数理解析研究所講究録
第 1948 巻 2015 年 108-123

Generalized sub-Riemannian manifold and abnormal extremals of generic driftless control-affine systems

Wataru Yukuno

Hokkaido University

Abstract

In order to study length minimizers on a generalized sub-Riemannian manifold, we consider the optimal control problem associated to the polynomial driftless control-affine systems on a finite dimensional smooth manifold with the Euclidean topology such that the formulation coincides with the ordinary normal Hamiltonian formalism in sub-Riemannian geometry in ordinary meaning. Then we have the following theorem: for generic polynomial driftless control-affine systems such that every degree of polynomial vector fields is sufficiently high and that the number of polynomial vector fields is two or more, any non-trivial abnormal extremal is strictly abnormal.

1 Introduction

A sub-Riemannian manifold is a triple such that a finite dimensional smooth manifold, a subbundle of the tangent bundle on its manifold, and a Riemannian metric on the fibres of its subbundle. In a sub-Riemannian manifold, there can exist length-minimizers not depending on the metric but depending only the subbundle. These geodesics never rise in Riemannian geometry, which are called abnormal geodesics.

By a rigorous application of the Pontryagin maximal principle of optimal control theory, every length-minimizer associated to a sub-Riemannian structure is either a normal extremal or an abnormal extremal. Note that, abnormal extremals do not depend on the metric but may be geodesic, and the two possibilities of normal and abnormal extremals are not mutually exclusive. It may happen that an extremal is both normal and abnormal. An abnormal extremal that is not the projection of a normal bi-extremal is called strictly abnormal.

Until recently, it was not clear whether strictly abnormal extremals that actually are length-minimizers can exist. Montgomery ([11]) and Kupka ([9]) seperately gave an example of length minimizer and strictly abnormal extremal for a two-dimensional subbundle of the tangent subbundle in $\mathbb{R}^3$. Moreover, since different authors had written false proofs of the fact that an abnormal extremal cannot be length-minimizer associated to a sub-Riemannian structure, Montgomery gave in [11] the list of several false proofs by different authors. After that, Liu and Sussmann constructed in [10], [14] more examples of strictly abnormal and length-minimizer by more simply proof.

Bellïaiche widely generalized in [2] a sub-Riemannian structure. The metric in a generalized sub-Riemannian structure is defined by using the system of vector fields on a finite dimensional manifold. Note that, the metric can be defined even if the system of vector fields is not always linearly independent everywhere on a finite dimensional manifold. If linearly independent everywhere, then the system generates a subbundle of the tangent bundle of the manifold and the metric in a generalized sub-Riemannian structure is the same as the sub-Riemannian metric in ordinary meaning.

Bonnard and Heutte showed in the preprint of [3], that for a generic linearly independent driftless control-affine system, any non-trivial abnormal extremal associated to the sub-Riemannian metric in ordinary meaning is always strictly abnormal. After that, Chitour, Jean and Trélat gave a more complete proof in the Appendix in [5] and had generalized the result of Bonnard and Heutte in [6].

In our paper, we consider the length-minimizer on a generalized sub-Riemannian structure by Bellïaiche as an analogy of a sub-Riemannian geometry and generalize the result of [5] to driftless control-affine systems including possibly linearly-dependent systems of vector fields.

Let X be an n dimensional smooth manifold M. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on an X-admissible control with the fixed initial point and the end point. Our formulation coincides with the ordinary normal Hamiltonian formalism in sub-Riemannian geometry (see §2.2).

To formulate the main theorem, we reintroduce the important concept of an X-strictly abnormal extremal (see Definition 3.1). An X-abnormal extremal $x : [0, T] \to M$ is called *strictly* if it is not the projection of a normal X-bi-extremal. Let $\mathrm{VF}(M)^m$ denote the set of systems of smooth vector fields $X = (X_1, \cdots, X_m)$ over M. We endow $\mathrm{VF}(M)^m$ with the Whitney smooth topology. Then, the following holds (see Theorem 3.2) :

Theorem A (Y. Chitour, F. Jean, and E. Trélat, [6]) *Suppose* $2 \leqq m \leqq n$. *Then, there exists an open dense subset* $G \subset \mathrm{VF}(M)^m$ *such that, if* $X \in G$ *and if an* X-*abnormal extremal* $x : [0, T] \to M$ *is non-trivial for the fixed initial point* $x_0 \in M$ *and end point* $x_1 \in M$, *then* x *is strictly abnormal.*

Theorem A is the special case of Proposition 2.19 in [6] and Thom's transversality theorem (for instance see [8]) is used in the proofs of the result. However, since the proof of Proposition 2.19 in [6] is hard to read and in particular, as an important part, the concrete construction of G seems to be not written and the codimension of its complement as a semi-algebraic set is not computed. Therefore, in this paper, we give more complete proof of Theorem A in this paper, independently from Proposition 2.19 in [6]. Note that the idea of the proofs of Theorem A are performed basically following the ideas from [5] and Thom's transversality theorem (for instance see [8]) is used in the proof.

Note that an abnormal trajectory is of corank one if and only if it admits a unique (up to scalar normalization) abnormal extremal lift. It is strictly abnormal and of corank one if and only if it admits a unique extremal lift which is abnormal.

On the other hand, we consider abnormal extremals for a generic polynomial driftless control-affine system: Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ denote an m-tuple of integers, and $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ denotes the product space of m-tuple systems of polynomial vector fields over $\mathbb{R}^n, Q = (Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ with the Euclidean topology.

