delta^2\mathrm{W}}{\sin\alpha}\dfrac{\varepsilon\eta_{xx}}{(1+(\varepsilon\delta\eta_x)^2)^{\frac{3}{2}}}\Big)\boldsymbol{n}^\delta & \text{on } \Gamma_\varepsilon(t),\ t>0, \\ \eta_t + \big(1 - (\varepsilon\eta)^2 + \varepsilon u\big)\eta_x - v = 0 & \text{on } \Gamma_\varepsilon(t),\ t>0, \end{cases}$$

$$\boldsymbol{u}^\delta = \mathbf{0} \quad \text{on} \quad \Sigma,\ t>0, \tag{2.3}$$

where $\boldsymbol{u}^\delta = (u, \delta v)^T$, $\boldsymbol{U} = (\bar{u}, 0)^T$, $\bar{u} = 2y - y^2$, $\boldsymbol{D}_\delta \boldsymbol{f} = \frac{1}{2}\{\nabla_\delta(\boldsymbol{f}^T) + (\nabla_\delta(\boldsymbol{f}^T))^T\}$, $\boldsymbol{n}^\delta = (-\varepsilon\delta\eta_x, 1)^T$, $\mathrm{R} = \rho U_0 h_0/\mu$ is the Reynolds number, and $\mathrm{W} = \sigma/\rho g h_0^2$ is the Weber number. In this scaling, the liquid domain $\Omega_\varepsilon(t)$ and the liquid surface $\Gamma(t)$ are of the forms

$$\begin{cases} \Omega_\varepsilon(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < 1 + \varepsilon\eta(x,t)\}, \\ \Gamma_\varepsilon(t) = \{(x,y) \in \mathbb{R}^2 \mid y = 1 + \varepsilon\eta(x,t)\}. \end{cases}$$

Next, we transform the problem in the moving domain $\Omega_\varepsilon(t)$ to a problem in the fixed domain Ω by using an appropriate diffeomorphism $\Phi : \Omega \to \Omega_\varepsilon(t)$ defined by

$$\Phi(x,y,t) = \big(x, y(1 + \varepsilon\tilde{\eta}(x,y,t))\big),$$

where $\tilde{\eta}$ is an extension of η to Ω. We need to choose the extension $\tilde{\eta}$ carefully and in this paper we adopt the following extension. For $\phi \in H^s(\Gamma)$, we define its extension $\tilde{\phi}$ to Ω by

$$\tilde{\phi}(x,y) = \begin{cases} \displaystyle\sum_{n\in\mathbb{Z}} \frac{\hat{\phi}_n}{1 + (\delta n(1-y)y)^4} \mathrm{e}^{2\pi i n x} & \text{in the case } \mathbb{G} = \mathbb{T}, \\ \displaystyle\int_{\mathbb{R}} \frac{\hat{\phi}(\xi)}{1 + (\delta\xi(1-y)y)^4} \mathrm{e}^{2\pi\mathrm{i}\xi x} \mathrm{d}\xi & \text{in the case } \mathbb{G} = \mathbb{R}. \end{cases}$$

As usual, this extension operator has a regularizing effect so that $\tilde{\phi} \in H^{s+\frac{1}{2}}(\Omega)$. However, if we use such a regularizing property, then we need to pay the cost of a power of δ. Moreover, in this extension, ∂_y corresponds to $\delta\partial_x$.

The solenoidal condition on the velocity field is destroyed in general by the transformation. To keep the condition, following Beale [2], we also change the dependent variables and introduce new unknown functions (u', v', p') defined in Ω by

$$u' = J(u \circ \Phi), \quad v' = v \circ \Phi - y\varepsilon\tilde{\eta}_x(u \circ \Phi), \quad p' = p \circ \Phi,$$

where $J = 1 + \varepsilon(y\tilde{\eta})_y$ is the Jacobian of the diffeomorphism Φ.

Combining the above transformations and dropping the prime sign in the notation, we transform (2.1)–(2.3) to

$$\begin{cases} \delta \boldsymbol{u}_t^\delta + (\boldsymbol{U}\cdot\nabla_\delta)\boldsymbol{u}^\delta + (\boldsymbol{u}^\delta\cdot\nabla_\delta)\boldsymbol{U} & \\ \qquad + \dfrac{2}{\mathrm{R}}(I + A_4)\nabla_\delta p - \dfrac{1}{\mathrm{R}}\{\delta^2 \boldsymbol{u}_{xx}^\delta + (I + A_3)\boldsymbol{u}_{yy}^\delta\} = \boldsymbol{f} & \text{in } \Omega,\ t>0, \\ u_x + v_y = 0 & \text{in } \Omega,\ t>0, \end{cases} \tag{2.4}$$

$$\begin{cases} \delta^2 v_x + u_y - (2 + b_3)\eta = h_1 & \text{on } \Gamma,\ t>0, \\ p - \delta v_y - \dfrac{1}{\tan\alpha}\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\eta_{xx} = h_2 & \text{on } \Gamma,\ t>0, \\ \eta_t + \eta_x - v = h_3 & \text{on } \Gamma,\ t>0, \end{cases} \tag{2.5}$$

$$u = v = 0 \quad \text{on} \quad \Sigma,\ t > 0. \tag{2.6}$$

Here, $A_3 = b_2 E_{11}$, A_4 is a symmetric matrix, and $b_2, b_3, \boldsymbol{f}, h_1, h_2, h_3$ are collections of nonlinear terms. Particularly, $h_3 = \varepsilon^2 \eta\eta_x$. For details, see [20]. In the following, we will consider the initial value problem to (2.4)–(2.6) under the initial conditions

$$\eta|_{t=0} = \eta_0 \quad \text{on} \quad \Gamma, \quad (u,v)^T|_{t=0} = (u_0, v_0)^T \quad \text{in} \quad \Omega. \tag{2.7}$$

Here we denote b_3 and h_1 determined from the initial data by $b_3^{(0)}$ and $h_1^{(0)}$, respectively.

Now, we are ready to state our main result in this paper.

Theorem 2.1. (*Uniform estimate*) *There exist positive constants* R_0 *and* α_0 *such that the following statement holds: Let* m *be an integer satisfying* $m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, *and* $0 < \alpha \leq \alpha_0$. *There exist positive constants* c_0 *and* T *such that if the initial data* (η_0, u_0, v_0) *and the parameters* δ, ε, R, *and* W *satisfy the compatibility conditions*

$$\begin{cases} u_{0x} + v_{0y} = 0 & \text{in} \ \ \Omega, \\ u_{0y} + \delta^2 v_{0x} - (2 + b_3^{(0)})\eta_0 = h_1^{(0)} & \text{on} \ \ \Gamma, \\ u_0 = v_0 = 0 & \text{on} \ \ \Sigma, \end{cases}$$

and

$$\begin{cases} |(1+\delta|D_x|)^2\eta_0|_2 + \|(1+|D_x|)^2(u_0, \delta v_0)^T\| + \|(1+|D_x|)^2 D_\delta (u_0, \delta v_0)^T\| \\ \qquad + \|(1+|D_x|)^2 D_\delta^2 (u_0, \delta v_0)^T\| + \delta^2 \mathrm{W}\{|(1+\delta|D_x|)^2 \eta_{0x}|_2 + \delta^2 \|(1+|D_x|)^2 v_{0xy}\|\} \leq c_0, \\ |(1+\delta|D_x|)^2\eta_0|_m + \|(1+|D_x|)^m(u_0, \delta v_0)^T\| + \|(1+|D_x|)^m D_\delta (u_0, \delta v_0)^T\| \\ \qquad + \|(1+|D_x|)^m D_\delta^2 (u_0, \delta v_0)^T\| + \delta^2 \mathrm{W}\{|(1+\delta|D_x|)^2 \eta_{0x}|_m + \delta^2 \|(1+|D_x|)^m v_{0xy}\|\} \leq M, \\ 0 < \delta, \varepsilon \leq 1, \quad \mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0, \quad \mathrm{W}_1 \leq \mathrm{W} \leq \delta^{-2}\mathrm{W}_2, \end{cases}$$

