数理解析研究所講究録
第 1948 巻 2015 年 28-37

AROUND DISTANCE-SQUARED MAPPINGS

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ABSTRACT. This is a survey article on distance-squared mappings and their related topics.

1. DISTANCE-SQUARED MAPPINGS

Distance-squared mappings were firstly investigated in [5].

Let n (resp., $\mathbb{R}^n$) be a positive integer (resp., the n-dimensional Euclidean space). The n-dimensional Euclidean distance is the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For any point $p \in \mathbb{R}^n$, the mapping $d_p : \mathbb{R}^n \to \mathbb{R}$, defined by $d_p(x) = d(p, x)$, is called a *distance function.*

Definition 1.1. Let $p_1, \ldots, p_\ell$ $(\ell \geq 1)$ be given points in $\mathbb{R}^n$. Then, the following mapping $d_{(p_1,\ldots,p_\ell)} : \mathbb{R}^n \to \mathbb{R}^\ell$ is called a *distance mapping*:

$$d_{(p_1,\ldots,p_\ell)}(x) = (d(p_1, x), \ldots, d(p_\ell, x)).$$

A distance mapping is one in which each component is a distance function. Distance mappings were firstly studied in the undergraduate-thesis of the first author, and the main result of the thesis is the following proposition. Proposotion 1.1 can be found also in [5] with a rigorous proof. The proof uses several geometric results in [2]. Let S^n be the n-dimensional unit sphere in $\mathbb{R}^{n+1}$

Proposition 1.1 ([5]). *Let $i : S^1 \to i(S^1) \subset \mathbb{R}^2$ be a homeomorphism. Then, there exist two points $p_1, p_2 \in i(S^1)$ such that $d_{(p_1,p_2)} \circ i : S^1 \to \mathbb{R}^2$ is homeomorphic to the image $d_{(p_1,p_2)} \circ i(S^1)$.*

Proposition 1.1 is applicable even if a mapping i is not differentiable anywhere. However, it seems quite difficult to derive higher-dimensional extensions of the proposition.

On the other hand, it is possible to obtain the differentiable version of higher-dimensional extensions as follows.

Definition 1.2. Let p_i $(1 \leq i \leq \ell)$ be a given point in $\mathbb{R}^n$. Then, the following mapping $D_{(p_1,\ldots,p_\ell)} : \mathbb{R}^n \to \mathbb{R}^\ell$ is called a *distance-squared mapping:*

$$D_{(p_1,\ldots,p_\ell)}(x) = (d^2(p_1, x), \ldots, d^2(p_\ell, x)).$$

Although $D_{(p_1,\ldots,p_\ell)}$ always has a singular point, the following Theorems 1.1 and 1.2 hold as follows.

Theorem 1.1 ([5]). *Let M be an m-dimensional closed C^∞ manifold* ($m \geq 1$)*, and let $i : M \to \mathbb{R}^\ell$ $(m+1 \leq \ell)$ be a C^∞ embedding. Then, there exist $p_1, \ldots, p_{m+1} \in i(M)$, $p_{m+2}, \ldots, p_\ell \in \mathbb{R}^\ell$such that $D_{(p_1,\ldots,p_\ell)} \circ i : M \to \mathbb{R}^\ell$ is a C^∞ embedding.*

For the definition of embedding, see [4].

Corollary 1.1 ([5]). *Let M be an m-dimensional closed C^∞ manifold* ($m \geq 1$)*, and let $i : M \to \mathbb{R}^{m+1}$ be a C^∞ embedding. Then, there exist $p_1, \ldots, p_{m+1} \in i(M)$ such that $D_{(p_1,\ldots,p_{m+1})} \circ i : M \to \mathbb{R}^{m+1}$ is a C^∞ embedding.*

Let M be an m-dimensional closed C^∞ manifold ($m \geq 1$), and let $i : M \to \mathbb{R}^{m+1}$ be a C^∞ embedding. Then, there exist $p_1, \ldots, p_{m+1} \in i(M)$ such that $D_{(p_1,\ldots,p_{m+1})} \circ i : M \to \mathbb{R}^{m+1}$ is a C^∞ embedding. Let M and N be C^∞ manifolds. A C^∞ immersion $f : M \to N$ (for the definition of immersion, see [4]) is said to be *with normal crossing at a point* $y \in N$ if $f^{-1}(y)$ is a finite set $\{x_1, x_2, \ldots, x_n\}$ and for any subsets $\{\lambda_1, \lambda_2, \ldots, \lambda_s\} \subset \{1, 2, \ldots, n\}$ $(s \leq n)$,