For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i\, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \mathbb{R}^m$. Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on a Q-admissible control with the fixed initial point and end point. Then, the following theorem holds (see Theorem 4.1):

Main theorem *Suppose* $2 \leqq m \leqq n$ *and suppose that, an* m*-tuple of integers* $\boldsymbol{D} = (d_1, \cdots, d_m)$ *satisfies the inequality* : $\min\{d_1, d_2, \cdots, d_m\} \geqq 3n+2$. *Then, there exists an open dense semi-algebraic subset* $H \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ *such that, if* $Q \in H$, *if a* Q*-abnormal extremal* $x : [0, T] \to \mathbb{R}^n$ *is non-trivial for a fixed initial point* $x_0 \in M$ *and end point* $x_1 \in M$, *then* x *is strictly abnormal.*

The ideas of the proof of the main theorem are performed basically following the ideas from [5] and Tarski-Seidenberg theorem (for instance see [7]) is used in the proof of the main Theorem.

In §2, we recall a generalized sub-Riemannian geometry by Bellïche and consider the necessary condition for length-minimizer on a generalized sub-Riemannian manifold. We show Theorem A in 3 and the main theorem in §4 repectively.

2 Generalized sub-Riemannian geometry and length minimizers

In §2.1, we recall the generalized sub-Riemannian geometry by Belliche (see [2]). In §2.2, we consider the geodesic on generalized sub-Riemannian manifold.

2.1 Generalized sub-Riemannian geometry

Let $X = (X_1, \cdots, X_m)$ be a system of vector fields over an n-dimensional smooth manifold M. Given a point $x \in M$, let $L_x \subset T_xM$ be the vector space over $\mathbb{R}$ generated by $X_1(x), \cdots, X_m(x)$, namely $L_x = \langle X_1(x), \cdots, X_m(x)\rangle_{\mathbb{R}}$. Let $L \subset TM$ be the union of the sets L_x with $x \in M$. In particar, if the system of vector fields $X = (X_1, \cdots, X_m)$ is linearly independent, then $L \subset TM$ is a subbundle of TM, and $L \subset TM$ is called a *distribution* of TM.

Definition 2.1 Let $X = (X_1, \cdots, X_m)$ be a system of vector fields over an n-dimensional smooth manifold M. Let $L \subset TM$ be the union of the sets $L_x = \langle X_1(x), \cdots, X_m(x)\rangle_{\mathbb{R}}$ with $x \in M$. Then $g : L \to \mathbb{R}$ is called a *generalized sub-Riemannian metric* or a *generalized sub-Riemannian metric* if for $w = (x, v) \in L$,

$$g(w) = g(x, v) = \min\{(u_1)^2 + \cdots + (u_m)^2 \mid u_1X_1(x) + \cdots + u_rX_m(x) = v\},$$

where $w = (x, v)$ is canonical coordinates of $L \subset TM$, namely, $x \in M, v \in L_x$.

Note that, if $X = (X_1, \cdots, X_m)$ is linearly independent everywhere on M, then the system generates a distribution and the metric g in a generalized sub-Riemannian structure is the same as the sub-Riemannian metric in ordinary meaning.

Let $x : [0, T] \to M$ be an absolutely continuous curve. Then, $x : [0, T] \longrightarrow M$ is called *X-admissible* (or *L-admissible*) if for a.e. $t \in [0, T]$,

$$\dot{x}(t) \in L_{x(t)} = \langle X_1(x(t)), \cdots, X_m(x(t))\rangle_{\mathbb{R}}.$$

Then, *generalized Carnot-Carathéodry distance* $d_{CC} : M \times M \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined by the following:
for $p, q \in M$,

$$d_{CC}(p, q) = \inf\left\{ \int_{[0,T]} \sqrt{g(x(t), \dot{x}(t))}dt \;\middle|\; \begin{array}{l} x : [0,T] \longrightarrow M : X\text{-admissible} \\ x(0) = p,\ x(T) = q \end{array} \right\}.$$

Definition 2.2 An X-admissible curve $x : [0, T] \to M$ is called a *length-minimizer* if the length of x is equal to $d_{CC}(x(0), x(T))$:

$$d_{CC}(x(0), x(T)) = \int_0^T \sqrt{g(x(t), \dot{x}(t))}dt.$$

2.2 Necessary condition of length-minimizer

Let X be an n dimensional manifold. Let $x_0 \in M$ and $T > 0$. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

with the control parameter $u \in \mathbb{R}^m$. We denote by $\mathcal{U}_{x_0,x_1,T}$ the set of admissible X-controls from $[0,T]$ to $\mathbb{R}^m$ such that the corresponding trajectory to u has a fixed initial point $x_0 \in M$ and end point $x_1 \in M$.

We define an *energy* function $e : \mathbb{R}^m \to \mathbb{R}$ by

$$e(u) = \frac{1}{2}\sum_{i=1}^{m} u_i^2, \text{ for } u \in \mathbb{R}^m.$$

Consider the optimal control problem to minimize the *energy* functional $C_e : \mathcal{U}_{x_0,x_1,T} \to \mathbb{R}$

$$C_e(u) = \int_{[0,T]} e(u(t))dt = \int_{[0,T]} \frac{1}{2}\sum_{i=1}^{m} u_i(t)^2 dt, \text{ for } u \in \mathcal{U}_{x_0,x_1,T}.$$

It is known that the problem is equivalent to minimizing the *length* :

$$\ell(u) = \int_{[0,T]} \sqrt{\sum_{i=1}^{m} u_i(t)^2 dt}, \text{ for } u \in \mathcal{U}_{x_0,x_1,T}.$$

If $X_1, \cdots, X_m$ are linearly independent everywhere, then the optimal problem (X, e) is exactly to minimise the Carnot-Carathéodory distances in sub-Riemannian geometry (see [12]).