then the initial value problem (2.4)–(2.7) *has a unique solution* (η, u, v, p) *on the time interval* $[0, T/\varepsilon]$ *and the solution satisfies the estimate*

$$\begin{aligned} &|(1+\delta|D_x|)^2\eta(t)|_m^2 + \delta^2|\eta_t(t)|_m^2 + \delta^2\mathrm{W}\{|(1+\delta|D_x|)^2\eta_x(t)|_m^2 + \delta^2|\eta_{tx}(t)|_m^2\} \\ &\quad + \|(1+|D_x|)^m(1+\delta|D_x|)^2 \boldsymbol{u}^\delta(t)\|^2 + \|(1+|D_x|)^m \boldsymbol{u}_y^\delta(t)\|^2 + \delta^2\|(1+|D_x|)^m \boldsymbol{u}_t^\delta(t)\|^2 \\ &+ \int_0^t \{\delta|\eta_x(\tau)|_m^2 + \delta|(1+\delta|D_x|)^{\frac{5}{2}}\eta_t(\tau)|_m^2 \\ &\quad + (\delta^2\mathrm{W})\delta|\eta_{xx}(\tau)|_m^2 + (\delta^2\mathrm{W})^2\{\delta|\eta_{xxx}(\tau)|_m^2 + \delta^2||D_x|^{\frac{7}{2}}\eta(\tau)|_m^2\} \\ &\quad + \delta\|(1+|D_x|)^m \boldsymbol{u}_x^\delta(\tau)\|^2 + \delta\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta \boldsymbol{u}_x^\delta(\tau)\|^2 + \delta\|(1+|D_x|)^m \nabla_\delta \boldsymbol{u}_t^\delta(\tau)\|^2 \\ &\quad + +\delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)\boldsymbol{u}_{yy}^\delta(\tau)\|^2 + \delta\|(1+|D_x|)^m \partial_y^{-1} p_x(\tau)\|^2 \\ &\quad + \delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta p(\tau)\|^2 + \delta\|(1+|D_x|)^{m-1}\nabla_\delta p_t(\tau)\|^2\}\mathrm{d}\tau \leq C \end{aligned}$$

for $0 \leq t \leq T/\varepsilon$ *with a constant* $C = C(\mathrm{R}_1, \mathrm{W}_1, \mathrm{W}_2, \alpha, M)$ *independent of* δ, ε, R, *and* W. *Moreover, the following uniform estimate holds.*

$$\begin{aligned} &|\eta(t)|_m + \|(1+|D_x|)^{m-1}u(t)\|_1 + \|\partial_x^m u_y(t)\| \\ &\quad + \|(1+|D_x|)^{m-2}v(t)\|_1 + \|\partial_x^{m-1} v_{yy}(t)\| \leq C \end{aligned} \tag{2.8}$$

for $0 \le t \le T/\varepsilon$. If, in addition, $0 \le \varepsilon \lesssim \delta$, then the solution can be extended for all $t \ge 0$ and the above estimates hold for $t \ge 0$.

Remark 2.1. In the case $\varepsilon \simeq 1$, this theorem gives a uniform boundedness of the solution only for a short time interval $[0, T]$. However, this is essential and we cannot extend this uniform estimate for all $t \ge 0$ in general, because by (1.5) we see that the limiting equation for η as $\delta \to 0$ becomes a nonlinear hyperbolic conservation law of the form

$$\eta_t + 2(1+\varepsilon\eta)^2\eta_x = 0,$$

whose solution will have a singularity in finite time in general.

3 Energy estimates

We recall two fundamental inequalities which have a key role in this paper.

Lemma 3.1. (*Korn's inequality*) *There exists a constant K independent of δ such that for any $0 < \delta \le 1$ and $\boldsymbol{u} = (u, v)^T$ satisfying*

$$\begin{cases} u_x + v_y = 0 & in \quad \Omega, \\ u = v = 0 & on \quad \Sigma, \end{cases}$$

we have

$$\iint_\Omega (\delta^2 u_x^2 + u_y^2 + \delta^4 v_x^2 + \delta^2 v_y^2)\mathrm{d}x\mathrm{d}y \le K \iint_\Omega (2\delta^2 u_x^2 + (u_y + \delta^2 v_x)^2 + 2\delta^2 v_y^2)\mathrm{d}x\mathrm{d}y.$$

Remark 3.1. Teramoto and Tomoeda [18] proved that the best constant of K is 3. Note that in the case of $\delta = 1$, this inequality is well-known.

Lemma 3.2. (*Trace theorem*) *For $0 < \delta \le 1$, we have*

$$|f|_0^2 + \delta||D_x|^{\frac{1}{2}} f|_0^2 \lesssim \|f\|^2 + \delta^2\|f_x\|^2 + \|f_y\|^2.$$

Remark 3.2. This trace theorem is also well-known in the case of $\delta = 1$.

We omit the proofs of the above lemmas because we only have to modify slightly the proofs in the case of $\delta = 1$.

The following proposition is a slight modification of the energy estimate obtained in [14].

Proposition 3.3. *There exists a positive constant* R_0 *such that if* $0 < \mathrm{R} \le \mathrm{R}_0$, *then the solution* (η, u, v, p) *of* (2.4)–(2.6) *satisfies*

$$
\begin{aligned}
(3.1)\qquad & \frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\boldsymbol{u}^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\} + \frac{1}{4KR}\|\nabla_\delta \boldsymbol{u}^\delta\|^2 \\
&\le \frac{4K}{\mathrm{R}}(|\eta|_0^2 + |b_3\eta|_0^2) + \frac{1}{\mathrm{R}}(h_1, u)_\Gamma - \frac{2}{\mathrm{R}}(h_2, \delta v)_\Gamma \\
&\quad + \frac{2}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta h_3)_\Gamma + (\boldsymbol{F}_1, \boldsymbol{u}^\delta)_\Omega,
\end{aligned}
$$

where K *is the constant in Korn's inequality and*

$$
(3.2)\qquad \boldsymbol{F}_1 = \boldsymbol{f} - \frac{2}{\mathrm{R}}A_4\nabla_\delta p + \frac{1}{\mathrm{R}}\begin{pmatrix} b_2 u_{yy} \\ 0 \end{pmatrix}.
$$

Proof. Note that Lemma 3.1 implies

$$
(3.3)\qquad \|\nabla_\delta \boldsymbol{u}^\delta\|^2 \le K|||\boldsymbol{u}^\delta|||^2,
$$

where $|||\boldsymbol{u}^\delta|||^2 = 2\|\delta u_x\|^2 + \|u_y + \delta^2 v_x\|^2 + 2\|\delta v_y\|^2$. Taking the inner product of $\boldsymbol{u}^\delta$ with the first equation in (2.4), we have

$$
(3.4)\qquad \frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^\delta\|^2 + (u, \bar{u}_y\delta v)_\Omega + \frac{1}{\mathrm{R}}(2\nabla_\delta p - \Delta_\delta \boldsymbol{u}^\delta, \boldsymbol{u}^\delta)_\Omega = (\boldsymbol{F}_1, \boldsymbol{u}^\delta)_\Omega.
$$

Using the second equation in (2.4) and integration by parts in x and y, we see that

$$
\begin{aligned}
&(2\nabla_\delta p - \Delta_\delta \boldsymbol{u}^\delta, \boldsymbol{u}^\delta)_\Omega \\
&= 2(p, \delta v)_\Gamma - (2\delta^2 u_{xx} + \delta^2 v_{xy} + u_{yy}, u)_\Omega - (\delta^3 v_{xx} + 2\delta v_{yy} + \delta u_{xy}, \delta v)_\Omega \\
&= 2(p, \delta v)_\Gamma + 2\|\delta u_x\|^2 + (\delta^2 v_x + u_y, u_y)_\Omega - (\delta^2 v_x + u_y, u)_\Gamma \\
&\quad + 2\|\delta v_y\|^2 - 2(\delta v_y, \delta v)_\Gamma + (\delta^2 v_x + u_y, \delta^2 v_x)_\Omega \\
&= |||\boldsymbol{u}^\delta|||^2 + 2(p - \delta v_y, \delta v)_\Gamma - (\delta^2 v_x + u_y, u)_\Gamma.
\end{aligned}
$$

