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An Essay on A Statistical Theory of Turbulence

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Abstract

This note is based on an on-going joint work with Prof. Sakajo (Kyoto Univ) and Prof. Matsumoto (Kyoto Univ). Revisiting Kolmogorov's statistical laws (appearing in so-called Kolmogorov's Theory of 1941) and Onsager's conjecture (1949), we make an assessment of their mathematical relevance from the view point of stochastic processes. Then we need to examine the exact meaning of Kolmogorov's fundamental hypothesis, so that we introduce a "new energy dissipation rate", which is inspired by the Kármán-Howarth-Monin relation. Our mathematical strategy viewing turbulence may not be a conventional one: we don't assume any fluid equations describing the turbulence at first, but we regard turbulence as an infinite dimensional probability measure on an ensemble of appropriate time-dependent vector fields on the flat torus $\mathbb{T}^3$, which describes (a part of) Kolmogorov's statistical laws. We then consider necessary properties of the ensemble in which the desired probability should be constructed. Now we have a speculation that a family of incompressible Euler flows could be our candidate, according to a number of mathematical results on the Euler flows, e.g., Constantin-E-Titi (1994), Duchon-Robert (2000), Eyink (2003), De Lellis-Székelyhidi (2012), Isett (2012), Buckmaster (2013), Buckmaster- De Lelis-Székelyhidi (2014). This speculation could lead us to a (pseudo) Gibbs measure on the ensemble.

1 Description of Fluid flows

Mathematically, fluid flow in a domain of $\mathbb{R}^3$ is described by a time-dependent vector field on the domain. Of course it is natural to ask ourselves what equation governs the fluid when we know the background of the fluid clearly. For the time being, we do not care about the governing equation of the fluid however.

The three adjectives:homogeneous, isotropic, steady that quantify turbulence do not represent the attribute of the single vector fields but do the one of an ensemble of vector fields. That is, these three properties are understood to be a statistical attribute of turbulence. Then we prepare some concepts and notation from probability.

For simplicity, we suppose that the fluid flows in a $3D$-cube with periodic boundary condition. Hence, our domain is the flat torus $\mathbb{T}^3$.

(i) $\Omega := C(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3)$, the time-dependent vector field over $3D$-cube with periodic boundary condition.*[1]

(ii) $\mathfrak{B}$: the Borel algebra of Ω with the maximum topology.

(iii) P: the probability measure defined over $\mathfrak{B}$.

Therefore, the triplet $(\Omega, \mathfrak{B}, P)$ is a probability space, and we can safely say that the statistical law of the fluids is determined by the support of P; the peculiarity of the set of flows which appear with probability 1.

The ultimate goal of mathematics is to construct a probability measure on Ω which gives us the statistical laws we expect for turbulence. But it seems that we have a long way to go. Now, supposing we have such a probability measure P on Ω, we proceed further.

We define a family of random variables $\{V_{x,t}\}_{(x,t)\in\mathbb{T}^3\times[0,T]}$, each of which is considered to be apparatus observing vector fields:

$$\boxed{\begin{array}{cccc} V_{x,t}: & \Omega & \longrightarrow & T_x\mathbb{T}^3 \cong \mathbb{R}^3 \\ & \Psi & & \Psi \\ & u & \longmapsto & u(x,t) =: V_{x,t}(u) \end{array}}$$

This is just an evaluation map (or a projection) defined on Ω which gives us the

*[1] We may assume that the vector field is also periodic in time variable t; $\Omega = C(\mathbb{T}^4)$

direction of the vector field at space-time (x,t); $V_{x,t}$ observes the direction of the vector field at (x,t).

Using the family $\{V_{x,t}\}_{(x,t)\in\mathbb{T}^3\times[0,T]}$, we can express the three basic property of turbulence as follows:

- Homogeneity: for any $x, y \in \mathbb{T}^3$,

$$\mathbb{E}[V_{x,t}] = \mathbb{E}[V_{y,t}].$$

  More strongly, for any $f,\ g \in C\cap L^\infty(\mathbb{R}^3)$, (we can take f and g from wider class function spaces, if the random variables have better integrability conditions) there exists $F_{f,g} \in C(\mathbb{R}^3)$ such that

$$\mathbb{E}[f(V_{x,t})g(V_{y,t})] = F_{f,g}(x-y).$$

  That is, the quantity in the left hand side (the two-point correlation) is irrelevant to the choice of the origin.
- Isotropy: for any $f,\ g \in C \cap L^\infty(\mathbb{R}^3)$ (as we wrote above, we can reduce the conditions on f and g, if the random variables have better integrability conditions), there exists $G_{f,g} \in C([0,\infty))$ with the property of $G_{f,g}(0) = 0$ such that

$$\mathbb{E}[f(V_{x,t})g(V_{y,t})] = G_{f,g}(|x-y|).$$

- Steadiness: for any $t,\ s \in [0,\ T]$,

$$\mathbb{E}[V_{x,t}] = \mathbb{E}[V_{x,s}].$$

  More strongly, for any $f,\ g \in C\cap L^\infty(\mathbb{R}^3)$, (we can take f and g from wider class function spaces, if the random variables have better integrability conditions) there exists $H_{f,g} \in C(\mathbb{R})$ such that

$$\mathbb{E}[f(V_{x,t})g(V_{x,s})] = H_{f,g}(t-s).$$

  That is, the quantity in the left hand side is irrelevant to the choice of the origin.