The Hamiltonian function $H = H_{(X,e)} : (T^*M \times \mathbb{R}^m) \times \mathbb{R} \to \mathbb{R}$ of the optimal control problem (X, e) is given by

$$H(x, p, u\,;\, p_0) = \sum_{i=1}^{m} \langle p, u_i X_i(x)\rangle + \frac{1}{2} p_0 (\sum_{i=1}^{m} u_i^2).$$

where $(x, p, u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ is the local coordinate of $T^*M \times \mathbb{R}^m$ with a canonical coordinate of (x, p) of T^*M. Then the constraint $\frac{\partial H}{\partial u} = 0$ is equivalent to the following

$$p_0\, u_j = -\langle p, X_j(x)\rangle, (1 \leqq j \leqq m).$$

For an X-normal extremal, we have $p_0 < 0$. Then we have

$$u_j = -\frac{1}{p_0}\langle p, X_j(x)\rangle, (1 \leqq j \leqq m).$$

Then

$$H = -\frac{1}{2p_0}\sum_{i=1}^{m}\langle p, X_i(x)\rangle^2.$$

From the linearity of Hamiltonian function on (p, p_0), we can normalize p_0, so that

$$H = \frac{1}{2}\sum_{i=1}^{m}\langle p, X_i(x)\rangle^2.$$

Thus our formulation coincides with the ordinary normal Hamiltonian formalism in sub-Riemannian geometry.

Therefore, by the Pontryagin maximum principle (see [13],[1]), the following property holds:

Proposition 2.3 *Let $x_0, x_1 \in M$. Let $u : [0,T] \to \mathbb{R}^m$ be an admissible X-controls and $x : [0,T] \to M$ be the corresponding trajectory with a fixed initial point $x_0 \in M$ and end point $x_1 \in M$. Then, if u is optimal, namely, x is length-minimizer, then there exists a pair (z, p_0) of an absolute continuous curve $z : [0,T] \to T^*M$ and a real number $p_0 \leqq 0$ such that, $x = \pi \circ z$, and that the following equations hold: for any local canonical coordinates $(x,p,u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ of $T^*M \times \Omega$ with a canonical coordinate of (x,p) of T^*M :*

$$\begin{cases} (1)\ \dot{x}_i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t), u(t)\,;\, p_0)\ (1 \leqq i \leqq n) & for\, a.e.\, t \in [0,T] \\ (2)\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}(x(t), p(t); u(t)\,;\, p_0)\ (1 \leqq i \leqq n) & for\, a.e.\, t \in [0,T] \\ (3)\ \frac{\partial H}{\partial u_j}(x(t), p(t); u(t)\,;\, p_0) = 0\ (1 \leqq j \leqq m) & for\, a.e.\, t \in [0,T] \\ (4)\ (p(t), p_0) \neq 0. & for\, every\, t \in [0,T] \end{cases},$$

with $H(x,p,u) = H_X(x,p,u) = \langle p, \sum_{i=1}^{m} u_i\, X_i(x)\rangle$.

A curve $z : [0,T] \to T^*M$ is called an *X-normal bi-extremal* (resp. an *X-abnormal bi-extremal*) if $p_0 < 0$ (resp. $p_0 = 0$). A curve $x : [0,T] \to T^*M$ is called an *X-normal extremal* (resp. an *X-abnormal extremal*) if it possesses an X-normal bi-extremal lift (resp. an X-abnormal bi-extremal lift).

Definition 2.4 *An X-abnormal extremal $x : [0,T] \to M$ is called strictly abnormal if it is not the projection of an X-normal bi-extremal.*

Note that it may happen that an X-extremal $x : [0,T] \to M$ is both normal and abnormal.

3 Abnormal extremals of generic driftless control-affine system in generalized sub-Riemannian geometry

We prove Theorem A (Theorem 3.2). In order to formulate the Theorem A, we recall the strictly abnormal extremal: Let X be an n dimensional manifold M. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over M. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x).$$

Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on an X-admissible control with the initial point $x_0 \in M$. The Hamiltonian function $H = H_{(X,e)} : (T^*M \times \mathbb{R}^m) \times \mathbb{R} \to \mathbb{R}$ of the optimal control problem (X, e) is given by

$$H(x, p, u\,; p_0) = \sum_{i=1}^{m} \langle p, u_i X_i(x) \rangle + \frac{1}{2} p_0 (\sum_{i=1}^{m} u_i^2).$$

where $(x, p, u) = (x_1, \cdots, x_n, p_1, \cdots, p_n, u_1, \cdots, u_m)$ is the local coordinate of $T^*M \times \mathbb{R}^m$ with a canonical coordinate of (x, p) of T^*M.

Recall the definition of a strictly abnormal extremal (see 2.4).

Definition 3.1 An X-abnormal extremal $x : [0, T] \to M$ is called *strictly* if it is not the projection of a normal X-bi-extremal.

Let $\mathrm{VF}(M)^m$ denote the set of systems of smooth vector fields $X = (X_1, \cdots, X_m)$ over M. We endow $\mathrm{VF}(M)^m$ with the Whitney smooth topology. Then, the following Theorem 3.2 holds:

Theorem 3.2 (Y. Chitour, F. Jean, and E. Trélat, [6]) *Suppose $2 \leqq m \leqq n$. Then, there exists an open dense subset $G \subset \mathrm{VF}(M)^m$ such that, if $X \in G$ and if an X-abnormal extremal $x : [0, T] \to M$ is non-trivial for the fixed initial point $x_0 \in M$ and end point $x_1 \in M$, then x is strictly abnormal.*

This Theorem 3.2 is the special case of Proposition 2.19 of [6]. However the proof of Proposition 2.19 is hard to read , because the construction of G is not written. We will improve the proof of Proposition 2.19 clearly.