By (2.5) and integration by parts in x, the boundary terms in the right-hand side of the above equality are calculated as

$$
\begin{aligned}
(3.5)\qquad 2(p - \delta v_y, \delta v)_\Gamma &= 2(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta(\eta_t + \eta_x - h_3))_\Gamma + 2(h_2, \delta v)_\Gamma \\
&= \delta\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right\} + 2(h_2, \delta v)_\Gamma \\
&\quad - 2(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta h_3)_\Gamma
\end{aligned}
$$

and $-(\delta^2 v_x + u_y, u)_\Gamma = -((2+b_3)\eta, u)_\Gamma - (h_1, u)_\Gamma$. Moreover, by the Cauchy–Schwarz and Poincaré's inequalities we see that $|(u, \bar{u}_y\delta v)_\Omega| \le 2\|u\|\|\delta v\| \le \|\boldsymbol{u}^\delta\|^2 \le \|\boldsymbol{u}_y^\delta\|^2 \le \|\nabla_\delta \boldsymbol{u}^\delta\|^2$

and that $\frac{2}{\mathrm{R}}|(\eta,u)_\Gamma| \le \frac{2}{\mathrm{R}}|\eta|_0\|u_y\| \le \frac{1}{4K\mathrm{R}}\|u_y\|^2 + \frac{4K}{\mathrm{R}}|\eta|_0^2$. Here, we used the inequality $|u(\cdot,1)|_0 = |u(\cdot,1)-u(\cdot,0)|_0 \le \|u_y\|$ thanks to the boundary condition (2.6). In the following, we use frequently this type of inequality without any comment. Thus we can rewrite (3.4) as

$$\begin{aligned}
&\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\{\|\boldsymbol{u}^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\} + \frac{1}{2K\mathrm{R}}\|\nabla_\delta \boldsymbol{u}^\delta\|^2 \\
&\le \|\nabla_\delta \boldsymbol{u}^\delta\|^2 + \frac{4K}{\mathrm{R}}(|\eta|_0^2 + |b_3\eta|_0^2) + \frac{1}{\mathrm{R}}(h_1,u)_\Gamma - \frac{2}{\mathrm{R}}(h_2,\delta v)_\Gamma \\
&\quad + \frac{2}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta h_3)_\Gamma + (\boldsymbol{F}_1, \boldsymbol{u}^\delta)_\Omega,
\end{aligned}$$

where we used Korn's inequality (3.3). Therefore, taking R_0 sufficiently small so that $4K\mathrm{R}_0 \le 1$, for $0 < \mathrm{R} \le \mathrm{R}_0$ we obtain the desired energy estimate. □

Note that we can take the tangential and time derivatives of the boundary conditions. Applying ∂_x, ∂_x^2, and ∂_t to (2.4)–(2.6) and using the above proposition, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\{\delta^2\|\boldsymbol{u}_x^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_x|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{xx}|_0^2\right)\right\} + \frac{1}{4K\mathrm{R}}\delta\|\nabla_\delta \boldsymbol{u}_x^\delta\|^2 \\
&\le \frac{4K}{\mathrm{R}}(\delta|\eta_x|_0^2 + \delta|(b_3\eta)_x|_0^2) + \frac{1}{\mathrm{R}}\delta(h_{1x},u_x)_\Gamma - \frac{2}{\mathrm{R}}\delta(h_{2x},\delta v_x)_\Gamma \\
&\quad + \frac{2}{\mathrm{R}}\delta(\frac{1}{\tan\alpha}\eta_x - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xxx}, \delta h_{3x})_\Gamma + \delta(\boldsymbol{F}_{1x}, \boldsymbol{u}_x^\delta)_\Omega,
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\{\delta^4\|\boldsymbol{u}_{xx}^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^4|\eta_{xx}|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^4|\eta_{xxx}|_0^2\right)\right\} + \frac{1}{4K\mathrm{R}}\delta^3\|\nabla_\delta \boldsymbol{u}_{xx}^\delta\|^2 \\
&\le \frac{4K}{\mathrm{R}}(\delta^3|\eta_{xx}|_0^2 + \delta^3|(b_3\eta)_{xx}|_0^2) + \frac{1}{\mathrm{R}}\delta^3(h_{1xx},u_{xx})_\Gamma - \frac{2}{\mathrm{R}}\delta^3(h_{2xx},\delta v_{xx})_\Gamma \\
&\quad + \frac{2}{\mathrm{R}}\delta^3(\frac{1}{\tan\alpha}\eta_{xx} - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xxxx}, \delta h_{3xx})_\Gamma + \delta^3(\boldsymbol{F}_{1xx}, \boldsymbol{u}_{xx}^\delta)_\Omega,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\{\delta^2\|\boldsymbol{u}_t^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{tx}|_0^2\right)\right\} + \frac{1}{4K\mathrm{R}}\delta\|\nabla_\delta \boldsymbol{u}_t^\delta\|^2 \\
&\le \frac{4K}{\mathrm{R}}(\delta|\eta_t|_0^2 + \delta|(b_3\eta)_t|_0^2) + \frac{1}{\mathrm{R}}\delta(h_{1t},u_t)_\Gamma - \frac{2}{\mathrm{R}}\delta(h_{2t},\delta v_t)_\Gamma \\
&\quad + \frac{2}{\mathrm{R}}\delta(\frac{1}{\tan\alpha}\eta_t - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{txx}, \delta h_{3t})_\Gamma \\
&\quad + \delta(\boldsymbol{f}_t, \boldsymbol{u}_t^\delta)_\Omega - \frac{2}{\mathrm{R}}\delta((A_4\nabla_\delta p)_t, \boldsymbol{u}_t^\delta)_\Omega + \frac{1}{\mathrm{R}}\delta((b_2u_{yy})_t, u_t)_\Omega.
\end{aligned} \tag{3.8}$$

For later use, we will compute $-\frac{2}{\mathrm{R}}\delta(\partial_x^k(A_4\nabla_\delta p)_t, \partial_x^k\boldsymbol{u}_t^\delta)_\Omega$ for nonnegative integer k. Applying $\delta\partial_t$ to the first equation in (2.4), we have

$$\delta^2\boldsymbol{u}_{tt}^\delta = -\frac{2}{\mathrm{R}}\delta(I+A_4)\nabla_\delta p_t - \frac{2}{\mathrm{R}}\delta A_{4t}\nabla_\delta p + \delta\boldsymbol{F}_{3t}, \tag{3.9}$$

where

$$\boldsymbol{F}_3 = -(\boldsymbol{U}\cdot\nabla_\delta)\boldsymbol{u}^\delta - (\boldsymbol{u}^\delta\cdot\nabla_\delta)\boldsymbol{U} + \frac{1}{\mathrm{R}}\big(\delta^2\boldsymbol{u}^\delta_{xx} + (I+A_3)\boldsymbol{u}^\delta_{yy}\big) + \boldsymbol{f}. \tag{3.10}$$