$$\operatorname{codim}\Big(\bigcap_{j=1}^{s} df_{x_{\lambda_j}}(T_{x_{\lambda_j}}M)\Big) = \sum_{j=1}^{s} \operatorname{codim}(df_{x_{\lambda_j}}(T_{x_{\lambda_j}}M)),$$

where $\operatorname{codim} H = \dim T_yN - \dim H$ for a linear subspace $H \subset T_yN$. A C^∞ immersion $f : M \to N$ is said to be *with normal crossing* if f is a C^∞ immersion with normal crossing at any point $y \in N$.

Theorem 1.2 ([5]). *Let M be an m-dimensional closed C^∞ manifold* ($m \geq 1$)*, and let $i : M \to \mathbb{R}^\ell$ $(m+1 \leq \ell)$ be a C^∞ immersion with normal crossing. Then, there exist $p_1, \ldots, p_{m+1} \in i(M)$, $p_{m+2}, \ldots, p_\ell \in \mathbb{R}^\ell$ such that $D_{(p_1,\ldots,p_\ell)} \circ i : M \to \mathbb{R}^\ell$ is a C^∞ immersion with normal crossing.*

Corollary 1.2 ([5]). *Let M be an m-dimensional closed C^∞ manifold* ($m \geq 1$)*, and let $i : M \to \mathbb{R}^{m+1}$ be a C^∞ immersion with normal crossing. Then, there exist $p_1, \ldots, p_{m+1} \in i(M)$ such that $D_{(p_1,\ldots,p_{m+1})} \circ i : M \to \mathbb{R}^{m+1}$ is a C^∞ immersion with normal crossing.*

We say that ℓ-points $p_1, \ldots, p_\ell \in \mathbb{R}^n$ $(1 \leq \ell \leq n+1)$ are *in general position* if $\ell = 1$ or $\overrightarrow{p_1p_2}, \ldots, \overrightarrow{p_1p_\ell}$ $(2 \leq \ell \leq n+1)$ are linearly independent, where $\overrightarrow{p_ip_j}$ stands for $(p_{j1} - p_{i1}, \ldots, p_{jn} - p_{in})$ $(p_i = (p_{i1}, \ldots, p_{in}), p_j = (p_{j1}, \ldots, p_{jn}) \in \mathbb{R}^n)$. A mapping $f : \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be *$\mathcal{A}$-equivalent* to a mapping $g : \mathbb{R}^n \to \mathbb{R}^\ell$ if there exist C^∞ diffeomorphisms $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\psi \circ f \circ \varphi = g$. For any two positive integers ℓ, n satisfying $\ell \leq n$, the following mapping $\Phi_\ell : \mathbb{R}^n \to \mathbb{R}^\ell$ $(\ell \leq n)$ is called *the normal form of definite fold mappings*:

$$\Phi_\ell(x_1, \ldots, x_n) = (x_1, \ldots, x_{\ell-1}, x_\ell^2 + \cdots + x_n^2).$$

The properties of distance-squared mappings, especially (I) and (II) of the following Theorem 1.3, are essential in the proofs of Theorems 1.1 and 1.2. Thus, in this sense, the following Theorem 1.3 may be regarded as the main result in [5].

Theorem 1.3 ([5]).
(I) Let ℓ,n be integers such that $2 \leq \ell \leq n$, and let $p_1, \ldots, p_\ell \in \mathbb{R}^n$ be in general position. Then, $D_{(p_1,\ldots,p_\ell)} : \mathbb{R}^n \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the normal form of definite fold mappings.
(II) Let ℓ,n be integers such that $2 \leq n < \ell$, and let $p_1, \ldots, p_{n+1} \in \mathbb{R}^n$ be in general position. Then, $D_{(p_1,\ldots,p_\ell)} : \mathbb{R}^n \to \mathbb{R}^\ell$ is $\mathcal{A}$-equivalent to the inclusion $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$.

All results in this section have been rigorously proved in [5].

2. Lorentzian distance-squared mappings

Lorentzian distance-squared mappings were firstly studied in [6].