Here, $\mathbb{E}[\,\cdot\,]$ represents the operation taking expectation of the random variable appearing in the square brackets with respect to P. Precisely, for "nice" function f defined on $\mathbb{R}^3$,

$$\mathbb{E}[f(V_{x,t})] := \int_\Omega f(V_{x,t}(u))P(\mathfrak{D}u)$$
$$= \int_{\mathbb{R}^3} f(v)P_{V_{x,t}}(dv).$$

If $P_{V_{x,t}}$ is absolute continuous with respect to the Lebesgue measure $\mathcal{L}^3$, i.e., there exists an integrable function $p(v;x,t)$ such that $\frac{P_{V_{x,t}}(dv)}{\mathcal{L}^3(dv)} = p(v;x,t)$, then we have:

$$\int_{\mathbb{R}^3} f(v)P_{V_{x,t}}(dv) = \int_{\mathbb{R}^3} f(v)p(v;x,t)\mathcal{L}^3(dv)$$

Here, we should note that v denotes the independent variable, while (x,t) is just a label for random variables.*2 P is a measure on the infinite dimensional space Ω, and $P_{V_{x,t}}$ is the distribution of the random variable $V_{x,t}$ defined on $\mathbb{R}^3$. In what follows, we also use the following symbol $\langle \cdot \rangle$ for simplisity to denote the expectation, which is frequently used in physics literatures:*3

$$\langle f(u(x,t)) \rangle := \mathbb{E}[f(V_{x,t})].$$

2 Kolmogorov's and Onsager's conjectures

Two well-known statements on turbulence are Kolmogorov' conjecture and Onsager's one. In this section, we consider some relations between these two conjectures.

In Kolmogorov's famous theory so-called K41 which was developed in a series of his papers [13, 14, 15], the incompressible Navier-Stokes equations are employed as the governing equations of the fluid:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = \nu \triangle u - \nabla p + f, \\ \nabla \cdot u = 0. \end{cases} \tag{2.1}$$

Here, $\nu > 0$ is a constant denoting the kinematical viscosity and f an external force. A candidate for the turbulent vector fields over $\mathbb{T}^3 \times [0,T]$ $(T \gg 1)$ is the velocity fields

*2 In the sequel, we may simply denote the Lebesgue measure $\mathcal{L}^d(dx)$ by dx, etc, for any dimension d.

*3 Of course, this is a conventional and useful way to denote the expectation. But it seems appropriate to introduce random variables and describe the mathematical concept and quantity by using the random variables.

u solving the incompressible Navier-Stokes equations (2.1) with very small $\nu > 0$. In Kolmogorov theory K41, the following assumption of non-vanishing energy dissipation rate (dissipation anomaly) is fundamental:

$$\langle \epsilon \rangle := \liminf_{\nu \to 0} \nu \left\langle \|\nabla u\|^2 \right\rangle > 0, \tag{2.2}$$

where*4

$$\|\nabla u\|^2 = \int_{\mathbb{T}^3} \sum_{1 \le i,j \le 3} \left(\frac{\partial u_i}{\partial x_j} \right)^2 dx.$$

Instead of (2.2), we may declare that the dissipation anomaly assumed in K41 is:

$$\langle \epsilon \rangle := \liminf_{\nu \to 0} \nu \left\langle \frac{1}{T} \int_0^T \|\nabla u(t)\|^2 dt \right\rangle > 0. \tag{2.3}$$

We cannot find the explicit formulae either (2.2) or (2.3) in the series of papers [13, 14, 15] however. This speculation is based on what properties of turbulence we expect for our theory; local or global homogeneity, steadiness, etc.*5

Under the hypothesis of (2.2) or (2.3), Kolmogorov derived several statistical laws of turbulence. According to Kolmogorov theory K41, we introduce the p-th order structure function S_p for $p \in \mathbb{N}$:

$$\begin{aligned} S_p[u](x,t) : &= \mathbb{E}\left(\left[(V_{x+h,t}(u) - V_{x,t}(u)) \cdot \frac{h}{|h|} \right]^p \right) \\ &= \left\langle \left[(u(x+h,t) - u(x,t)) \cdot \frac{h}{|h|} \right]^p \right\rangle. \end{aligned} \tag{2.4}$$

So-called K41 theory, by a kind of similarity assumption in h together with homogeneity and steadiness of turbulence, tells us that for any $p \in \mathbb{N}$ there exists a constant C_p such that $|h| \ll 1$ yields

$$S_p[u](x,t) = C_p \left(\langle \epsilon \rangle |h| \right)^{p/3} \tag{2.5}$$

for any $(x,t) \in \mathbb{T}^3 \times [0,T]$.*6 This is Kolmogorov's conjecture.

*4 We use the fact that $\nabla \cdot u = 0$.

*5 Kolmogorov reformulated the definition of $\langle \nu \rangle$ in [16] answering Landau's criticism. In this note, we do not discuss his theory of so-called K62 developed in that paper.

*6 When $p = 3$, we have $C_3 = -\frac{4}{5}$. This is known as Kolmogorov's four- fifth law.

Homogeneity and steadiness yield that C_p is independent of (x,t). When $p = 2$, this expression is the "dual" of well-known energy-cascade relation in the Fourier space. We do not consider the energy-cascade issue in this note, while the property is supposed to be connected to the smoothness of the vector fields.