Moreover, we can rewrite (2.4) as

$$\frac{2}{\mathrm{R}}A_4\nabla_\delta p = -\delta A_5\boldsymbol{u}^\delta_t + A_5\boldsymbol{F}_3, \tag{3.11}$$

where $A_5 = A_4(I+A_4)^{-1}$. Note that A_5 is a symmetric matrix due to the symmetry of A_4. Applying $\delta\partial_x^k\partial_t$ to the above equation, we have

$$\frac{2}{\mathrm{R}}\delta\partial_x^k(A_4\nabla_\delta p)_t = -\delta^2A_5\partial_x^k\boldsymbol{u}^\delta_{tt} - \delta^2\partial_x^k(A_{5t}\boldsymbol{u}^\delta_t) - \delta^2[\partial_x^k, A_5]\boldsymbol{u}^\delta_{tt} + \delta\partial_x^k(A_5\boldsymbol{F}_3)_t.$$

This together with (3.9) yields

(3.12)

$$\begin{aligned}-\frac{2}{\mathrm{R}}\delta(\partial_x^k(A_4\nabla_\delta p)_t, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2(A_5\partial_x^k\boldsymbol{u}^\delta_t, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega \\ &\quad + \delta(\partial_x^k\{\frac{1}{2}\delta A_{5t}\boldsymbol{u}^\delta_t - (A_5\boldsymbol{F}_3)_t\}, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega + \delta(\boldsymbol{G}_k, \partial_x^k\boldsymbol{u}^\delta_t)_\Omega,\end{aligned}$$

where

$$\boldsymbol{G}_k = [\partial_x^k, A_5]\Big\{-\frac{2}{\mathrm{R}}(I+A_4)\nabla_\delta p_t - \frac{2}{\mathrm{R}}A_{4t}\nabla_\delta p + \boldsymbol{F}_{3t}\Big\} + \frac{1}{2}\delta[\partial_x^k, A_{5t}]\boldsymbol{u}^\delta_t. \tag{3.13}$$

Particularly, in the case of $k=0$, we have

$$-\frac{2}{\mathrm{R}}\delta((A_4\nabla_\delta p)_t, \boldsymbol{u}^\delta_t)_\Omega = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2(A_5\boldsymbol{u}^\delta_t, \boldsymbol{u}^\delta_t)_\Omega + \delta(\frac{1}{2}\delta A_{5t}\boldsymbol{u}^\delta_t - (A_5\boldsymbol{F}_3)_t, \boldsymbol{u}^\delta_t)_\Omega.$$

By substituting this into (3.8), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\delta^2((I-A_5)\boldsymbol{u}^\delta_t, \boldsymbol{u}^\delta_t)_\Omega + \frac{2}{\mathrm{R}}\Big(\frac{1}{\tan\alpha}\delta^2|\eta_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{tx}|_0^2\Big)\Big\} + \frac{1}{4K\mathrm{R}}\delta\|\nabla_\delta\boldsymbol{u}^\delta_t\|^2 \\ &\le \frac{4K}{\mathrm{R}}(\delta|\eta_t|_0^2 + \delta|(b_3\eta)_t|_0^2) + \frac{1}{\mathrm{R}}\delta(h_{1t}, u_t)_\Gamma - \frac{2}{\mathrm{R}}\delta(h_{2t}, \delta v_t)_\Gamma \\ &\quad + \frac{2}{\mathrm{R}}\delta(\frac{1}{\tan\alpha}\eta_t - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{txx}, \delta h_{3t})_\Gamma + \delta(\boldsymbol{F}_2, \boldsymbol{u}^\delta_t)_\Omega,\end{aligned} \tag{3.14}$$

where

$$\boldsymbol{F}_2 = \boldsymbol{f}_t + \frac{1}{\mathrm{R}}\begin{pmatrix}(b_2u_{yy})_t \\ 0\end{pmatrix} + \frac{1}{2}\delta A_{5t}\boldsymbol{u}^\delta_t - (A_5\boldsymbol{F}_3)_t. \tag{3.15}$$

Note that $I - A_5$ is positive definite for small solutions.

The lowest order energy obtained in (3.1) is not appropriate in order to get the uniform estimate in δ, which is our goal in this paper. We thereby need to modify the lowest energy estimate. Now it follows from the first and second equations in (2.4) that

$$\delta^2 v_t + \bar{u}\delta^2 v_x + \frac{2}{\mathrm{R}}p_y - \frac{1}{\mathrm{R}}\delta(\delta^2 v_x + u_y)_x - \frac{2}{\mathrm{R}}\delta v_{yy} = f_1,$$

where

$$f_1 = \left(\boldsymbol{f} - \frac{2}{\mathrm{R}}A_4\nabla_\delta p\right)\cdot \boldsymbol{e}_2. \tag{3.16}$$

Taking the inner product of δv with the above equation, we obtain

$$\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2\|v\|^2 - \frac{2}{\mathrm{R}}(p,\delta v_y)_\Omega + \frac{1}{\mathrm{R}}(\delta^2 v_x + u_y, \delta^2 v_x)_\Omega + \frac{2}{\mathrm{R}}\delta^2\|v_y\|^2 + \frac{2}{\mathrm{R}}(p - \delta v_y, \delta v)_\Gamma = (f_1, \delta v)_\Omega.$$

Thus using the second equation in (2.4) and integration by parts in x, we have

$$\begin{aligned}\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2\|v\|^2 &+ \frac{2}{\mathrm{R}}(p - \delta v_y, \delta v)_\Gamma + \frac{1}{\mathrm{R}}\delta^4\|v_x\|^2 + \frac{2}{\mathrm{R}}\delta^2\|v_y\|^2 \\ &= \frac{2}{\mathrm{R}}(\delta p_x, u)_\Omega + \frac{1}{\mathrm{R}}(\delta u_{xy}, \delta v)_\Omega + (f_1, \delta v)_\Omega.\end{aligned} \tag{3.17}$$

Lemma 3.4. *The following inequality holds.*

$$\begin{aligned}&\frac{2}{\mathrm{R}}(\delta p_x, u)_\Omega + \frac{1}{3\mathrm{R}}\left(\frac{1}{\tan^2\alpha}\delta^2|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta^2|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta^2|\eta_{xxx}|_0^2\right) + \frac{1}{\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 \\ &\quad\le I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{cases} I_1 = -\dfrac{2}{\mathrm{R}}(\delta\partial_y^{-1}p_x, (2+b_3)\eta)_\Omega, \\ I_2 = -\dfrac{2}{\mathrm{R}}(\delta\partial_y^{-1}p_x, -\delta^2 v_x(\cdot,1) + h_1 + \partial_y^{-1}(u_{yy} - 2\delta p_x))_\Omega, \\ I_3 = \dfrac{1}{\mathrm{R}}(2\delta^4|u_{xx}|_0^2 + 2\delta^2|h_{2x}|_0^2 + 3\delta^2\|\partial_y^{-2}p_{xy}\|^2). \end{cases}$$

Proof. By the first equation in (2.5) and (2.6), we see that

$$\begin{aligned}\frac{2}{\mathrm{R}}(\delta p_x, u)_\Omega &= -\frac{2}{\mathrm{R}}(\partial_y^{-1}\delta p_x, u_y)_\Omega = -\frac{2}{\mathrm{R}}(\partial_y^{-1}\delta p_x, u_y(\cdot,1) + \partial_y^{-1}u_{yy})_\Omega \\ &= -\frac{2}{\mathrm{R}}(\partial_y^{-1}\delta p_x, (2+b_3)\eta - \delta^2 v_x(\cdot,1) + h_1 + 2\partial_y^{-1}\delta p_x + \partial_y^{-1}(u_{yy} - 2\delta p_x))_\Omega \\ &= -\frac{4}{\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 + I_1 + I_2.\end{aligned} \tag{3.18}$$

On the other hand, it follows from the second equations in (2.4) and (2.5) that

$$\begin{aligned}p(x,y) &= p(x,1) + (\partial_y^{-1}p_y)(x,y) \\ &= -\delta u_x(x,1) + \frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} + h_2 + (\partial_y^{-1}p_y)(x,y).\end{aligned} \tag{3.19}$$