As same as in Section 1, we let n be a positive integer. Let x, y be two vectors of $\mathbb{R}^{n+1}$. Then, the *Lorentzian inner product* is the following qudratic form:

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n,$$

where $x = (x_0, x_1, \ldots, x_n), y = (y_0, y_1, \ldots, y_n)$. If the role of the Euclidean inner product $x \cdot y = \sum_{i=0}^{n} x_iy_i$ is replaced by the Lorentzian inner product, then the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ is called *Lorentzian* $(n+1)$*-space*, and it is denoted by $\mathbb{R}^{1,n}$. For a vector x of Lorentzian $(n+1)$-space $\mathbb{R}^{1,n}$, *Lorentzian length* of x is $\sqrt{\langle x, x\rangle}$. Notice that a pure imaginary value may be taken as the Lorentzian length and thus $\sqrt{\langle x, x\rangle}$ does not give a real-valued function. On the other hand, its square $x \mapsto \langle x, x\rangle$ is always a real value. For a non-zero vector $x \in \mathbb{R}^{1,n}$, it is called *space-like*, *light-like* or *time-like* if its Lorentzian length is positive, zero or pure imaginary respectively. The following is the definition of the likeness of the vector subspace.

Definition 2.1 ([20]). Let V be a vector subspace of $\mathbb{R}^{1,n}$. Then V is said to be

(1) *time-like* if V has a time-like vector,
(2) *space-like* if every nonzero vector in V is space-like, or
(3) *light-like* otherwise.

The *light cone* of Lorentzian $(n+1)$-space $\mathbb{R}^{1,n}$, denoted by LC, is the set of $x \in \mathbb{R}^{1,n}$ such that $\langle x, x\rangle = 0$. For more details on Lorentzian space, refer to [20]. Recently, Singularity Theory has been very actively applied to geometry of submanifolds in Lorentzian space (for instance, see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23]). In [6], it is given a different application of Singularity Theory to the study of Lorentzian space from these researches.

Let p be a point of $\mathbb{R}^{1,n}$. The *Lorentzian distance-squared function* is the following function $\ell_p^2 : \mathbb{R}^{1,n} \to \mathbb{R}$ ([9]):

$$\ell_p^2(x) = \langle x - p, x - p\rangle.$$

Let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ $(1 \leq k)$ be finitely many points. For any $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$, the *Lorentzian distance-squared mapping*, denoted by $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$, is defined as follows:

$$L_{(p_0,\ldots,p_k)}(x) = \left(\ell_{p_0}^2(x), \ldots, \ell_{p_k}^2(x)\right).$$

For finitely many points $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ $(1 \leq k)$, a vector subspace V is called the *recognition subspace* and is denoted by $V(p_0, \ldots, p_k)$ of $\mathbb{R}^{1,n}$ if the following is satisfied:

$$V = \sum_{i=1}^{k} \mathbb{R}\, \overrightarrow{p_0p_i}.$$

For any two positive integers k, n satisfying $k < n$, the following mapping $\Psi_k : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is called the *normal form of Lorentzian indefinite fold mapping*:

$$\Psi_k(x_0, x_1, \ldots, x_n) = \left(x_1, \ldots, x_k, -x_0^2 + \sum_{i=k+1}^{n} x_i^2\right).$$

Let j, k be two positive integers satisfying $j \leq k$ and let $\tau_{(j,k)} : \mathbb{R}^{j+1} \to \mathbb{R}^{k+1}$ be the inclusion:

$$\tau_{(j,k)}(X_0, X_1, \ldots, X_j) = (X_0, X_1, \ldots, X_j, 0, \ldots, 0).$$

Theorem 2.1 ([6]). (1) *Let k, n be two positive integers and let $p_0, \ldots, p_k \in \mathbb{R}^{n,1}$ be the same point (i.e. $\dim V(p_0, \ldots, p_k) = 0$). Then, the Lorentzian distance-squared mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{n,1} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to the mapping*

$$(x_0, \ldots, x_n) \mapsto \left(-x_0^2 + \sum_{i=1}^{n} x_i^2, 0, \ldots, 0\right).$$

(2) *Let j, k, n be three positive integers satisfying $j < n, j \leq k$, and let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points such that two recognition subspaces $V(p_0, \ldots, p_k)$ and $V(p_0, \ldots, p_j)$ have the same dimension j. Then, the following hold:*

(a) *The mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to $\tau_{(j,k)} \circ \Phi_{j+1}$ if and only if $V(p_0, \ldots, p_k)$ is time-like.*

(b) *The mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to $\tau_{(j,k)} \circ \Psi_j$ if and only if $V(p_0, \ldots, p_k)$ is space-like.*

(c) *The mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to*

$$(x_0, \ldots, x_n) \mapsto \left(x_1, \ldots, x_j, x_0x_1 + \sum_{i=j+1}^{n} x_i^2, 0, \ldots, 0\right)$$

if and only if $V(p_0, \ldots, p_k)$ is light-like.