On the other hand, Onsager [19] considers the incompressible Euler equations as the governing equations of turbulence:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0, \end{cases} \tag{2.6}$$

and he claims that the sample space of turbulent flows Ω consists of weak solutions (in the sense of distributions) of (2.6) which do not conserve their L^2-norms. Such solutions are recently called dissipative weak solutions.*7

Onsager conjectures in [19] that if the Hölder continuous solution u of (2.6) is dissipative, then its Hölder exponent is less than 1/3:

$$\left|(u(x+h,t) - u(x,t)) \cdot \frac{h}{|h|}\right| \lesssim |h|^\alpha, \quad 0 < \alpha \leq 1/3. \tag{2.7}$$

In other words, if u satisfies

$$\left|(u(x+h,t) - u(x,t)) \cdot \frac{h}{|h|}\right| \lesssim |h|^\alpha, \quad \alpha > 1/3, \tag{2.8}$$

then u conserves its L^2 norm:

$$\int_{\mathbb{T}^3} |u(x,t)|^2 dx = \int_{\mathbb{T}^3} |u(x,0)|^2 dx, \quad t \in [0,T]. \tag{2.9}$$

It is worthwhile noting here that Onsager's conjecture is a statement for each sample flow (vector field) belonging to Ω in contrast with Kolmogorov's one which is a set of statistical laws of sample space Ω.

Mathematically, Onsager's conjecture concerns the relation between the smoothness of the solution u of (2.6) and the conservation of (kinetic) energy (2.9). Such a problem has been known for solutions in Besov spaces (see, e.g., [3, 4]), and recently the problem have been studying for solutions in Hölder spaces, so that the Hölder exponent to ensure the energy conservation is getting closer and closer to 1/3 as Onsager's conjecture says (see, e.g. [5, 10, 2, 1]).

*7 We do not put any external forces in the equation. Even for the case, we know that there exist solutions of (2.6) such that they do not conserve their L^2-norms, see, e.g., [20, 21] and [5, 10, 2, 1].

Now looking back on two conjectures, we summarize here that, aside the problem of identifying Ω,[*8] Kolmogorov's conjecture is the one on the modulus of continuity for $\mathbb{E}[|V_{x,t} - V_{y,t}|]$ and Onsager's is on that for $|u(x,t) - u(y,t)|$ with $u \in \Omega$.

Observing Kolmogorov's conjecture and Onsager's from the point of view above, these two are not independent of each other; they are related through the concept of stochastic processes. Here we introduce another random variable: for any unit vector $\hat{r}$ in $\mathbb{R}^3$, we define:

$$X^{x,t}_{\hat{r},s}(u) := V_{x+s\hat{r},t}(u) \cdot \hat{r} = u(x + s\hat{r}, t) \cdot \hat{r}, \qquad s > 0.$$

Regarding this as a function of $s \in [0,1]$ for each fixed point (x,t), we study the stochastic process $\{X^{x,t}_{\hat{r},s}\}_{s\in[0,1]}$ on $(\Omega, \mathfrak{B}, P)$, so that we know from (2.5) that for any even number $p \in \mathbb{N}$

$$\mathbb{E}\left[\left|X^{x,t}_{\hat{r},s+r} - X^{x,t}_{\hat{r},s}\right|^p\right] \lesssim |r|^{p/3} \tag{2.10}$$

Now we recall Kolmogorov - Čentsov Theorem (see, e.g., [11]):

Theorem 1 (Kolmogorov - Čentsov). *Suppose that $X = \{X_t \,|\, 0 \le t \le T\}$ is a stochastic process on a probability space, say, $(\Omega, \mathfrak{B}, P)$, and assume that, for some positive constants, α, β and C, we have:*

$$\mathbb{E}[\,|X_t - X_s|^\alpha] \le C|t-s|^{1+\beta}, \qquad 0 \le s,\, t \le T.$$

Then there exists a continuous modification $\widetilde{X} = \left\{\widetilde{X}_t \;|\; 0 \le t \le T\right\}$ of X, which is locally Hölder continuous with some exponent γ; precisely we have, for every $\gamma \in (0, \beta/\alpha)$,

$$P\left[\omega \in \Omega \;\middle|\; \sup_{\substack{0<t-s<h(\omega)\\ s,\,t\in[0,T]}} \frac{\left|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)\right|}{|t-s|^\gamma} \le \delta\right] = 1,$$

where $h(\omega)$ is an almost surely positive random variable and $\delta > 0$ is an appropriate constant.

This theorem tells us that the modulus of continuity of a stochastic process in the mean yields the Hölder continuity of each sample path of in Ω. We apply this theorem to our process $\{X^{x,t}_{\hat{r},s}\}_{s\in[0,1]}$; if we have (2.10) for any even number $p \in \mathbb{N}$ (which is

[*8] From another perspective, we may say that Kolmogorov's theory K41 is also based on (2.6) as Onsager's mentioned in [19].

obtained from (2.5)), then there exists a Hölder continuous modification $\widetilde{X}_{\hat{r},\cdot}^{x,t}$ of $X_{\hat{r},\cdot}^{x,t}$ with its Hölder exponent $\gamma \in (0, 1/3)$:

$$\left|(u(x+h,t)-u(x,t))\cdot\frac{h}{|h|}\right| \lesssim |h|^{\alpha}, \quad 0<\alpha \lneqq 1/3. \tag{2.11}$$