Outline of proof Let $d \geqq 1$ be an integer. Put $N = d + 1$. We denote the space of all N-jets of vector fields $X \in \mathrm{VF}(M)$ by $J^N(\mathrm{VF}(M))$, and the fibre product over M of

m-tuple spaces of $J^N(\mathrm{VF}(M))$, by $J^N(\mathrm{VF}(M))^m$. Then, we will show Theorem 3.2 by the following procedures:
[Step1] Construct the "bad set" with respect to minimal order, $B_{sa}(d) \subset J^N(\mathrm{VF}(M))^m$.
[Step2] Show that, if $X \in \mathrm{VF}(M)^m$ satisfies the condition that any $x \in M$, $j_x X \notin B_{sa}(d)$ and if an X-abnormal extremal $x : [0,T] \to M$ is non-trivial, then x is of strictly abnormal.
[Step3] Compute the codimension of $B_{sa}(d)$ in $J^N(\mathrm{VF}(M))^m$.
[Step4] For $N > 3n+1\,(d > 3n)$, let G be the set of $X \in \mathrm{VF}(M)^m$ such that the jet $j_x^N X$ is not included in the closure of $B_{sa}(d)$ in $J^N(\mathrm{VF}(M))^m$. Then, show that, G is an open dense subset of $\mathrm{VF}(M)^m$ in the sense of Whitney smooth topology by Thom transversality theorem (for instance see [8]).

3.1 Construction of bad set with respect to strictly abnormal

Let $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$ and $x = \pi(z^{[n]}) = \pi(z^{[a]})$. For every muliti index I of $\{1,\cdots,m\}$, set

$$H_I^{[n]}(z^{[n]}, z^{[a]}) = H_I(z^{[n]}) \text{ and } H_I^{[a]}(z^{[n]}, z^{[a]}) = H_I(z^{[a]}).$$

and define inductively the following functions in $\mathcal{F}$, depending on $(z^{[n]}, z^{[a]})$

$$\begin{cases} \beta_{i,0} = H_i^{[a]} \\ \beta_{i,s+1} = \sum\limits_{j=1}^{m} H_j^{[n]} \mathcal{L}_{\overrightarrow{H_j}} \beta_{i,s}, \ (s = 1, 2, \cdots), \end{cases}$$

where $\mathcal{F}$ and $\mathcal{L}_{\overrightarrow{H_j}}$ are defined in before section.

Definition 3.3 Let d be a positive integer. Let $N = d+1$. For every integer $i\,(1 \leqq i \leqq m)$ and $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$, we define $\hat{B}(d, i, z^{[n]}, z^{[a]})$ by the set of $j_x^N X \in J^N(\mathrm{VF}(M))^m$ such that the following conditions hold:

1)$X_i(x) \neq 0$;

2)$H_i^{[n]}(z^{[n]}, z^{[a]}) \neq 0$;

3)$\beta_{i,s}(z^{[n]}, z^{[a]}) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

$\hat{B}((d, z^{[n]}, z^{[a]}) \subset J^N(\mathrm{VF}(M))^m$ is the union of $\hat{B}(d, i, z^{[n]}, z^{[a]})$ with $i\,(1 \leqq i \leqq m)$.

Definition 3.4 Let d be a positive integer. Let $N = d+1$. we define $\hat{B}_{sa}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M$ by

$$\hat{B}_{sa}(d) = \{(j_x^N X, z^{[n]}, z^{[a]}) | j_x^N X \in \hat{B}((d, z^{[n]}, z^{[a]})\}.$$

Definition 3.5 Let d be a positive integer. Let $N = d+1$. we define the bad set with respect to strictly abnormal $B_{sa}(d)$ by the canonical projection of $\hat{B}_{sa}(d)$ on $J^N(\mathrm{VF}(M))^m$.

3.2 The property of abnormal bi-extremals avoiding bad set with respect to strictly abnormal

Lemma 3.6 *Suppose that, $2 \leqq m \leqq n$. Let d be a positive integer and $N = d+1$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X \notin B_{sa}(d)$. Then, if an X-abnormal bi-extremal $x : [0,T] \to M$ is non-trivial, then x is of strictly abnormal.*

Proof: By contradiction, assume that there exists a nontrivial abnormal X-trajectory $x : [0,T] \to M$ with an X-abnormal control $u : [0,T] \to \mathbb{R}^m$ such that $x = \pi \circ z^{[n]} = \pi \circ z^{[a]}$, where $z^{[n]}$ is a normal X-bi-extremal lift of x, and $z^{[a]}$ is an X-abnormal bi-extremal lift of x.

For every multi-index $I \subset \{0, \cdots, m\}$ and $t \in [0,T]$, set

$$H_I(z^{[n]}(t)) = \langle z^{[n]}(t), X_I(x(t)) \rangle, H_I(z^{[a]}(t)) = \langle z^{[a]}(t), X_I(x(t)) \rangle.$$

After time differentiation, we have on $[0,T]$,

$$\begin{cases} \frac{d}{dt} H_I(z^{[n]}(t)) = \sum_{i=1}^m u_i(t) H_{Ii}(z^{[n]}(t)), \\ \frac{d}{dt} H_I(z^{[a]}(t)) = \sum_{i=1}^m u_i(t) H_{Ii}(z^{[a]}(t)). \end{cases}$$

By Pontryagin maximum principle,

$$\begin{cases} u_i(t) = H_i(z^{[n]}(t)) = H_i^{[n]}(z^{[n]}(t), z^{[a]}(t)), \\ H_i^{[a]}(z^{[n]}(t), z^{[a]}(t)) = H_i(z^{[a]}(t)) = 0 \cdots (\star) \end{cases} \quad \text{for every integer } i\,(1 \leqq i \leqq m), t \in [0,T]$$