Thus applying $\delta \mathrm{R}^{-\frac{1}{2}}\partial_y^{-1}\partial_x$ to the above equation, we obtain

$$\begin{aligned}
&\frac{y-1}{\mathrm{R}^{\frac{1}{2}}}\left(\frac{1}{\tan\alpha}\delta\eta_x - \frac{\delta^2 \mathrm{W}}{\sin\alpha}\delta\eta_{xxx}\right) \\
&= \frac{\delta}{\mathrm{R}^{\frac{1}{2}}}(\partial_y^{-1}p_x)(x,y) + \frac{y-1}{\mathrm{R}^{\frac{1}{2}}}(\delta^2 u_{xx}(x,1) - \delta h_{2x}) - \frac{\delta}{\mathrm{R}^{\frac{1}{2}}}(\partial_y^{-2}p_{xy})(x,y).
\end{aligned}$$

Squaring both sides of the above equation and integrating the resulting equality on Ω, we have

$$\frac{1}{3\mathrm{R}}\left(\frac{1}{\tan^2\alpha}\delta^2|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta^2|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta^2|\eta_{xxx}|_0^2\right) \le \frac{3}{\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 + I_3,$$

where we used integration by parts in x. This and (3.18) lead to the desired inequality. □

This lemma together with (3.5) and (3.17) implies that

(3.20)
$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\delta^2\|v\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right)\right\} + \frac{1}{\mathrm{R}}(\delta^3\|v_x\|^2 + 2\delta\|v_y\|^2 + \delta\|\partial_y^{-1}p_x\|^2) \\
&\quad + \frac{1}{3\mathrm{R}}\left(\frac{1}{\tan^2\alpha}\delta|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta|\eta_{xxx}|_0^2\right) \\
&\le -\frac{2}{\mathrm{R}}(h_2,v)_\Gamma + \frac{1}{\mathrm{R}}\delta(u_{xy},v)_\Omega + (f_1,v)_\Omega + \frac{2}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, h_3)_\Gamma \\
&\quad + \delta^{-1}(I_1 + I_2 + I_3).
\end{aligned}$$

The first three terms in the right-hand side are estimated as

$$-\frac{2}{\mathrm{R}}(h_2,v)_\Gamma + \frac{1}{\mathrm{R}}\delta(u_{xy},v)_\Omega + (f_1,v)_\Omega \le \frac{1}{\mathrm{R}}\delta\|v_y\|^2 + \frac{1}{\mathrm{R}}(2\delta^{-1}|h_2|_0^2 + \delta\|u_{xy}\|^2) + \mathrm{R}\|f_1\|^2$$

and the first term in the right-hand side can be absorbed in the left-hand side of (3.20). We proceed to estimate I_1, I_2, and I_3. By (3.19) and integration by parts in x, I_1 is rewritten as

$$\begin{aligned}
(3.21)\qquad I_1 &= -\frac{2}{\mathrm{R}}(\delta\partial_y^{-1}\Big(-\delta u_x(\cdot,1) + \frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx} + h_2 + \partial_y^{-1}p_y\Big)_x, (2+b_3)\eta)_\Omega \\
&= I_4 + I_5,
\end{aligned}$$

where

$$\begin{aligned}
I_4 &= \frac{2}{\mathrm{R}}((y-1)(-\delta u_x(\cdot,1) + h_2) + \partial_y^{-2}p_y, \delta\big((2+b_3)\eta\big)_x)_\Omega, \\
(3.22)\qquad I_5 &= -\frac{1}{\mathrm{R}}(\frac{1}{\tan\alpha}\eta - \frac{\delta^2\mathrm{W}}{\sin\alpha}\eta_{xx}, \delta(b_3\eta)_x)_\Gamma.
\end{aligned}$$

Here we used identities $(\eta,\eta_x)_\Gamma = (\eta_{xx},\eta_x)_\Gamma = 0$. We estimate I_2, I_3, and I_4 as follows.

Lemma 3.5. *There exists a positive constant C independent of δ, R, W, and α such that the following estimates hold.*

$$
\begin{aligned}
|I_2| \le{}& \frac{1}{2\mathrm{R}}\delta^2\|\partial_y^{-1}p_x\|^2 + C\Big\{\frac{1}{\mathrm{R}}(\delta^4\|v_{xy}\|^2 + |h_1|_0^2 + \delta^4\|u_{xx}\|^2) \\
&+ \mathrm{R}(\delta^2\|u_{ty}\|^2 + \delta^2\|u_x\|^2 + \delta^2\|v_y\|^2 + \|f_2\|^2)\Big\}, \\
|I_3| \le{}& C\Big\{\frac{1}{\mathrm{R}}(\delta^4\|u_{xxy}\|^2 + \delta^2|h_{2x}|_0^2 + \delta^8\|v_{xxx}\|^2 + \delta^4\|v_{xyy}\|^2) \\
&+ \mathrm{R}(\delta^6\|v_{tx}\|^2 + \delta^6\|v_{xx}\|^2 + \delta^2\|f_{1x}\|^2)\Big\}, \\
|I_4| \le{}& \frac{1}{6\mathrm{R}\tan^2\alpha}(\delta^2|\eta_x|_0^2 + \delta^2|(b_3\eta)_x|_0^2) \\
&+ C\Big\{\frac{\tan^2\alpha}{\mathrm{R}}(\delta^2\|u_{xy}\|^2 + \delta^6\|v_{xx}\|^2 + \delta^2\|v_{yy}\|^2 + |h_2|_0^2) \\
&+ R\tan^2\alpha(\delta^4\|v_{ty}\|^2 + \delta^4\|v_x\|^2 + \|f_1\|^2)\Big\},
\end{aligned}
$$

where

$$
f_2 = -\frac{b_2}{1+b_2}\Big(\delta u_t + \bar{u}\delta u_x + \bar{u}_y\delta v - \frac{1}{\mathrm{R}}\delta^2 u_{xx}\Big) - \frac{2b_2}{\mathrm{R}(1+b_2)}\delta p_x - \frac{1}{1+b_2}f_3 \tag{3.23}
$$

and $f_3 = (\boldsymbol{f} - \frac{2}{\mathrm{R}}A_4\nabla_\delta p)\cdot \boldsymbol{e}_1$.

Proof. We can easily estimate I_3 and I_4 by using the second component of the first equation in (2.4) so as to eliminate p_y. As for I_2, by the first component of the first equation in (2.4), we have

$$
\frac{1}{\mathrm{R}}\Big(u_{yy} - \frac{2}{1+b_2}\delta p_x\Big) = \frac{1}{1+b_2}\Big(\delta u_t + \bar{u}\delta u_x + \bar{u}_y\delta v - \frac{1}{\mathrm{R}}\delta^2 u_{xx}\Big) - \frac{1}{1+b_2}f_3.
$$

Substituting the above equation into I_2, we easily obtain the desired estimate. □

Combining (3.20), (3.21), and Lemma 3.5, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\delta^2\|v\|^2 + \frac{2}{\mathrm{R}}\Big(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\Big)\Big\} + \frac{1}{\mathrm{R}}\Big(\delta\|\boldsymbol{u}_x^\delta\|^2 + \frac{1}{2}\delta\|\partial_y^{-1}p_x\|^2\Big) \\
&\quad + \frac{1}{3\mathrm{R}}\Big(\frac{1}{2\tan^2\alpha}\delta|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta|\eta_{xxx}|_0^2\Big) \\
&\le C_1\Big\{\frac{1}{\mathrm{R}}\big((1+\tan^2\alpha)\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + \delta^3\|\nabla_\delta\boldsymbol{u}_{xx}^\delta\|^2 \\
&\qquad + \delta^{-1}|h_1|_0^2 + (1+\tan^2\alpha)\delta^{-1}|h_2|_0^2 + \delta|h_{2x}|_0^2\big) \\
&\quad + \mathrm{R}\big(\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + (1+\tan^2\alpha)\delta\|\nabla_\delta\boldsymbol{u}_t^\delta\|^2 \\
&\qquad + (1+\tan^2\alpha)\delta^{-1}\|f_1\|^2 + \delta^{-1}\|f_2\|^2 + \delta\|f_{1x}\|^2\big)\Big\} \\
&\quad + \frac{2\delta^2\mathrm{W}}{\mathrm{R}\sin\alpha}\delta^{-1}|(\eta_{xx},\delta h_3)_\Gamma| + \frac{1}{6\mathrm{R}\tan^2\alpha}\delta|(b_3\eta)_x|_0^2 + \delta^{-1}I_5,
\end{aligned}
\tag{3.24}
$$

where we used the second equation in (2.4) and $(\eta, h_3)_\Gamma = (\eta, \varepsilon^2\eta^2\eta_x)_\Gamma = 0$. Here the constant C_1 does not depend on δ, R, W, nor α. This is the modified energy estimate. In the left-hand side, we have a new term $\delta\|\partial_y^{-1}p_x\|^2$, which plays an important role in this paper.