This is almost the same statement as (2.7) that Onsager states in [19].*9

Nevertheless there arises a problem here: The governing equation of the fluid flow is supposed to be the incompressible Navier-Stokes equation (2.1) in Kolmogorov theory K41, and to be the incompressible Euler equation (2.6) in Onsager's paper [19]. In order to exclude this inconsistency, we once put aside the issue of model equations generating the turbulent flows. As we mentioned in §1, we just begin with the family of continuous (but non-smooth) time-dependent vector fields Ω, and seek out a probability measure P giving us the desired statistical laws of turbulence. In the course of the study, we shall shed light on the support of the desired probability measure. However, to proceed the scenario, we find that the conventional definition of the energy dissipation rate $\langle\epsilon\rangle$ defined by (2.2) or (2.3) is inconvenient, since it contains the viscous coefficient $\nu > 0$ which is presumed to be from Navier-Stokes equations (2.1). Accordingly, we need to introduce a new energy dissipation rate which should be considered to be equivalent to the conventional one. In the next section, we consider this problem.

Remark (Historical Contingency). According to [11], Kolmogorov proved Theorem 1 in 1933 which state just "there exists a continuous modification", and later Čentsov added the statement about Hölder continuity to it in 1956. While Kolmogorov's turbulence theory K41 was developed in the series of papers [13, 14, 15] published in 1941 literary and Onsager's conjecture was a statement declared in several lines in [19] without proof, Eyink-Sreenivasan [8] reports that Onsager's private, handwritten notes of the 1940s contain similar results to Kolmogorov's four- fifth law and the statement of Onsager's conjecture with "proof" (see footnote *12).

*9 We just put $\alpha = p$ and $\beta = \frac{p}{3} - 1$ in Theorem 1.

For the case of Brownian motion, we can take $\alpha = n$ and $\beta = \frac{n}{2} - 1$ for any $n \in \mathbb{N}$ in Theorem 1, so that we have $\gamma \in (0, 1/2)$. Actually, it is well-known that the Hölder exponent of sample paths of Brownian motion is strictly less than 1/2 almost surely, and that we need a log log correction term at the right end of $(0, 1/2)$ (the law of Iterated logarithm).

In this sense, Theorem 1 gives us the best possible Hölder exponent of the sample path. As to our process $X_{\hat{r},\cdot}^{x,t}$, it is not clear whether 1/3 is the best possible exponent or not, however. Onsager conjectures that the Hölder exponent does not exceed 1/3.

3 Energy dissipation rate of vector fields

We would like to define a quantity which is consistent with the conventional energy dissipation rate (2.2) or (2.3) for $\Omega := C(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3)$ without any governing equations.

Now we recall the following Kármán-Howarth-Monin relation for a family of solutions of (2.1): defining $\delta_\xi u := u(x+\xi,t) - u(x,t)$, we have that there exists $\eta > 0$ such that

$$\langle \epsilon \rangle = -\frac{1}{4}\mathrm{div}_\xi \left\langle |\delta_\xi u|^2 \delta_\xi u \right\rangle \tag{3.1}$$

for $\eta < |\xi| \ll 1$. This $\langle \epsilon \rangle$ is independent of (x,t), provided that the fluid flows subjected to (2.1) forms a steady and homogeneous emsamble. The constant $\eta > 0$ is called the Kolmogorov dissipation length (or the Kolmogorov scale) which is considered to be related to the "resolution" limit of the fluid model under consideration.*10 It is said that this relation is due to Monin, who proved (3.1) for a family of solutions of the incompressible Navier-Stokes equations (2.1) under the assumption of homogeneity and steadiness without isotropy. For the derivation and the origin of the name of this relation, see, e.g., [9].

Fortunately, ν does not appear in the right hand side of (3.1). So, taking the Kolmogorov scale infinitely small,*11 we broadly define our local energy dissipation rate for each $u \in \Omega$ by

$$\begin{aligned} \epsilon[u](x,t) :=& -\frac{1}{4}\mathrm{div}_\xi \left(|\delta_\xi u(x,t)|^2 \, \delta_\xi u(x,t) \right) |_{\xi=0} \\ =& -\frac{1}{4}\mathrm{div}_\xi \left(|\Delta_\xi V_{x,t}[u]|^2 \, \Delta_\xi V_{x,t}[u] \right) |_{\xi=0}, \end{aligned} \tag{3.2}$$

where

$$\Delta_\xi V_{x,t}[u] := V_{x+\xi,t}[u] - V_{x,t}[u] \;= u(x+\xi,t) - u(x,t) =: \delta_\xi u(x,t).$$

Thus, this $\epsilon[\,\cdot\,](x,t)$ is also a random variable on Ω.

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*10 Nevertheless, in Duchon-Robert [6], we can find the following expression

$$\langle \epsilon \rangle := -\frac{1}{4}\mathrm{div}_\xi \left\langle |\delta_\xi u|^2 \delta_\xi u \right\rangle |_{\xi=0}.$$

*11 Mathematics is free!