Since $x : [0,T] \to \mathbb{R}^n$ is nontrivial, there exists an open subset $J \subset [0,T]$ and an integer $i_0\,(1 \leqq i_0 \leqq m)$ such that $u_{i_0}(t) X_{i_0}(x(t)) \neq 0$ on J. Therefore, $u_{i_0}(t) \neq 0$ and $X_{i_0}(x(t)) \neq 0$ on J. Since $u_{i_0}(t) = H_{i_0}(z^{[n]}(t))$,

$$H_{i_0}(z^{[n]}(t)) \neq 0.$$

on the other hand, by differentiating $(\star)$ with respect to $t \in [0,T]$,

$$\begin{aligned} 0 &= \frac{d}{dt} H_{i_0+1}^{[a]}(z^{[n]}(t), z^{[a]}(t)) \\ &= \sum_{j=1}^m u_j(t) H_{(i_0+1)j}^{[a]}(z^{[n]}(t), z^{[a]}(t)) \\ &= \sum_{j=1}^m H_j^{[n]}(z^{[n]}(t), z^{[a]}(t))\, H_{(i_0+1)j}^{[a]}(z^{[n]}(t), z^{[a]}(t)) \\ &= \beta_{i_0,1}(z^{[n]}(t), z^{[a]}(t)) \end{aligned}$$

For every $t \in [0,T]$, by induction,

$$\beta_{i_0,s}(z^{[n]}(t), z^{[a]}(t)) = 0.$$

for every $s\,(0 \leqq s \leqq d-1)$ and $t \in J$. Hence, $j_x^N X \in \hat{B}(d, i_0, z^{[n]}, z^{[a]})$ for $t \in J$, which contradicts the hypothesis. □

3.3 Codimension of bad set with respect to strictly abnormal

Lemma 3.7 $\mathrm{codim}(\overline{B_{sa}(d)}; J^N(VF(M))^m) \geqq d - 2n$.

Proof: We describe only the outline of the proof of Lemma 4.6. Let $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

Step1: Construct the typical fiber $G_{sa}(d)$ of $B_{sa}(d)$.

Typical fiber $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T_0^*\mathbb{R}^m \times T_0^*\mathbb{R}^m)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G_{sa}(d; T_0^*M \times_M T_0^*M)$ is defined by the set of $(Q, p^{[n]}, p^{[a]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $i\,(1 \leqq i \leqq m)$ such that $(Q, p^{[n]}, p^{[a]})$ satisfies the following conditions 1) to 4):

1)$Q_i(0)$ are linearly independent;

2)$H_i^{[n]}(z_0^{[n]}, z_0^{[a]}) \neq 0$;

3)$\beta_{i,s}(z_0^{[n]}, z_0^{[a]}) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step2: Construct the mapping $\phi_i : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then we define the mapping $\phi_i : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p_1, p_2) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_i(Q, p_1, p_2) = \beta_{i,s}(z_0^{[n]}, z_0^{[a]}),$$

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step3: Construct the open subset $V_i \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then V_i is the defined by the set of $(Q, p_1, p_2) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that (Q, p_1, p_2) satisfies the following condition:

$$H_i^{[n]}(z_0^{[n]}, z_0^{[a]}) \neq 0,$$

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$. Then, V_i is an open subset of $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Step4: $G_{sa}(d; T_0^*\mathbb{R}^m \times T_0^*\mathbb{R}^m)$ is the union of the kernel of restriction to V_i of the mapping ϕ_i with $i\,(1 \leqq i \leqq m)$.

Step5: Let Ω_0^i be the set of $Q \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_i \neq 0$. It is well-known that the local coordinate systems on Ω_0^i can be constructed (see Coordinate systems in [4],[5]) Then, for every integer $i\,(1 \leqq i \leqq m)$, the restriction to the intersection $V_i \cap \hat{V}$ of the mapping ϕ_i is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

Step6: $\mathrm{codim}(\overline{B_{sa}(d)}; J^N(\mathrm{VF}(M))^m) \geqq d - 2n$.

By step 4,5, $\mathrm{codim}(G_{sa}(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n); \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n) = d$. On the other hand, $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T_0^*\mathbb{R}^m \times T_0^*\mathbb{R}^m)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G_{sa}(d); \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 2n$. Since $G_{sa}(d)$ is the typical fiber of $B_{sa}(d)$,

$$\mathrm{codim}(B_{sa}(d); J^N(\mathrm{VF}(M))^m) = \mathrm{codim}(G_{sa}(d); \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 2n.$$

Since the dimensions of $B_{sa}(d)$ and $\overline{B_{sa}(d)}$ are equal,

$$\operatorname{codim}(\overline{B_{sa}(d)}; J^N(\mathrm{VF}(M))^m) \geqq d - 2n$$

□

3.4 Proof of Theorem A

Let $d > 3n$ be an integer. Let $N = d + 1(> 3n + 1)$. Let G be the set of $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X$ is not included in the closure of $B_{sa}(d)$ in $J^N(\mathrm{VF}(M))^m$:

$$G = \left\{ X \in \mathrm{VF}(M))^m \,\middle|\, j_x^N X \notin \overline{B_{sa}(d)} \text{ for any } x \in M. \right\}.$$

By Lemma 3.7,

$$\operatorname{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - 2n > n.$$

Then G is an open dense subset of $\mathrm{VF}(M)^m$ by using the transversality theorem (see [8]).