(3) *Let k, n be two positive integers satisfying $n \leq k$ and let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points such that $\dim V(p_0, \ldots, p_k) = \dim V(p_0, \ldots, p_n) = n$. Then, the following hold:*

(a) *The mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to $\tau_{(n,k)} \circ \Phi_{n+1}$ if and only if $V(p_0, \ldots, p_k)$ is time-like or space-like.*

(b) *The mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to*

$$(x_0, \ldots, x_n) \mapsto (x_1, \ldots, x_n, x_0x_1, 0 \ldots, 0)$$

if and only if $V(p_0, \ldots, p_k)$ is light-like.

(4) *Let k, n be two positive integers satisfying $n < k$ and let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points such that $\dim V(p_0, \ldots, p_k) = \dim V(p_0, \ldots, p_{n+1}) = n + 1$. Then, $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is always $\mathcal{A}$-equivalent to the inclusion $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_n, 0, \ldots, 0)$.*

Let $p_0, \ldots, p_k \in \mathbb{R}^{n,1}$ be given $(k+1)$ points. We say that $p_0, \ldots, p_k$ are *in general position* if the dimension of $V(p_0, \ldots, p_k)$ is k. For $(k+1)$ points $q_0, \ldots, q_k \in \mathbb{R}^{1,n}$ in general position $(k \leq n)$, the singular set of $L_{(q_0,\ldots,q_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is clearly the k-dimensional affine subspace spanned by these points. Since $\tau_{(k.k)}$ is the identity mapping, we have the following corollary.

Corollary 2.1 ([6]). (1) *Let k, n be two positive integers satisfying $k < n$ and let $p_0, \ldots, p_k$ belonging to $\mathbb{R}^{1,n}$ be $(k+1)$ points in general position. Then, the following hold:*

(a) *The mapping $L_{(p_0,\ldots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ is $\mathcal{A}$-equivalent to Φ_{k+1} if and only if $V(p_0, \ldots, p_k)$ is time-like.*

(b) *The mapping* $L_{(p_0,\dots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ *is* $\mathcal{A}$*-equivalent to* Ψ_k *if and only if* $V(p_0,\dots,p_k)$ *is space-like.*

(c) *The mapping* $L_{(p_0,\dots,p_k)} : \mathbb{R}^{1,n} \to \mathbb{R}^{k+1}$ *is* $\mathcal{A}$*-equivalent to*

$$(x_0,\dots,x_n) \mapsto \left(x_1,\dots,x_k, x_0x_1 + \sum_{i=k+1}^{n} x_i^2\right)$$

if and only if $V(p_0,\dots,p_k)$ *is light-like.*

(2) *Let* n *be a positive integer and let* $p_0,\dots,p_n \in \mathbb{R}^{1,n}$ *be* $(n+1)$ *points in general position. Then, the following hold:*

(a) *The mapping* $L_{(p_0,\dots,p_n)} : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ *is* $\mathcal{A}$*-equivalent to* Φ_{n+1} *if and only if* $V(p_0,\dots,p_n)$ *is time-like or space-like.*

(b) *The mapping* $L_{(p_0,\dots,p_n)} : \mathbb{R}^{1,n} \to \mathbb{R}^{n+1}$ *is* $\mathcal{A}$*-equivalent to*

$$(x_0,\dots,x_n) \mapsto (x_1,\dots,x_n, x_0x_1)$$

if and only if $V(p_0,\dots,p_n)$ *is light-like.*

The following are clear:

(1) Any non-singular fiber of Φ_n is a circle.
(2) Any non-singular fiber of Ψ_{n-1} is an equilateral hyperbola.
(3) Any non-singular fiber of $(x_0,\dots,x_n) \mapsto (x_1,\dots,x_{n-1}, x_0x_1 + x_n^2)$ is a parabola (possibly at infinity).