According to Onsager's conjecture, the support of desired probability measure is supposed to be on a set of Hölder continuous vector fields with their Hölder exponents being less than 1/3. Hence, we cannot directly evaluate our new local energy dissipation rate $\epsilon[\,\cdot\,](x,t)$ defined by (3.2). Accordingly, we propose here two ways of evaluating it as follows:

- We can compute (3.2) in the sense of distributions. Let φ_ε ($\varepsilon > 0$) be a family of non negative, radially symmetric functions in $C_0^\infty(\mathbb{T}^3) =: \mathcal{D}(\mathbb{T}^3)$ such that we have $\varphi_\varepsilon \to \delta_0$ ($\varepsilon \downarrow 0$) in $\mathcal{D}'(\mathbb{T}^3)$ (= the dual space of $\mathcal{D}(\mathbb{T}^3)$). We define

$$\epsilon[u](x,t) := \frac{1}{4}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\delta_\xi u|^2\, \delta_\xi u \cdot \nabla\varphi_\varepsilon(\xi)\, \mathcal{L}^3(d\xi). \tag{3.3}$$

- We employ the method of integral mean, so that

$$\begin{aligned} \epsilon[u](x,t) &:= -\frac{1}{4}\lim_{r\to 0}\frac{1}{\mathcal{L}^3(B(0;r))}\int_{|\xi|=r} |\delta_\xi u|^2 \delta_\xi u \cdot \frac{\xi}{|\xi|}\, \mathcal{H}^2(d\xi) \\ &= -\frac{3}{4}\lim_{r\to 0}\frac{1}{4\pi r}\int_{|\hat{\omega}|=1} |\delta_{r\hat{\omega}} u|^2 \delta_{r\hat{\omega}} u \cdot \hat{\omega}\, \mathcal{H}^2(d\hat{\omega}). \end{aligned} \tag{3.4}$$

We should note that if u is sufficiently smooth, we have

$$\int_{B(0;r)} \operatorname{div}_\xi\left(|\delta_\xi u|^2 \delta_\xi u\right)\, \mathcal{L}^3(d\xi) = \int_{|\xi|=r} |\delta_\xi u|^2 \delta_\xi u \cdot \frac{\xi}{|\xi|}\, \mathcal{H}^2(d\xi).$$

Since our vector field $u \in \Omega$ is continuous, the integrals appearing in both (3.3) and (3.4) are all well-defined. We expect that each limit appearing in (3.3) and (3.4) exists in a certain sense (even in the sense of ditributions), not necessary point wise, so that taking the integral of $\epsilon[u](x,t)$ over a space-time domain or making the duality pairing of it with some nice functions will give us a quantity which would be consistent with the original energy dissipation rate (2.2) or (2.3).

We have just proposed two ways of evaluating $\epsilon[u](x,t)$ defined by (3.2), which will give rise to a question: Do these two evaluations (3.3) and (3.4) coincide? Under appropriate conditions, It must be:

$$\begin{aligned} &\frac{1}{4}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\delta_\xi u|^2 \delta_\xi u \cdot \nabla\varphi_\varepsilon(\xi)\, \mathcal{L}^3(d\xi) \\ &= -\frac{3}{4}\lim_{r\to 0}\frac{1}{4\pi r}\int_{|\hat{\omega}|=1} |\delta_{r\hat{\omega}} u|^2 \delta_{r\hat{\omega}} u \cdot \hat{\omega}\, \mathcal{H}^2(d\hat{\omega}). \end{aligned} \tag{3.5}$$

This relation is pointed out by Duchon-Robert [6] and they prove the equality for a class of weak solutions to (2.6), putting aside the existence problem of such solutions.

4 Incompressible Euler flows and dissipation anomaly

Duchon-Robert [6] proves the following: if $u \in L^3(\mathbb{T}^3 \times (0,T))$ is a weak solution of the incompressible Euler equations (2.6), then we have the limit in the left hand side of (3.5), i.e.,

$$\begin{aligned} D[u](x,t) &:= \frac{1}{4}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} |\delta_\xi u|^2 \delta_\xi u \cdot \nabla\varphi_\varepsilon(\xi)\,\mathcal{L}^3(d\xi) \\ &=: \lim_{\varepsilon\to 0} D_\varepsilon[u](x,t), \end{aligned} \tag{4.1}$$

exists in the sense of distributions (in the simple topology of $\mathcal{D}'(\mathbb{T}^3 \times [0,T])$), and we have the following equality:*12

$$D[u] = -\frac{\partial}{\partial t}\left(\frac{|u|^2}{2}\right) - \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right). \tag{4.2}$$

These means that (1) $D[u]$ is a manifestation of $\epsilon[u]$ defined by (3.3) in $\mathcal{D}'(\mathbb{T}^3 \times [0,T])$; (2) $D[u]$ is determined independently of the choice of φ_ε; (3) $D[u]$ is the defect term for the energy conservation law (2.9).

Furthermore, Duchon-Robert [6] shows that (3.5) holds true as long as the limit in

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*12 Taking the convolution of u with φ_ε seems to work as a low pass filter for u. The convolution of "nice function" f and φ_ε will be denoted by f^ε, i.e., $f^\varepsilon = \varphi_\epsilon * f$ for an appropriate f. Putting $u = (u_1, u_2, u_3)$ and $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1,2,3)$, we have, by $\nabla \cdot u = \partial_i u_i = 0$, that

$$D_\varepsilon[u] = \frac{1}{2}E_\varepsilon[u] - \frac{1}{4}\partial_i(u_i u_j u_j)^\varepsilon + \frac{1}{4}u_i\,\partial_i(u_j u_j)^\varepsilon$$

where

$$E_\varepsilon[u] := u_j\,\partial_i(u_i u_j)^\varepsilon - u_i u_j\,\partial_j(u_i)^\varepsilon.$$

Here, we have used Einstein summation convention.