In view of the energy estimates obtained in this section, we define an energy function E_0, a dissipation function F_0, and a collection of the nonlinear terms N_0 by

$$
\begin{aligned}
(3.25)\qquad E_0(\eta, \boldsymbol{u}^\delta) &= \delta^2\|v\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}|\eta|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}|\eta_x|_0^2\right) \\
&+ \beta_1\left\{\delta^2\|\boldsymbol{u}_x^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_x|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{xx}|_0^2\right)\right\} \\
&+ \beta_2\left\{\delta^4\|\boldsymbol{u}_{xx}^\delta\|^2 + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^4|\eta_{xx}|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^4|\eta_{xxx}|_0^2\right)\right\} \\
&+ \beta_3\left\{\delta^2((I - A_5)\boldsymbol{u}_t^\delta, \boldsymbol{u}_t^\delta)_\Omega + \frac{2}{\mathrm{R}}\left(\frac{1}{\tan\alpha}\delta^2|\eta_t|_0^2 + \frac{\delta^2\mathrm{W}}{\sin\alpha}\delta^2|\eta_{tx}|_0^2\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
(3.26)\qquad F_0(\eta, \boldsymbol{u}^\delta, p) &= \frac{1}{2\mathrm{R}}\left(\delta\|\boldsymbol{u}_x^\delta\|^2 + \frac{1}{2}\delta\|\partial_y^{-1}p_x\|^2\right) \\
&+ \frac{1}{6\mathrm{R}}\left(\frac{1}{2\tan^2\alpha}\delta|\eta_x|_0^2 + \frac{2\delta^2\mathrm{W}}{\tan\alpha\sin\alpha}\delta|\eta_{xx}|_0^2 + \frac{(\delta^2\mathrm{W})^2}{\sin^2\alpha}\delta|\eta_{xxx}|_0^2\right) \\
&+ \frac{1}{8K\mathrm{R}}(\beta_1\delta\|\nabla_\delta\boldsymbol{u}_x^\delta\|^2 + \beta_2\delta^3\|\nabla_\delta\boldsymbol{u}_{xx}^\delta\|^2 + \beta_3\delta\|\nabla_\delta\boldsymbol{u}_t^\delta\|^2),
\end{aligned}
$$

$$
\begin{aligned}
(3.27)\quad N_0(Z) &= \delta^{-1}|h_1|_0^2 + \delta^{-1}|h_2|_0^2 + \delta|h_{1x}|_0^2 + \delta|h_{2x}|_0^2 \\
&+ \delta|h_3|_0^2 + \delta^3|h_{3t}|_0^2 + \delta^3|h_{3x}|_0^2 + \delta^5|h_{3xx}|_0^2 \\
&+ \delta^2||D_x|^{\frac{1}{2}}h_{1x}|_0^2 + \delta^2||D_x|^{\frac{1}{2}}h_{2x}|_0^2 + \delta|(h_{1t}, u_t)_\Gamma| + \delta|(h_{2t}, \delta v_t)_\Gamma| \\
&+ \delta|(b_3\eta)_x|_0^2 + \delta^3|(b_3\eta)_{xx}|_0^2 + \delta|(b_3\eta)_t|_0^2 + |(\eta, (b_3\eta)_x)_\Gamma| \\
&+ \delta^2\mathrm{W}\{\delta^{-1}|(\eta_{xx}, \delta h_3 + \delta(b_3\eta)_x)_\Gamma| + \delta^3|(\eta_{xxxx}, \delta h_{3xx})_\Gamma| + \delta|(\eta_{xxt}, \delta h_{3t})_\Gamma|\} \\
&+ \delta^{-1}\|f_1\|^2 + \delta^{-1}\|f_2\|^2 + \delta\|f_{1x}\|^2 \\
&+ \delta|(\boldsymbol{F}_{1x}, \boldsymbol{u}_x^\delta)_\Omega| + \delta^3|(\boldsymbol{F}_{1xx}, \boldsymbol{u}_{xx}^\delta)_\Omega| + \delta|(\boldsymbol{F}_2, \boldsymbol{u}_t^\delta)_\Omega|,
\end{aligned}
$$

where $Z = (\eta, \boldsymbol{u}^\delta, h_1, h_2, h_3, b_3\eta, f_1, f_2, \boldsymbol{F}_1, \boldsymbol{F}_2)$ and we will determine the constants β_1, β_2, and β_3 later. Note that the terms $|(\eta, (b_3\eta)_x)_\Gamma|$ and $(\delta^2\mathrm{W})\delta^{-1}|(\eta_{xx}, \delta(b_3\eta)_x)_\Gamma|$ come from I_5. Summarizing our energy estimates, we obtain the following proposition.

Proposition 3.6. *Let* W_1 *is a positive constant. There exists a positive constant* α_0 *such that if* $0 < \mathrm{R}_1 \le \mathrm{R} \le \mathrm{R}_0$, $\mathrm{W}_1 \le \mathrm{W}$, *and* $0 < \alpha \le \alpha_0$, *then the solution* (η, u, v, p) *of* (2.4)–(2.6) *satisfies*

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_0 + F_0 \le C_2 N_0,
$$

where R_0 *is the constant in Proposition 3.3 and the constant* $C_2(\mathrm{R}_1, \mathrm{W}_1, \alpha)$ *is independent of* δ, R, *and* W.

Proof. Multiplying (3.6), (3.7), and (3.14) by β_1, β_2, and β_3, respectively, and adding these and (3.24), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0 + 2F_0 \le L + C(N + N_0),$$

where

$$\begin{aligned} L &= \frac{4K}{\mathrm{R}}((\beta_1 + 3\beta_3)\delta|\eta_x|_0^2 + \beta_2\delta^3|\eta_{xx}|_0^2) + \left\{C_1\left(\frac{1+\tan^2\alpha}{\mathrm{R}} + \mathrm{R}\right) + \frac{12K}{\mathrm{R}}\beta_3\right\}\delta\|\nabla_\delta \boldsymbol{u}_x^\delta\|^2 \\ &\quad + \frac{C_1}{\mathrm{R}}\delta^3\|\nabla_\delta \boldsymbol{u}_{xx}^\delta\|^2 + C_1\mathrm{R}(1+\tan^2\alpha)\delta\|\nabla_\delta \boldsymbol{u}_t^\delta\|^2, \\ N &= \delta|(h_{1x}, u_x)_\Gamma| + \delta|(h_{2x}, \delta v_x)_\Gamma| + \delta|(\eta_x, \delta h_{3x})_\Gamma| + |((\delta^2\mathrm{W})\delta^{1/2}\eta_{xxx}, \delta^{3/2}h_{3x})_\Gamma| \\ &\quad + \delta^3|(h_{1xx}, u_{xx})_\Gamma| + \delta^3|(h_{2xx}, \delta v_{xx})_\Gamma| + \delta^3|(\eta_{xx}, \delta h_{3xx})_\Gamma| + \delta|(\eta_t, \delta h_{3t})_\Gamma| + \delta^{-1}|I_5|. \end{aligned}$$