Therefore, by the case $k = n-1$ in Corollary 2.1, we have the following:

Corollary 2.2 ([6]). *Let* n *be a positive integer such that* $2 \le n$ *and let* $p_0,\dots,p_{n-1}$ *belonging to* $\mathbb{R}^{1,n}$ *be* n *points in general position. Then, the following hold:*

(1) *There exists a* C^∞ *diffeomorphism* $h : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ *by which any non-singular fiber* $L^{-1}_{(p_0,\dots,p_{n-1})}(y)$ *is mapped to a circle if and only if the recognition subspace* $V(p_0,\dots,p_{n-1})$ *is time-like.*
(2) *There exists a* C^∞ *diffeomorphism* $h : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ *by which any non-singular fiber* $L^{-1}_{(p_0,\dots,p_{n-1})}(y)$ *is mapped to an equilateral hyperbola if and only if* $V(p_0,\dots,p_{n-1})$ *is space-like.*
(3) *There exists a* C^∞ *diffeomorphism* $h : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ *by which any non-singular fiber* $L^{-1}_{(p_0,\dots,p_{n-1})}(y)$ *is mapped to a parabola if and only if the recognition subspace* $V(p_0,\dots,p_{n-1})$ *is light-like.*

In [6], it is remarked that an affine diffeomorphism can be chosen as the diffeomorphism $h : \mathbb{R}^{1,n} \to \mathbb{R}^{1,n}$ in Corollary 2.2.

The motivation to classify Lorentzian distance-squared mappings in [6] is the classification results on distance-squared mappings, namely Theorem 1.3. It is natural to ask how Theorem 1.3 changes if distance-squared functions are replaced with Lorentzian distance-squared functions. Combining Theorem 1.3 and Corollary 2.1, we have the following:

Corollary 2.3 ([6]). (1) *Let* k, n *be two positive integers satisfying* $k < n$ *and let* $p_0,\dots,p_k$ *belonging to* $\mathbb{R}^{1,n}$ *be* $(k+1)$ *points in general position. Then,* $L_{(p_0,\dots,p_k)}$ *is* $\mathcal{A}$*-equivalent to* $D_{(p_0,\dots,p_k)}$ *if and only if* $V(p_0,\dots,p_k)$ *is time-like.*

(2) *Let* n *be a positive integer and let* $p_0,\dots,p_n \in \mathbb{R}^{1,n}$ *be* $(n+1)$ *points in general position. Then,* $L_{(p_0,\dots,p_n)}$ *is* $\mathcal{A}$*-equivalent to* $D_{(p_0,\dots,p_n)}$ *if and only if* $V(p_0,\dots,p_n)$ *is time-like or space-like.*

(3) *Let k, n be two positive integers satisfying $n < k$ and let $p_0, \ldots, p_k \in \mathbb{R}^{1,n}$ be $(k+1)$ points such that the $(n+2)$ points $p_0, \ldots, p_{n+1}$ are in general position. Then, $L_{(p_0,\ldots,p_k)}$ is always $\mathcal{A}$-equivalent to $D_{(p_0,\ldots,p_k)}$.*

All results in this section have been rigorously proved in [6].

3. Generalized distance-squared mappings of $\mathbb{R}^2$ into $\mathbb{R}^2$

Generalized distance-squared mappings were firstly studied in [8]. For any two positive integers k, n, we let $p_0, p_1, \ldots, p_k$ be $(k+1)$ points of $\mathbb{R}^{n+1}$. We set $p_i = (p_{i0}, p_{i1}, \ldots, p_{in})$ $(0 \le i \le k)$. We let $A = (a_{ij})_{0 \le i \le k, 0 \le j \le n}$ be a $(k+1) \times (n+1)$ matrix with non-zero entries. Then, we consider the following mapping $G_{(p_0,p_1,\ldots,p_k,A)} : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$:

$$G_{(p_0,p_1,\ldots,p_k,A)}(x) = \left(\sum_{j=0}^{n} a_{0j}(x_j - p_{0j})^2, \sum_{j=0}^{n} a_{1j}(x_j - p_{1j})^2, \ldots, \sum_{j=0}^{n} a_{kj}(x_j - p_{kj})^2 \right),$$

where $x = (x_0, x_1, \ldots, x_n)$ The mapping $G_{(p_0,p_1,\ldots,p_k,A)}$ is called a *generalized distance-squared mapping.* Notice that a distance-squared mapping $D_{(p_0,p_1,\ldots,p_k)}$ defined in Section 1 is the mapping $G_{(p_0,p_1,\ldots,p_k,A)}$ in the case that each entry of A is 1, and a Lorentzian distance-squared mapping $L_{(p_0,p_1,\ldots,p_k)}$ defined in Section 2 is the mapping $G_{(p_0,p_1,\ldots,p_k,A)}$ in the case of $a_{i0} = -1$ and $a_{ij} = 1$ if $j \neq 0$. Notice also that in these cases, the rank of A is 1. In the applications of singularity theory to differential geometry, generalized distance-squared mappings are a useful tool. Information on the contacts amongst the families of quadrics defined by the components of $G_{(p_0,p_1,\ldots,p_k,A)}$ is given by their singularities. Hence, it is natural to classify maps $G_{(p_0,p_1,\ldots,p_k,A)}$.