On the other hand, the equality obtained by taking convolution of both sides of (2.6) with φ_ϵ yields that E_ε is twice as much as the right hand side of (4.2). Consequently, we have $\lim_{\varepsilon\to 0} E_\varepsilon = 2\lim_{\varepsilon\to 0} D_\varepsilon$ in $\mathcal{D}'(\mathbb{T}^3 \times [0,T])$.

According to Eyink-Sreenivasan [8], one can find such arguments above in Onsager's private, handwritten notes of the 1940s to obtain (4.2) with (4.1). It is easy to imagine that (4.1) and (4.2) lead to Onsager's conjecture.

the right hand side of (3.5) exists in appropriate topology, so that we have:

$$-\frac{4}{3}D[u] = \lim_{r\to 0}\frac{1}{4\pi r}\int_{|\hat\omega|=1}|\delta_{r\hat\omega}u|^2\delta_{r\hat\omega}u\cdot\hat\omega\,\mathcal{H}^2(d\hat\omega). \tag{4.3}$$

Duchon-Robert [6] claims that if the defect $D[u]$ is positive, then (4.3) is the manifestation of the four-thirds law of the isotropic turbulence owing to Kármán-Howarth-Monin:

$$-\frac{4}{3}\langle\epsilon\rangle|\xi| = \left\langle|\delta_\xi u|^2\delta_\xi u\cdot\frac{\xi}{|\xi|}\right\rangle,\quad 0<|\xi|\ll 1$$

where $\langle\epsilon\rangle$ is defined by (2.3), while the right hand side of (4.3) is the spherical mean of $|\delta_\xi u|^2\delta_\xi u\cdot\frac{\xi}{|\xi|}$. We should note that (4.3) is proved without any assumption on homogeneity, isotropy, or steadiness of solutions; Duchon-Robert [6] only assume that $u\in L^3(\mathbb{T}^3\times(0,T))$ is a weak solution of (2.6).

Eyink [7] shows a similar result to Duchon-Robert [6]: He proves that the following equality holds true as long as the limit of the right hand side exists in an appropriate topology:

$$\begin{aligned}\frac{1}{4}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}&|\delta_\xi u|^2\delta_\xi u\cdot\nabla\varphi_\varepsilon(\xi)\,\mathcal{L}^3(d\xi)\\&=-\frac{5}{4}\lim_{r\to 0}\frac{1}{4\pi r}\int_{|\hat\omega|=1}[\delta_{r\hat\omega}u\cdot\hat\omega]^3\,\mathcal{H}^2(d\hat\omega).\end{aligned} \tag{4.4}$$

Here, $D[u]$ in the left hand side is the same one in (4.1). Thus (4.4) in disguise is

$$-\frac{4}{5}D[u] = \lim_{r\to 0}\frac{1}{4\pi r}\int_{|\hat\omega|=1}[\delta_{r\hat\omega}u\cdot\hat\omega]^3\,\mathcal{H}^2(d\hat\omega), \tag{4.5}$$

which corresponds to the Kolmogorov's conjecture (2.5) with $p=3$ as known as the Kolmogorov four-fifth law:

$$-\frac{4}{5}\langle\epsilon\rangle|\xi| = \left\langle\left[\delta_\xi u\cdot\frac{\xi}{|\xi|}\right]^3\right\rangle,\quad 0<|\xi|\ll 1.$$

Eyink [7] regards (4.5) as the manifestation of the celebrated, Kolmogorov's four-fifth law as well as Duchon-Robert [6] does, provided that we have $D[u]>0$.

These results above due to Duchon-Robert [6] and Eyink [7] are proved without the assumption of homogeneity, isotropy, and steadiness for weak solutions of (2.6). Nevertheless, their results seem strongly to suggest that a set of weak solutions of

(2.6) with $D[u] > 0$ could be the support of the desired probability measure P on Ω. Now, there arise two questions: Do we have

(1) the existence of dissipative weak solutions of (2.6) with $D[u] > 0$?

(2) the existence of the limits in the right hand side of both (4.3) and (4.5)?

Recently, a great progress on the problem (1) has been made by De Lellis-Székelyhidi (2012), Isett (2012), Buckmaster (2013), Buckmaster- De Lelis-Székelyhidi (2014) (see [5, 10, 2, 1]), in which Hölder continuous weak solutions of (2.6) with obeying any given continuous behavior of L^2 norm have been constructed.*13 Hence, we have infinitely many weak solutions with both $D[u] > 0$ and $D[u] < 0$.

The problem (2) concerns the Hölder exponent of those solutions constructed in [5, 10, 2, 1]. It seems apparent from the structure of the integrands in the right hand side of (4.3) and (4.5) that, if the Hölder exponent is bigger than 1/3, then we see $D[u] = 0$: We have thus proved Onsager's conjecture in disguise. For the existence of (dissipative) weak solutions with Hölder exponent being 1/3, Buckmaster-De Lelis-Székelyhidi [2] has succeeded in proving, for any $\varepsilon > 0$, the existence of compactly supported solutions in $L^1([0,T]; C^{1/3-\varepsilon}(\mathbb{T}^3))$ to (2.6). But it is not enough to insist the existence of negative limits in the right hand side of both (4.3) and (4.5).