Here we used $|\eta_t|_0 \le |\eta_x|_0 + \|u_{xy}\| + |h_3|_0$, which comes from the second equation in (2.4), the third equation in (2.5), and Poincaré's inequality. Moreover, it is easy to see that for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that $N \le \epsilon F_0 + C_\epsilon N_0$. Therefore, if we take $(\beta_1, \beta_2, \beta_3)$ so that

(3.28)
$$\begin{cases} \dfrac{4K}{\mathrm{R}}(\beta_1 + 3\beta_3) < \dfrac{1}{12\mathrm{R}\tan^2\alpha}, \quad \dfrac{4K}{\mathrm{R}}\beta_2 < \dfrac{\mathrm{W}}{3\mathrm{R}\tan\alpha\sin\alpha}, \\ C_1\left(\dfrac{1+\tan^2\alpha}{\mathrm{R}} + \mathrm{R}\right) + \dfrac{12K}{\mathrm{R}}\beta_3 < \dfrac{\beta_1}{8K\mathrm{R}}, \quad \dfrac{C_1}{\mathrm{R}} < \dfrac{\beta_2}{8K\mathrm{R}}, \quad C_1\mathrm{R}(1+\tan^2\alpha) < \dfrac{\beta_3}{8K\mathrm{R}}, \end{cases}$$

and if we choose $\epsilon > 0$ sufficiently small, then we obtain $L + CN \le F_0 + C_\epsilon N_0$. Here taking $(\beta_1, \beta_2, \beta_3)$ as

$$\beta_2 := 16KC_1, \quad \beta_3 := 16KC_1\mathrm{R}_0^2(1+\tan^2\alpha), \quad \beta_1 := 16K\{C_1(1+\tan^2\alpha + \mathrm{R}_0^2) + 12K\beta_3\},$$

we see that (3.28) is equivalent to

$$48K(\beta_1 + 3\beta_3)\tan^2\alpha < 1, \quad 12K\beta_2\tan\alpha\sin\alpha < \mathrm{W}_1.$$

Thus there exists a small constant α_0 which depends on W_1 such that (3.28) is fulfilled and we obtain the desired energy inequality. □

Hereafter, m is an integer satisfying $m \ge 2$. We define a higher order energy and a dissipation functions E_m and F_m and a collection of the nonlinear terms N_m by

$$E_m = \sum_{k=0}^{m} E_0(\partial_x^k\eta, \partial_x^k\boldsymbol{u}^\delta), \quad F_m = \sum_{k=0}^{m} F_0(\partial_x^k\eta, \partial_x^k\boldsymbol{u}^\delta, \partial_x^k p), \tag{3.29}$$

$$(3.30) \qquad N_m = \sum_{k=0}^{m} N_0(\partial_x^k Z) + \sum_{k=1}^{m}\big(\delta|(\boldsymbol{G}_k, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega| + |(\partial_x^k \eta, \partial_x^k h_3)_\Gamma|\big).$$

Here, we note that $\delta|(\boldsymbol{G}_k, \partial_x^k \boldsymbol{u}_t^\delta)_\Omega|$ is the term appearing in (3.12) and that $(\eta, h_3)_\Gamma = 0$. Under an appropriate assumption of the solution, we have the following equivalence uniformly in δ.

$$\begin{aligned}
E_m &\simeq |(1+\delta|D_x|)^2\eta|_m^2 + \delta^2|\eta_t|_m^2 + +\delta^2\mathrm{W}\big\{|(1+\delta|D_x|)^2\eta_x|_m^2 + \delta^2|\eta_{tx}|_m^2\big\} \\
&\quad + \delta^2\|(1+|D_x|)^m v\|^2 + \delta^2\|(1+|D_x|)^m(1+\delta|D_x|)\boldsymbol{u}_x^\delta\|^2 + \delta^2\|(1+|D_x|)^m \boldsymbol{u}_t^\delta\|^2 \\
&\simeq |\eta|_m^2 + \delta^2\big\{|(\eta_x, \eta_t)|_m^2 + \|(1+|D_x|)^m(v, u_x, u_t)\|^2\big\} \\
&\quad + \delta^4\big\{|(\eta_{xx}, \eta_{tx})|_m^2 + \|(1+|D_x|)^m(v_x, u_{xx}, v_t)\|^2\big\} + \delta^6\|(1+|D_x|)^m v_{xx}\|^2 \\
&\quad + \delta^2\mathrm{W}\big\{|\eta_x|_m^2 + \delta^2|(\eta_{xx}, \eta_{tx})|_m^2 + \delta^4|\eta_{xxx}|_m^2\big\},
\end{aligned}$$

$$\begin{aligned}
F_m &\simeq \delta|\eta_x|_m^2 + (\delta^2\mathrm{W})\delta|\eta_{xx}|_m^2 + (\delta^2\mathrm{W})^2\delta|\eta_{xxx}|_m^3 + \delta\|(1+|D_x|)^m\partial_y^{-1}p_x\|^2 \\
&\quad + \delta\|(1+|D_x|)^m\boldsymbol{u}_x^\delta\|^2 + \delta\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta \boldsymbol{u}_x^\delta\|^2 + \delta\|(1+|D_x|)^m\nabla_\delta \boldsymbol{u}_t^\delta\|^2 \\
&\simeq \delta\big\{|\eta_x|_m^2 + \|(1+|D_x|)^m(v_y, u_x, u_{xy}, u_{ty}, \partial_y^{-1}p_x)\|^2\big\} \\
&\quad + \delta^3\|(1+|D_x|)^m(v_x, v_{xy}, v_{ty}, u_{xx}, u_{xxy}, u_{tx})\|^2 \\
&\quad + \delta^5\|(1+|D_x|)^m(v_{xx}, v_{xxy}, v_{tx}, u_{xxx})\|^2 + \delta^7\|(1+|D_x|)^m v_{xxx}\|^2 \\
&\quad + (\delta^2\mathrm{W})\delta|\eta_{xx}|_m^2 + (\delta^2\mathrm{W})^2\delta|\eta_{xxx}|_m^2.
\end{aligned}$$

Applying ∂_x^k to (2.4)–(2.6), using Proposition 3.6, and adding the resulting inequalities for $0 \le k \le m$, we obtain a higher order energy estimate

$$(3.31) \qquad \frac{\mathrm{d}}{\mathrm{d}t}E_m + F_m \le C_2 N_m.$$

We modify the energy and the dissipation functions E_m and F_m defined by (3.29) as

$$\begin{aligned}
&(3.32) \qquad \tilde{E}_m = E_m + \|(1+|D_x|)^m u\|^2 + \|(1+|D_x|)^m u_y\|^2, \\
&(3.33) \qquad \tilde{F}_m = F_m + \delta|(1+\delta|D_x|)^{\frac{5}{2}}\eta_t|_m^2 + (\delta^2\mathrm{W})^2\delta^2||D_x|^{\frac{7}{2}}\eta|_m^2 \\
&\qquad\qquad + \delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)\nabla_\delta p\|^2 + \delta\|(1+|D_x|)^{m-1}\nabla_\delta p_t\|^2.
\end{aligned}$$

We also introduce another energy function D_m by

$$\begin{aligned}
(3.34) \quad D_m =& |(1+\delta|D_x|)^2\eta|_m^2 + \|(1+|D_x|)^m\boldsymbol{u}^\delta\|^2 + \|(1+|D_x|)^m D_\delta \boldsymbol{u}^\delta\|^2 \\
&+ \|(1+|D_x|)^m D_\delta^2 \boldsymbol{u}^\delta\|^2 + \delta^2\mathrm{W}\big\{|(1+\delta|D_x|)^2\eta_x|_m^2 + \delta^2\|(1+|D_x|)^m v_{xy}\|^2\big\},
\end{aligned}$$

which does not include any time derivatives. Moreover, we have the following equivalence uniformly in δ.

$$
\begin{aligned}
D_m \simeq & |\eta|_m^2 + \|(1+|D_x|)^m(u,u_y,u_{yy})\|^2 \\
& + \delta^2\{|\eta_x|_m^2 + \|(1+|D_x|)^m(v,v_y,u_x,u_{xy},v_{yy})\|^2\} \\
& + \delta^4\{|\eta_{xx}|_m^2 + \|(1+|D_x|)^m(v_x,v_{xy},u_{xx})\|^2\} \\
& + \delta^6\{|\eta_{xxx}|_m^2 + \|(1+|D_x|)^m v_{xx}\|^2\} \\
& + \delta^2\mathrm{W}\{|\eta_x|_m^2 + \delta^2|\eta_{xx}|_m^2 + \delta^4|\eta_{xxx}|_m^2 + \delta^2\|(1+|D_x|)^m v_{xy}\|^2\}.
\end{aligned}
$$

4 Proof of the main theorem

We will prove the following proposition.