Let $X = (X_1, \cdots, X_m) \in G$. Then, for any $x \in M$, $j_x^N X \notin B_{sa}(d)$. Therefore, by using Lemma 3.6, if $x : [0, T] \to M$ is X-abnormal extremal then $x : [0, T] \to T^*M$ is strictly abnormal. □

4 Abnormal extremals on generic polynomial system in generalized sub-Riemannian geometry

We prove the main theorem (Theorem 4.1). In order to formulate the main theporem, recall that, $\boldsymbol{D} = (d_1, \cdots, d_m)$ denotes an m-tuple of integers, and $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ denotes the product space of m-tuples of polynomial vector fields over $\mathbb{R}^n$: $(Q_1, \cdots, Q_m)$, such that the degree of Q_i satisfiies $\deg Q_i \leqq d_i$ for every integer $i\,(1 \leqq i \leqq m)$, and we endow $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ with the Euclidean topology.

For $Q = (Q_1, \cdots, Q_m) \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$, consider the polynomial driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i\, Q_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \mathbb{R}^m$. Moreover, consider the optimal control problem associated to the driftless control-affine systems to minimize the energy functional

$$e(u) = \frac{1}{2} \sum_{i=1}^{m} u_i^2.$$

on Q-admissible controls with the fixed initial point $x_0 \in M$ and the fixed end point x_1 Then, the following holds:

Theorem 4.1 *Suppose $2 \leqq m \leqq n$ and suppose that, an m-tuple of integers $\boldsymbol{D} = (d_1, \cdots, d_m)$ satisfies the inequality* : $\min\{d_1, d_2, \cdots, d_m\} \geqq 3n+2$. *Then, there exists an open dense semi-algebraic subset $H \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that, if $Q \in H$, if a Q-abnormal extremal $x : [0,T] \to \mathbb{R}^n$ is non-trivial for a fixed initial point $x_0 \in M$ and end point $x_1 \in M$, then x is strictly abnormal.*

Outline of proof Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be an m-tuple Let $d = \min\{d_1, \cdots, d_m\}$. Then, we will show 4.1 by the following procedures:
[Step1] Construct the "bad set" with respect to minimal order, $B_{sa}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.
[Step2] Show that, if $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ satisfies the condition that any $x \in Q$, $(Q, x) \notin B_{sa}(\boldsymbol{D})$ and if a Q-abnormal extremal $x : [0,T] \to \mathbb{R}^n$ is non-trivial, then x is of strictly abnormal.
[Step3] Compute the codimension of $\pi(B_{sa}(\boldsymbol{D}))$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ by $\pi : \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$.
[Step4] For $d > 3n-1$, let H be the set of $Q \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that (Q, x) is not included in the closure of $\pi(B_{sa}(\boldsymbol{D}))$ in $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$. Then, show that, by Tarski-Seidenberg theorem, H is an open dense semi-algebraic subset of $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n)$ in the sense of Euclidean topology.

4.1 Construction of bad set

Let $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$ and $x = \pi(z^{[n]}) = \pi(z^{[a]})$. For every muliti index I of $\{1, \cdots, m\}$, set

$$H_I^{[n]}(z^{[n]}, z^{[a]}) = H_I(z^{[n]}) \text{ and } H_I^{[a]}(z^{[n]}, z^{[a]}) = H_I(z^{[a]}).$$

and define inductively the following functions in $\mathcal{F}$, depending on $(z^{[n]}, z^{[a]})$

$$\begin{cases} \beta_{i,0} = H_i^{[a]} \\ \beta_{i,s+1} = \sum\limits_{j=1}^{m} H_j^{[n]} \mathcal{L}_{\overrightarrow{H_j}} \beta_{i,s}, \ (s = 1, 2, \cdots), \end{cases}$$

where $\mathcal{F}$ and $\mathcal{L}_{\overrightarrow{H_j}}$ are defined in before section.

Definition 4.2 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $d = \min\{d_1, \cdots, d_m\} - 1$. For every integer $i\,(1 \leqq i \leqq m)$ and $(z^{[n]}, z^{[a]}) \in T^*M \times_M T^*M$, we define $\hat{B}(\boldsymbol{D}, i, z^{[n]}, z^{[a]})$ by the set of $(Q, x) \in \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$ such that the following conditions hold:

1)$X_i(x) \neq 0$;

2)$H_i^{[n]}(z^{[n]}, z^{[a]}) \neq 0$;

3)$\beta_{i,s}(z^{[n]}, z^{[a]}) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

$\hat{B}((\boldsymbol{D}, z^{[n]}, z^{[a]}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$ is the union of $\hat{B}(\boldsymbol{D}, i, z^{[n]}, z^{[a]})$ with $i\,(1 \leqq i \leqq m)$.

Definition 4.3 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i geqq2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $d = \min\{d_1, \cdots, d_m\} - 1$. we define $\hat{B}_{sa}(\boldsymbol{D}) \subset \mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times_M T^*M \times_M T^*M$ by

$$\hat{B}_{sa}(\boldsymbol{D}) = \{((Q, x), z^{[n]}, z^{[a]}) | (Q, x) \in \hat{B}((\boldsymbol{D}, z^{[n]}, z^{[a]})\}.$$

Definition 4.4 Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. we define the bad set with respect to strictly abnormal $B_{sa}(\boldsymbol{D})$ by the canonical projection of $\hat{B}_{sa}(d)$ on $\mathrm{VF}^{\boldsymbol{D}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$.

4.2 The property of abnormal bi-extremals avoiding bad set with respect to strictly abnormal

Lemma 4.5 *Suppose that, $2 \leqq m \leqq n$. Let $\boldsymbol{D} = (d_1, \cdots, d_m)$ be a pair of positive integers such that $d_i \geqq 2$ for every integer $i\,(1 \leqq i \leqq m)$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $(Q, x) \notin B_{sa}(\boldsymbol{D})$. Then, if $x : [0, T] \to \mathbb{R}^n$ is a Q-abnormal bi-extremal, then x is of strictly abnormal.*

Proof: By contradiction, assume that there exists a nontrivial abnormal X-trajectory $x : [0, T] \to M$ with an Q-abnormal control $u : [0, T] \to \mathbb{R}^m$ such that $x = \pi \circ z^{[n]} = \pi \circ z^{[a]}$, where $z^{[n]}$ is a normal X-bi-extremal lift of x, and $z^{[a]}$ is a Q-abnormal bi-extremal lift of x.