We end with this section with the following significant remark:

Remark (Dissipation anomaly). As we considered in the end of §3, the defect $D[u]$ itself will not give us a physical quantity. It must be something like:

$$\int_0^T \int_{\mathbb{T}^3} D[u](x,t)\, dx\, dt > 0 \quad \text{or} \quad \mathbb{E}\left[D[u](x,t)\right] > 0,$$

because the turbulence is a property of vector fields belonging to the ensemble $(\Omega, \mathfrak{B}, P)$. Hence, $D[u](x,t)$ may change sign over $\mathbb{T}^3 \times [0,T]$, so that it does not seem appropriate to state $D[u] > 0$ when we refer to dissipative weak solutions of (2.6).

5 Energy dissipation rate and dissipation anomaly revisited

A weak solution of the incompressible Euler equation (2.6) is called *dissipative*, if its kinetic energy is not conserved. Physically, dissipation of the kinetic energy usually

*13 It is a surprise, isn't it?

means the "loss" of it. Thus, recalling (4.2) and Remark stated at the end of the previous section, we also assume

$$\int_0^T \int_{\mathbb{T}^3} D[u](x,t)\,dx\,dt > 0, \tag{5.1}$$

when we say that u is dissipative on $\mathbb{T}^3 \times [0,T]$.*14

We will discuss the relation between our defect $D[u]$ defined by (3.3) for (2.6) and the conventional definition of the energy dissipation rate defined by (2.3) for (2.1) with $f \equiv 0$.

Let u^ν be a Leray-Hopf weak solution of (2.1) with $f \equiv 0$; in the sequel, we always assume $f \equiv 0$ for simplicity. For u^ν, we can define the defect term $D_{NS}[u^\nu]$ as we did for a Euler flow in footnote*12; we have

$$2D_{NS}[u^\nu] = -\partial_t |u^\nu|^2 - \operatorname{div}(u^\nu(|u^\nu|^2 + 2p)) + \nu\triangle|u^\nu|^2 - 2\nu|\nabla u^\nu|^2.$$

Supposing that u is a velocity field of (2.6) belonging to $L^3(\mathbb{T}^3 \times (0,T))$ such that we have

$$\lim_{\nu\downarrow 0} \|u^\nu - u\|_{L^3(\mathbb{T}^3\times(0,T))} = 0, \tag{5.2}$$

we can easily show that

$$\lim_{\nu\downarrow 0}(D_{\rm NS}[u^\nu] + \nu|\nabla u^\nu|^2) = D[u] \quad \text{in} \quad \mathcal{D}'(\mathbb{T}^3 \times [0,T]).$$

Therefore, since Leray-Hopf solution u^ν satisfies $D[u^\nu] \geq 0$ (see [6]), if $D[u]$ is a measure on $\mathbb{T}^3 \times [0,T]$ (roughly speaking $D[u] > 0$), then we have

$$\limsup_{\nu\downarrow 0} \int_0^T \int_{\mathbb{T}^3} D_{NS}[u^\nu]\,dx\,dt > 0 \quad \text{or} \quad \limsup_{\nu\downarrow 0} \nu \int_0^T \|\nabla u^\nu(t)\|^2\,dt > 0.$$

Furthermore, if $\{u^\nu\}_{\nu>0}$ is a family of smooth solutions of (2.1),*15 we obtain

$$\limsup_{\nu\downarrow 0} \nu \int_0^T \|\nabla u^\nu(t)\|^2\,dt > 0. \tag{5.3}$$

Conversely, under the assumption of (5.2), if there exists a family of smooth solutions of (2.1) satisfying (5.3), then the limit function u solves (2.6) and satisfies (5.1).

*14 Since the defect $D[u]$ defined by (4.1) is generally a distribution (not necessary a function), we may understand the integral in (5.1) as the duality pair of $D[u]$ and the constant function 1.

*15 Then, $D_{NS}[u^\nu] = 0$.

Thus, the question we must concern is the validity of (5.2), and we do not know whether we can construct a dissipative weak solution of (2.6) from a family of solutions of (2.1) or not.

Here we mention the result of Kato [12]. This paper considers the initial boundary value problem for (2.1) and (2.6) on a bounded domain in $\mathbb{R}^3$, and study the convergence of a sequence of solutions $\{u^\nu\}_{\nu>0}$ of (2.1) as $\nu \downarrow 0$ under the assumption that (2.6) has a classical solution u which conserves its L^2 norm. He proved that

$$\limsup_{\nu\downarrow 0} \nu \int_0^T \|\nabla u^\nu(t)\|^2 \, dt = 0$$

if and only if

$$\lim_{\nu\downarrow 0} \sup_{t\in[0,T]} \|u^\nu(t) - u(t)\|_{L^2} = 0.$$

6 Local partition function of turbulent fields

This section will be very sketchy. For more details, please refer to [17, 18].

Mathematical analysis on time-dependent vector fields is made by using mathematical concepts and objects defined on space-time: The arguments in §4 are developed over space-time $\mathbb{T}^3 \times [0,T]$, and Kato's theory of vanishing viscosity for (2.1) in [12] we introduced in the previous section consider the space-time norm of ∇u.