From now on in this section, we concentrate on the case of $\mathbb{R}^2$ into $\mathbb{R}^2$. It is interesting to observe that new $\mathcal{A}$-classes occur even in this case.

Definition 3.1. (1) Let $\Phi_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the following mapping:

$$\Phi_{n+1}(x_0, x_1, \ldots, x_n) = \left(x_0, x_1, \ldots, x_{n-1}, x_n^2\right).$$

When a map-germ $f : (\mathbb{R}^{n+1}, q) \to (\mathbb{R}^{n+1}, f(q))$ is $\mathcal{A}$-equivalent to $\Phi_{n+1} : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$, the point $q \in \mathbb{R}^{n+1}$ is said to be a *fold point of f.*

(2) Let $\Gamma_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the following mapping:

$$\Gamma_{n+1}(x_0, x_1, \ldots, x_n) = \left(x_0, x_1, \ldots, x_{n-1}, x_n^3 + x_0 x_n\right).$$

When a map-germ $f : (\mathbb{R}^{n+1}, q) \to (\mathbb{R}^{n+1}, f(q))$ is $\mathcal{A}$-equivalent to $\Gamma_{n+1} : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$, the point $q \in \mathbb{R}^{n+1}$ is said to be a *cusp point of f.*

It is known that both Φ_{n+1}, Γ_{n+1} are proper and stable mappings (for instance see [1]).

Recall the special cases of Theorem 1.3 and Corollary 2.1 as follows:

Proposition 3.1 (special cases of Theorem 1.3 and Corollary 2.1). *Let $p_0, p_1, \ldots, p_n$ be $(n+1)$-points of $\mathbb{R}^{n+1}$ such that the dimension of $\sum_{i=1}^{n} \mathbb{R}\overrightarrow{p_0p_i}$ is n. Then, the following hold:*

(1) *The distance-squared mapping $D_{(p_0,p_1,\ldots,p_n)} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is $\mathcal{A}$-equivalent to Φ_{n+1}.*

(2) *The Lorentzian distance-squared mapping $L_{(p_0,p_1,\ldots,p_n)} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is $\mathcal{A}$-equivalent to Φ_{n+1}.*

For generalized distance-squared mappings, it is natural to expect that for generic $p_0, p_1, \ldots, p_n$, $G_{(p_0,p_1,\ldots,p_n,A)}$ is proper and stable, and the rank of A is a complete invariant of $\mathcal{A}$-types. Thus, we reach the following conjecture.

Conjecture 3.1 ([8]). *Let A_k be an $(n+1)\times(n+1)$ matrix of rank k with non-zero entries $(1 \le k \le (n+1))$. Then, there exists a subset $\Sigma \subset (\mathbb{R}^{n+1})^{n+1}$ of Lebesgue measure zero such that for any $(p_0, p_1, \ldots, p_n) \in (\mathbb{R}^{n+1})^{n+1} - \Sigma$, the following hold:*

(1) *For any k $(1 \le k \le (n+1))$, the generalized distance-squared mapping $G_{(p_0,p_1,\ldots,p_n,A_k)}$ is proper and stable.*
(2) *For any two integers k_1, k_2 such that $1 \le k_1 < k_2 \le (n+1)$, $G_{(p_0,p_1,\ldots,p_n,A_{k_2})}$ is not $\mathcal{A}$-equivalent to $G_{(p_0,p_1,\ldots,p_n,A_{k_1})}$.*
(3) *Let B_k be an $(n+1)\times(n+1)$ matrix of rank k with non-zero entries $(1 \le k \le (n+1))$ and let $(q_0, q_1, \ldots, q_n)$ be in $(\mathbb{R}^{n+1})^{n+1} - \Sigma$. Then, $G_{(p_0,p_1,\ldots,p_n,A_k)}$ is $\mathcal{A}$-equivalent to $G_{(q_0,q_1,\ldots,q_n,B_k)}$ for any k.*