For every multi-index $I \subset \{0, \cdots, m\}$ and $t \in [0, T]$, set

$$H_I(z^{[n]}(t)) = \langle z^{[n]}(t), X_I(x(t))\rangle, H_I(z^{[a]}(t)) = \langle z^{[a]}(t), X_I(x(t))\rangle.$$

After time differentiation, we have on $[0, T]$,

$$\begin{cases} \frac{d}{dt} H_I(z^{[n]}(t)) = \sum\limits_{i=1}^{m} u_i(t) H_{Ii}(z^{[n]}(t)), \\ \frac{d}{dt} H_I(z^{[a]}(t)) = \sum\limits_{i=1}^{m} u_i(t) H_{Ii}(z^{[a]}(t)). \end{cases}$$

By Pontryagin maximum principle,

$$\begin{cases} u_i(t) = H_i(z^{[n]}(t)) = H_i^{[n]}(z^{[n]}(t), z^{[a]}(t)), \\ H_i^{[a]}(z^{[n]}(t), z^{[a]}(t)) = H_i(z^{[a]}(t)) = 0 \cdots (\star) \end{cases} \quad \text{for every integer } i\,(1 \leqq i \leqq m), t \in [0, T]$$

Since $x : [0, T] \to \mathbb{R}^n$ is nontrivial, there exists an open subset $J \subset [0, T]$ and an integer $i_0\,(1 \leqq i_0 \leqq m)$ such that $u_{i_0}(t) X_{i_0}(x(t)) \neq 0$ on J. Therefore, $u_{i_0}(t) \neq 0$ and $X_{i_0}(x(t)) \neq 0$ on J. Since $u_{i_0}(t) = H_{i_0}(z^{[n]}(t))$,

$$H_{i_0}(z^{[n]}(t)) \neq 0.$$

on the other hand, by differentiating $(\star)$ with respect to $t \in [0,T]$,

$$\begin{aligned} 0 &= \tfrac{d}{dt} H^{[a]}_{i_0+1}(z^{[n]}(t), z^{[a]}(t)) \\ &= \sum_{j=1}^{m} u_j(t) H^{[a]}_{(i_0+1)j}(z^{[n]}(t), z^{[a]}(t)) \\ &= \sum_{j=1}^{m} H^{[n]}_j(z^{[n]}(t), z^{[a]}(t))\, H^{[a]}_{(i_0+1)j}(z^{[n]}(t), z^{[a]}(t)) \\ &= \beta_{i_0,1}(z^{[n]}(t), z^{[a]}(t)) \end{aligned}$$

For every $t \in [0,T]$, by induction,

$$\beta_{i_0,s}(z^{[n]}(t), z^{[a]}(t)) = 0.$$

for every $s\,(0 \leqq s \leqq d-1)$ and $t \in J$. Hence, $j^N_x X \in \hat{B}(d, i_0, z^{[n]}, z^{[a]})$ for $t \in J$, which contradicts the hypothesis. □

4.3 Codimension of bad set with respect to strictly abnormal

Lemma 4.6 $\operatorname{codim}(\overline{\pi(B_{sa}(\boldsymbol{D}))};\ VF^{\boldsymbol{D}}_{poly}(\mathbb{R}^n)) \geqq d - 3n.$

Proof: We describe only the outline of the proof of Lemma 4.6. Let $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

Step1: Construct the typical fiber $G_{sa}(d)$ of $B_{sa}(d)$.

Typical fiber $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d; T^*_0\mathbb{R}^m \times T^*_0\mathbb{R}^m)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G_{sa}(d; T^*_0 M \times_M T^*_0 M)$ is defined by the set of $(Q, p^{[n]}, p^{[a]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $i\,(1 \leqq i \leqq m)$ such that $(Q, p^{[n]}, p^{[a]})$ satisfies the following conditions 1) to 4):

1)$Q_i(0)$ are linearly independent;

2)$H^{[n]}_i(z^{[n]}_0, z^{[a]}_0) \neq 0$;

3)$\beta_{i,s}(z^{[n]}_0, z^{[a]}_0) = 0$ for every integer $s\,(0 \leqq s \leqq d-1)$.

where $z^{[n]}_0, z^{[a]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0,p_1), (0,p_2)$.

Step2: Construct the mapping $\phi_i : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then we define the mapping $\phi_i : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p_1, p_2) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n$,

$$\phi_i(Q, p_1, p_2) = \beta_{i,s}(z^{[n]}_0, z^{[a]}_0),$$

where $z^{[n]}_0, z^{[a]}_0$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0,p_1), (0,p_2)$.

Step3: Construct the open subset $V_i \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$

Let $i(1 \leqq i \leqq m)$ be a positive integer. Then V_i is the defined by the set of $(Q, p_1, p_2) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that (Q, p_1, p_2) satisfies the following condition:

$$H^{[n]}_i(z^{[n]}_0, z^{[a]}_0) \neq 0,$$

where $z_0^{[n]}, z_0^{[a]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0,p_1),(0,p_2)$. Then, V_i is an open subset of $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n$.

Step4: $G_{sa}(d;T_0^*\mathbb{R}^m\times T_0^*\mathbb{R}^m)$ is the union of the kernel of restriction to V_i of the mapping ϕ_i with $i\,(1\leqq i\leqq m)$.