We regard turbulence as an infinite dimensional probability measure on $\Omega = C(\mathbb{T}^3 \times [0,T]; \mathbb{R}^3)$, where $\mathbb{R}^3 \cong T_x\mathbb{T}^3$ for $x \in \mathbb{T}^3$. We employ $\epsilon[u](x,t)$ defined by (3.3) as our local energy dissipation rate, so that for each dissipative weak solutions u of (2.6) there exists $D[u]$ defined by (4.1) in the sense of the distribution (at least) by means of the theory of Duchon-Robert [6]. The set of dissipative weak solutions of (2.6) would be included in the support of the desired probability measure P.

Supposing that the sample space of turbulent flows is the set of dissipative weak solutions of (2.6), we proceed further. We divide $\mathbb{T}^3 \times [0,T]$ into small space-time cubes

$$(x+\square) \times \left[t - \frac{\Delta t}{2}, t + \frac{\Delta t}{2}\right], \quad (x,t) \in \mathbb{T}^3 \times [0,T],$$

and introduce the virtual energy dissipation rate as follows:

$$\epsilon_{x,t}[u] := \frac{1}{|\square \times \Delta t|} \int_{t-\Delta t/2}^{t+\Delta t/2} ds \int_{x+\square} D[u](y,s)\, dy.$$

Here we introduce the concept of MFU, which is an abbreviation for Minimal Flow Unit, to clarify the meaning of $4D$ small cubes: Taking large $N \in \mathbb{N}$, we divide $\mathbb{T}^3 \times [0,T]$ into small N equal space-time cubes so that the vector fields still have the nature of turbulent flows on each small $4D$ cube which is a translation of

$$\square \times \left[-\frac{\Delta t}{2}, \frac{\Delta t}{2}\right],$$

by appropriate $(x,t) \in \mathbb{T}^3 \times (0,T)$ such that we have

$$\bigcup_{(x,t)\in \mathrm{MFU}} \left\{(x+\square) \times \left[t-\frac{\Delta t}{2}, t+\frac{\Delta t}{2}\right]\right\} = \mathbb{T}^3 \times (0,T)$$

and for $(x,t) \neq (y,s)$

$$\left\{(x+\square) \times \left[t-\frac{\Delta t}{2}, t+\frac{\Delta t}{2}\right]\right\}^{\circ} \bigcap \left\{(y+\square) \times \left[s-\frac{\Delta t}{2}, s+\frac{\Delta t}{2}\right]\right\}^{\circ} = \varnothing \quad (6.1)$$

In well developed turbulent flows we may say that the vector field on each cube determined by as if rolling a "dice" independently. Precisely, we prepare an N-independent $\mathbb{R}^3$-valued random variables which are assigned to each MFU, and assume that they are independently, identically distributed (abbreviated to i.i.d.). For simplicity, we discretize the value of the random variables, and cut off the high velocity region. This is a kind of coarse graining procedure, so that we obtain a discretized model of turbulent vector fields.

Then, we can apply Shannon-McMillan's theorem*16 (see, e.g., [22]) to our situation to get a "statistical mechanics" by giving the expectation value of the local energy dissipation rate:

$$\bar{\epsilon} = \mathbb{E}\left[\,\epsilon(x,t)\,\right] = \mathbb{E}\left[\frac{1}{|\square \times \Delta t|}\int_{t-\Delta t/2}^{t+\Delta t/2} ds \int_{x+\square} D[v](y,s)dy\right].$$

This value should be positive*17 and is independent of (x,t) which labels each MFU, and is considered to be an intensive variable to identify the ensemble of turbulent flows. Principle of maximal entropy determines the distribution law of N random

*16 A refinement of the law of large numbers.

*17 This means that the local energy dissipation rate is "statistically" positive.

variables assigned to each MFU, so that the distribution of each discretized random variable should be something like that $e^{-\beta\epsilon_{x,t}}/Z_{x,t}(\beta)$ with some constant β which is designed to give us $\bar{\epsilon} = \mathbb{E}[\epsilon(x,t)]$. Here $Z_{x,t}(\beta) = \sum_{(x,t)\in \mathrm{MFU}} e^{-\beta\epsilon_{x,t}}$ is the partition function of our discretized model.

Boldly taking the continuous limit of our "statistical mechanics" above, we could obtain the following local partition function on each MFU:

$$Z_\beta^{x,t} := \int_\Omega \exp\left[\frac{-\beta}{|\Box\times\Delta t|}\int_{t-\Delta t/2}^{t+\Delta t/2} ds \int_{x+\Box} D[v](y,s)dy\right]\mathfrak{D}v, \tag{6.2}$$

where $\mathfrak{D}v$ is the infinite dimensional "flat measure" on Ω (mathematically meaningless). Unfortunately, the meaning of β is not clear now, which corresponds to the inverse temperature of a standard model of statistical mechanics.

In order to construct the desired probability measure P on Ω, we need a global partition function Z_β. The top term of a suitable approximation of Z_β may become

$$Z_\beta \sim \prod_{(x,t)\in \mathrm{MFU}} \exp\left[\frac{-\beta}{|\Box\times\Delta t|}\int_{t-\Delta t/2}^{t+\Delta t/2} ds \int_{x+\Box} D[v](y,s)dy\right] \tag{6.3}$$

$$= \exp\left[\frac{-N\beta}{|\mathbb{T}\times[0,T]|}\int_0^T ds\int_{\mathbb{T}^3} D[v](y,s)dy\right]. \tag{6.4}$$

Here, we recall that N denotes the number of MFUs. Watching at (6.4), one may feel that N would play a role of the volume parameter. It is just a speculation.

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