In [8], the affirmative answer to Conjecture 3.1 in the case $n = 1$ are given as follows:

Theorem 3.1 ([8]). *Let $((x_0, y_0), (x_1, y_1))$ be the standard coordinates of $(\mathbb{R}^2)^2$ and let Σ be the hypersurface in $(\mathbb{R}^2)^2$ defined by $(x_0 - x_1)(y_0 - y_1) = 0$. Let (p_0, p_1) be a point in $(\mathbb{R}^2)^2 - \Sigma$ and let A_k be a 2×2 matrix of rank k with non-zero entries (k=1, 2). Then, the following hold:*

(1) *The mapping $G_{(p_0,p_1,A_1)}$ is $\mathcal{A}$-equivalent to Φ_2.*
(2) *The mapping $G_{(p_0,p_1,A_2)}$ is proper and stable, and it is not $\mathcal{A}$-equivalent to $G_{(p_0,p_1,A_1)}$.*
(3) *Let B_2 be a 2×2 matrix of rank 2 with non-zero entries and let (q_0, q_1) be a point in $(\mathbb{R}^2)^2 - \Sigma$. Then, $G_{(p_0,p_1,A_2)}$ is $\mathcal{A}$-equivalent to $G_{(q_0,q_1,B_2)}$.*

There is another motivation for Theorem 3.1. Set $f_t(x) = x + tx^2$ $(t, x \in \mathbb{R})$. Then, the following two are easily observed.

(1) f_t is proper and stable for any $t \in \mathbb{R}$.
(2) f_t $(t \neq 0)$ is not $\mathcal{A}$-equivalent to f_0.

Notice that the mapping

$$F : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$$

defined by $F(t) = f_t$, is continuous nowhere. Here, $C^\infty(\mathbb{R}, \mathbb{R})$ is the topological space consisting of C^∞ mappings $\mathbb{R} \to \mathbb{R}$ endowed with the Whitney C^∞ topology. The one-parameter family f_t is a very simple example for preliminary phenomena of wall crossing phenomena. Theorem 3.1 gives an example for such phenomena in the case of the plane to the plane as follows. Let $M(2, \mathbb{R})$ be the set consisting of 2×2 matrices with real entries and let $P : \mathbb{R} \to (\mathbb{R}^2)^2 - \Sigma$ be a continuous mapping, where Σ is the set given in Theorem 3.1. Moreover, let $A : \mathbb{R} \to M(2, \mathbb{R})$ be a continuous mapping such that $\mathrm{rank}A(0) = 1$ and $\mathrm{rank}A(t) = 2$ if $t \neq 0$. Then, Theorem 3.1 implies the following interesting phenomenon:

(1) The mapping $G_{(P(s),A(t))}$ is proper and stable for any $(s, t) \in \mathbb{R}^2$.
(2) The mapping $G_{(P(s),A(t))}$ $(t \neq 0)$ is not $\mathcal{A}$-equivalent to $G_{(P(s),A(0))}$.

Notice that the mappings P and A induce the mapping

$$(\widetilde{P}, \widetilde{A}) : \mathbb{R}^2 \to C^\infty(\mathbb{R}^2, \mathbb{R}^2)$$

defined by $(\widetilde{P}, \widetilde{A})(s, t) = G_{(P(s),A(t))}$, where $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is the topological space consisting of C^∞ mappings $\mathbb{R}^2 \to \mathbb{R}^2$ endowed with the Whitney C^∞ topology.

Notice also that the mapping $(\widetilde{P}, \widetilde{A})$ is continuous nowhere. Therefore, $(\widetilde{P}, \widetilde{A})$ is useless for the proof of (3) of Theorem 3.1.

The keys for proving Theorem 3.1 are the following two propositions.

Proposition 3.2 ([8]). *Let A_2 be a 2×2 matrix of rank two with non-zero entries. Let p_0, p_1 be two points of $\mathbb{R}^2$ satisfying $(p_0, p_1) \in (\mathbb{R}^2)^2 - \Sigma$, where $\Sigma \subset (\mathbb{R}^2)^2$ is the hypersurface defined in Theorem 3.1. Then, the following hold:*

(1) *The singular set $S(G_{(p_0,p_1,A_2)})$ is a rectangular hyperbola.*
(2) *Any point of $S(G_{(p_0,p_1,A_2)})$ is a fold point except for one.*
(3) *The exceptional point given in (2) is a cusp point.*