Step5: Let Ω_0^i be the set of $Q\in\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ such that $Q_i\neq 0$. It is well-known that the local coordinate systems on Ω_0^i can be constructed (see Coordinate systems in [4],[5]) Then, for every integer $i\,(1\leqq i\leqq m)$, the restriction to the intersection $V_i\cap\hat{V}$ of the mapping ϕ_i is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0\times\mathbb{R}^n$.

Step6: $\mathrm{codim}(\overline{B_{sa}(d)};J^N(\mathrm{VF}(M))^m)\geqq d-2n$.

By step 4,5, $\mathrm{codim}(G_{sa}(d;T_0^*\mathbb{R}^n\times T_0^*\mathbb{R}^n);\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n)=d$. On the other hand, $G_{sa}(d)$ of $B_{sa}(d)$ is the canonical projection of $G_{sa}(d;T_0^*\mathbb{R}^m\times T_0^*\mathbb{R}^m)$ by $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n\to\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G_{sa}(d);\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n))\geqq d-2n$. Since $G_{sa}(d)$ is the typical fiber of $B_{sa}(d)$,

$$\mathrm{codim}(B_{sa}(d);J^N(\mathrm{VF}(M))^m)=\mathrm{codim}(G_{sa}(d);\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n))\geqq d-2n.$$

Since the dimensions of $B_{sa}(d)$ and $\overline{B_{sa}(d)}$ are equal,

$$\begin{aligned}\mathrm{codim}(\overline{\pi(B_{sa}(d))};\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)) &= \mathrm{codim}(\overline{\pi(B_{sa}(d))};\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n))\\ &\geqq \mathrm{codim}(\overline{B_{sa}(d))};\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\times\mathbb{R}^n)-n\\ &= \mathrm{codim}(\overline{G_{sa}(d))};\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n))-n\\ &\geqq d-3n\end{aligned}$$

□

4.4 Proof of main theorem

It is well-known that for every positive integer $K\geqq 1$, if $B\subset\mathbb{R}^K$ is semi-algebraic, then the complement of B in $\mathbb{R}^K$ is dense if and only if $\dim(\mathbb{R}^K,B)>0$. In particular, the complement of the closure of B in $\mathbb{R}^K$, $\mathbb{R}^K\backslash\overline{B}$ is open dense subset of $\mathbb{R}^K$.

Let $d>3n$ be an integer such that $\min\{D_1,D_2,\cdots,D_m\}=d+1(>3n+1)$. Let H be the set of $Q\in\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$ such that for any (Q,x) is not included in the closure of $\pi(B_{sa}(\boldsymbol{D}))$ by $\pi:\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\times\mathbb{R}^n\to\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$:

$$H=\left\{Q\in\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)\;\middle|\;(Q,x)\notin\overline{\pi(B_{sa}(\boldsymbol{D}))}\text{ for any }x\in M.\right\}.$$

By Lemma 4.6,

$$\mathrm{codim}(\overline{B_{mo}(d)},\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n))\geqq d-3n>0.$$

Then $\overline{\pi(B_{sa}(\boldsymbol{D}))}$ is an open dense semi-algebraic subset of $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{D}}(\mathbb{R}^n)$.

Let $Q=(Q_1,\cdots,Q_m)\in H$. Then, for any $x\in M$, $(Q,x)\notin B_{sa}(\boldsymbol{D})$. Therefore, by using Lemma 4.5, if $x:[0,T]\to\mathbb{R}^n$ is Q-abnormal extremal, then $x:[0,T]\to T^*\mathbb{R}^n$ is strictly abnormal. □

References

[1] A.A.Agrachev, Y.L.Sachkov, *Control Theory from Geometric Viewpoint*, Encyclopaedia of Mathematical Sciences 87, Springer-Verlag, (2004).

[2] A.Bellïaiche, *The tangent space in sub-Riemannian geometry*, in Sub-Riemannian Geometry, Progress in Math. 144, Birkhäuser, pp. 1-78, (1996).

[3] B. Bonnard, H. Heutte, *La propriété de stricte anormalité est générique*, Preprint de l'Univ. de Bourgogne, no. 77, (1995).

[4] B.Bonnard, I.Kupka, *Generic properties of singular trajectories*, Ann. Inst. H. Poincare Anal. Non Lineaire,14 (1997), pp 167-186.

[5] Y.Chitour, F.Jean, and E.Trélat, *Genericity results for singular curves*, J. Differential Geom., 73 (2006), pp. 45-73.

[6] Y.Chitour, F.Jean, and E.Trélat, *Singular trajectories of control-affine systems*, SIAM J.Control Optim.,Vol.47, No.2 (2008), pp.1078-1095.

[7] M. Coste, *An introduction to semialgebraic geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000)

[8] M.Golubitsky, V.Guillemin, *Stable Mappings and Their Singularities*, Graduate Texts in Mathematics no. 14. Springer-Verlag, Berlin (1974).

[9] I.Kupka, *Abnormal extremals,* preprint, 1992.

[10] W.S.Liu, H.J.Sussmann, *Shortest paths for sub-Riemannian metrics of rank two distributions*, Memoirs AMS 118, no.564, 1995.

[11] R.Montgomery, *Abnormal Minimizers*, SIAM J.Control and Optim. Vol.32, No.6 (1994), pp.1605-1620.

[12] R.Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Mathematical Surveys and Monographs, 91, Amer. Math. Soc. (2002).

[13] L.S.Pontryagin, V.G.Boltyanskii, R.V.Gamkrelidze, E.F.Mishchenko, *The Mathematical Theory of Optimal Processes*, Pergamon Press (1964).

[14] H.J.Sussmann, *A cornucopia of four-dimensional abnormal sub-Riemannian minimizers*, in Sub-Riemannian Geometry, Progress in Math. 144, Birkhäuser, pp. 1-78, (1996).