Proposition 4.1. *Let m be an integer satisfying $m \geq 2$, $0 < \mathrm{R}_1 \leq \mathrm{R}_0$, $0 < \mathrm{W}_1 \leq \mathrm{W}_2$, and $0 < \alpha \leq \alpha_0$, where R_0 and α_0 are constants in Propositions 3.3 and 3.6. There exist positive constants c_1, C_5, C_6, and C_7 such that if the solution (η, u, v, p) of (2.4)-(2.6) and the parameters δ, ε, R, and W satisfy*

$$\tilde{E}_2(t) \leq c_1, \quad 0 < \delta, \varepsilon \leq 1, \quad \mathrm{R}_1 \leq \mathrm{R} \leq \mathrm{R}_0, \quad \mathrm{W}_1 \leq \mathrm{W} \leq \delta^{-2}\mathrm{W}_2,$$

then we have

$$\text{(4.1)}\quad \tilde{E}_2(t) \leq C_7 E_2(0)\mathrm{e}^{C_6\varepsilon t}, \quad \tilde{E}_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau \leq C_7 E_m(0)\exp(C_5 E_2(0)\mathrm{e}^{C_6\varepsilon t} + C_5\varepsilon t)$$

Moreover, if $\varepsilon \lesssim \delta$, then we have

$$\tilde{E}_2(t) \leq C_7 E_2(0), \quad \tilde{E}_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau \leq C_7 E_m(0)\exp(C_5 E_2(0)).$$

In order to prove the above proposition, we prepare the following lemmas.

Lemma 4.2. *Under the same assumptions of Proposition 4.1, for any $\epsilon > 0$ there exists a positive constants C_ϵ such that the following estimate holds.*

$$N_m \leq \epsilon\tilde{F}_m + C_\epsilon\big(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m + \varepsilon\sqrt{\tilde{E}_2}\tilde{E}_m\big).$$

Moreover, if $\varepsilon \lesssim \delta$, then we have

$$N_m \leq \epsilon\tilde{F}_m + C_\epsilon(\tilde{E}_2\tilde{F}_m + \tilde{F}_2\tilde{E}_m).$$

Lemma 4.3. *Under the same assumptions of Proposition 4.1, the following estimates hold.*

$$\tilde{E}_m \lesssim E_m, \tag{4.2}$$

$$\tilde{F}_m \lesssim F_m + \tilde{F}_2\tilde{E}_m, \tag{4.3}$$

$$\|\partial_x^m \boldsymbol{f}\|^2 \lesssim D_2 D_m, \tag{4.4}$$

$$\|(1+|D_x|)^m\nabla_\delta p\|^2 \lesssim (1+D_2)^2 D_m. \tag{4.5}$$

These lemmas are proved in [20].

Proof of Proposition 4.1. Combining (3.31), Lemma 4.2, and (4.2) and (4.3) in Lemma 4.3 and taking ϵ and c_1 sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t) + \tilde{F}_m(t) \leq C_5(\tilde{F}_2(t) + \varepsilon)E_m(t) \tag{4.6}$$

for a positive constant C_5 independent of δ. Note that if $\varepsilon \lesssim \delta$, then we can drop the term $C_5\varepsilon E_m(t)$ from the above inequality. Now, let us consider the case where $m = 2$. By taking c_1 sufficiently small, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) + \tilde{F}_2(t) \leq C_6\varepsilon E_2(t)$$

for a positive constant C_6 independent of δ. Thus, Gronwall's inequality yields

$$E_2(t) + \int_0^t \exp\big(C_6\varepsilon(t-\tau)\big)\tilde{F}_2(\tau)\mathrm{d}\tau \leq E_2(0)\mathrm{e}^{C_6\varepsilon t}. \tag{4.7}$$

Particularly, we have $\int_0^t \tilde{F}_2(\tau)\mathrm{d}\tau \leq E_2(0)\mathrm{e}^{C_6\varepsilon t}$. By this, (4.6), and Gronwall's inequality, we see that

$$\begin{aligned} E_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau &\leq E_m(0)\exp\left(C_5\int_0^t(\tilde{F}_2(\tau)+\varepsilon)\mathrm{d}\tau\right) \\ &\leq E_m(0)\exp\big(C_5\tilde{E}_2(0)\mathrm{e}^{C_6\varepsilon t} + C_5\varepsilon t\big). \end{aligned}$$

This together with (4.7) and (4.2) in Lemma 4.3 gives the desired estimates in Proposition 4.1. The proof is complete. □

Proof of Theorem 2.1. Since the existence theorem of the solution locally in time is now classical, for example see [17, 14], it is sufficient to give a priori estimate of the solution. The first equation in (2.4) leads to

$$\delta^2\|\partial_x^k \boldsymbol{u}_t^\delta\|^2 \lesssim \|\partial_x^k \boldsymbol{u}^\delta\|^2 + \|\nabla_\delta\partial_x^k \boldsymbol{u}^\delta\|^2 + \|\Delta_\delta\partial_x^k \boldsymbol{u}^\delta\|^2 + \|\nabla_\delta\partial_x^k p\|^2 + \|\partial_x^k \boldsymbol{f}\|^2.$$

Thus, by (4.4) and (4.5) in Lemma 4.3, we have $\delta^2\|\partial_x^k \boldsymbol{u}_t^\delta\|^2 \lesssim (1+D_2)^2 D_m$. By this, the third equation in (2.5), and the definitions of E_m and D_m (see (3.29) and (3.34)), we obtain

$$E_m(0) \leq C_8\big(1 + D_2(0)\big)^2 D_m(0) \tag{4.8}$$

for a positive constant C_8 independent of δ. Thus considering the case of $m = 2$ in the above inequality, taking $D_2(0)$ and T sufficiently small so that $2C_7C_8\big(1+D_2(0)\big)^2 D_2(0) \leq c_1$ and $\mathrm{e}^{C_6T} \leq 2$, and using the first inequality in (4.1) in Proposition 4.1, we see that the solution satisfies

$$\tilde{E}_2(t) \leq c_1 \qquad \text{for} \quad 0 \leq t \leq T/\varepsilon.$$

Thus, using the second inequality in (4.1) in Proposition 4.1 together with (4.8), we obtain

$$\tilde{E}_m(t) + \int_0^t \tilde{F}_m(\tau)\mathrm{d}\tau \le C, \tag{4.9}$$

where the constant C depends on R_1, W_1, W_2, α, and M but not on δ, ε, R, nor W. By the first equation in (2.4), we easily obtain $\delta^{-1}\|(1+|D_x|)^m(1+\delta|D_x|)u_{yy}\|^2 \lesssim \tilde{F}_m$. Therefore, we obtain the desired estimate in Theorem 2.1. Moreover, in view of the explicit form of $\tilde{E}_m$, using the second equation in (2.4) and Poincaré's inequality, we easily obtain (2.8). The proof is complete. □

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