Proposition 3.3 ([8]). *Let A_2 be a 2×2 matrix of rank two with non-zero entries. Let p_0, p_1 be two points of $\mathbb{R}^2$ satisfying $(p_0, p_1) \in (\mathbb{R}^2)^2 - \Sigma$, where $\Sigma \subset (\mathbb{R}^2)^2$ is the hypersurface defined in Theorem 3.1. Then, for any positive real numbers a, b ($a \neq b$), there exists a point $q = (q_0, q_1) \in \mathbb{R}^2$ such that $(q, (0, 0)) \in (\mathbb{R}^2)^2 - \Sigma$ and $G_{(p_0,p_1,A_2)}$ is $\mathcal{A}$-equivalent to $F_q : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$F_q(x, y) = \left((x - q_0)^2 + (y - q_1)^2, ax^2 + by^2\right).$$

All results in this section have been rigorously proved in [8].

4. Generalized distance-squared mappings of $\mathbb{R}^{n+1}$ into $\mathbb{R}^{2n+1}$

Let n be a positive integer. Generalized distance-squared mappings $G_{(p_0,p_1,\ldots,p_{2n},A)} : \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1}$ were firstly studied in [7]. In this section, we survey results for generalized distance-squared mappings of $\mathbb{R}^{n+1}$ into $\mathbb{R}^{2n+1}$ obtained in [7].

In the case of $n = 2k$, a partial classification result for $G_{(p_0,\ldots,p_k,A)}$ is known as follows. A *distance-squared mapping* $D_{(p_0,p_1,\ldots,p_k)}$ (resp., *Lorentzian distance-squared mappings* $L_{(p_0,p_1,\ldots,p_k)}$) is the mapping $G_{(p_0,p_1,\ldots,p_k,A)}$ in the case that each entry of A is 1 (resp., in the case of $a_{i0} = -1$ and $a_{ij} = 1$ if $j \neq 0$).

Proposition 4.1 ([5, 6]). *There exists a closed subset $\Sigma \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, \ldots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma$, both $D_{(p_0,p_1,\ldots,p_{2n})}$ and $L_{(p_0,p_1,\ldots,p_{2n})}$ are $\mathcal{A}$-equivalent to an inclusion.*

The following Theorem 4.1 generalizes Proposition 4.1.

Theorem 4.1 ([7]). *Let $A = (a_{ij})_{0 \leq i \leq 2n, 0 \leq j \leq n}$ be a $(2n+1) \times (n+1)$ matrix with non-zero entries. Then, the following two hold:*

(1) *Suppose that the rank of A is $n + 1$. Then, there exists a closed subset $\Sigma_A \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, \ldots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$, $G_{(p,A)}$ is $\mathcal{A}$-equivalent to the following mapping:*

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0^2, x_0x_1, \ldots, x_0x_n, x_1, \ldots, x_n).$$

(2) *Suppose that the rank of A is less than $n + 1$. Then, there exists a closed subset $\Sigma_A \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, \ldots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$, $G_{(p,A)}$ is $\mathcal{A}$-equivalent to an inclusion.*

The mapping given in the assertion (1) of Theorem 4.1 is defined in [24] and is called *the normal form of Whitney umbrella.* It is clear that the normal form of Whitney umbrella is not $\mathcal{A}$-equivalent to an inclusion. Moreover, it is easily seen that these two mappings are proper and stable by the characterization theorem of stable mappings given in [19]. Thus, Theorem 4.1 may be regarded as a result of Theorem 3.1 type. On the other hand, it is desirable to improve Theorem 4.1 so that the bad set Σ_A given in Theorem 4.1 does not depend on the given matrix A. However, contrary to the case of $\mathbb{R}^2$ into $\mathbb{R}^2$, in this case it is impossible to expect the existence of such a universal bad set as follows.

Theorem 4.2 ([7]). *There does not exist a closed subset $\Sigma \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, \ldots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma$ the following two hold.*

(1) *Suppose that the rank of A is $n+1$. Then, $G_{(p,A)}$ is $\mathcal{A}$-equivalent to the following mapping:*

$$(x_0, x_1, \ldots, x_n) \mapsto (x_0^2, x_0x_1, \ldots, x_0x_n, x_1, \ldots, x_n).$$

(2) *Suppose that the rank of A is less than $n+1$. Then, $G_{(p,A)}$ is $\mathcal{A}$-equivalent to an inclusion.*

All results in this section have been rigorously proved in [7].

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