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CO-ORIENTABLE SINGULAR FIBERS OF STABLE MAPS OF 3-MANIFOLDS WITH BOUNDARY INTO SURFACES

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ABSTRACT. In [12] the authors classified the singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces, and computed the cohomology groups of the associated universal complex of singular fibers with coefficients in $\mathbb{Z}_2$. In this paper, we classify the co-orientable singular fibers of such stable maps and compute the cohomology groups of the associated universal complex with coefficients in $\mathbb{Z}$.

1. INTRODUCTION

Let M and N be smooth manifolds, where M may possibly have boundary, while N has no boundary. For a C^∞ map $f\colon M \to N$ and a point $q \in N$, we call the map germ along the pre-image $f^{-1}(q)$

$$f\colon (M, f^{-1}(q)) \to (N, q)$$

the *fiber* over q, adopting the terminology introduced in [6]. Furthermore, if a point $q \in N$ is a regular value of both f and $f|_{\partial M}$, then we call the fiber over q a *regular fiber*; otherwise, a *singular fiber*.

We define natural equivalence relations among fibers as follows. Let $f_i\colon M_i \to N_i$ be C^∞ maps with $q_i \in N_i$, $i = 0, 1$. The fibers over q_0 and q_1 are said to be C^∞ *equivalent* (or C^0 *equivalent*) if for some open neighborhoods U_i of q_i in N_i, there exist diffeomorphisms (resp. homeomorphisms) $\Phi\colon f_0^{-1}(U_0) \to f_1^{-1}(U_1)$ and $\varphi\colon U_0 \to U_1$ with $\varphi(q_0) = q_1$ that make the following diagram commutative:

$$\begin{array}{ccc} (f_0^{-1}(U_0), f_0^{-1}(q_0)) & \xrightarrow{\Phi} & (f_1^{-1}(U_1), f_1^{-1}(q_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\varphi} & (U_1, q_1). \end{array}$$

Denote by $C^\infty(M, N)$ the set of all C^∞ maps $M \to N$ equipped with the Whitney C^∞ topology. A C^∞ map $f\colon M \to N$ is *stable* (or more precisely, C^∞ *stable*) if there exists a neighborhood $N(f)$ of f in $C^\infty(M, N)$ such that every map $g \in N(f)$ is C^∞

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right-left equivalent to f [3], where two maps f and $g \in C^\infty(M,N)$ are C^∞ *right-left equivalent* if there exist diffeomorphisms $\Psi\colon M \to M$ and $\psi\colon N \to N$ such that $f \circ \Psi = \psi \circ g$.

The notion of singular fibers of C^∞ maps between manifolds without boundary was first introduced in [6], where the singular fibers of stable maps $M \to N$ with $(\dim M, \dim N) = (2,1), (3,2)$ and $(4,3)$ were classified up to the above equivalences. Later, singular fibers of stable maps of manifolds without boundary were studied in [6, 7, 10, 11, 15, 16, 17], especially in connection with cobordisms. The first author [6] established the theory of universal complex of singular fibers of C^∞ maps as an analogy of the Vassiliev complex for map germs [5, 14]. Its cohomology groups can be used for getting certain cobordism invariants of singular maps. For example, in [6], cobordism invariants for stable Morse functions on closed surfaces were obtained, and the authors obtained a complete cobordism invariant for closed oriented 4-dimensional manifolds in terms of singular fibers in [10].

In our previous paper [12], we studied singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary. By computing the cohomology groups of the associated universal complex with coefficients in $\mathbb{Z}_2$, we obtained a non-trivial $\mathbb{Z}_2$-valued cobordism invariant for admissible stable Morse functions on compact surfaces with boundary. Here, a map on a manifold with boundary is said to be *admissible* if it is submersive near the boundary. Two admissible stable Morse functions on surfaces with boundary are said to be $\mathcal{AS}_{\mathrm{pr}}$*-cobordant* if there exists a cobordism between them which is admissible (for details, see Definition 3.4). The above cohomology class, in fact, gives rise to a non-trivial $\mathcal{AS}_{\mathrm{pr}}$-cobordism invariant for admissible stable Morse functions on surfaces with boundary [12].

In this paper, we classify the co-orientable singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary. Then, we compute the cohomology groups with integer coefficients of the associated universal complex. It will turn out that the cohomology groups are non-trivial, but that the associated $\mathbb{Z}$-valued $\mathcal{AS}_{\mathrm{pr}}$-cobordism invariants are, unfortunately, trivial. This supports the conjecture that the $\mathcal{AS}_{\mathrm{pr}}$-cobordism group of admissible stable Morse functions on compact surfaces with boundary is isomorphic to $\mathbb{Z}_2$ (see Conjecture 6.4).

The paper is organized as follows. In §2, we recall the classification of fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary, with respect to the C^∞ equivalence. In §3, we briefly recall the theory of universal complex of singular fibers of a certain class of C^∞ maps. In §4, we formulate the fibers which are (strongly) co-orientable and construct the associated universal complex with integer coefficients. In §5, some specific classes of stable maps and admissible stable maps together with certain equivalence relations among their fibers are introduced. In §6, we compute the cohomology groups of the universal complex of co-orientable singular fibers of proper (admissible) C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary and discuss their associated cobordism invariants. In §7, we consider fibers of maps of orientable manifolds and compute the cohomology groups of the corresponding subcomplexes. Our results support the conjecture that the orientable $\mathcal{AS}_{\mathrm{pr}}$-cobordism group of admissible stable Morse functions on compact orientable surfaces with boundary is isomorphic to $\mathbb{Z}_2$ (see Conjecture 7.6).

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified. For a map $f\colon M \to N$ between manifolds, we denote by $S(f)$ the set of points in M where the differential of f does not have maximal rank $\min\{\dim M, \dim N\}$. For a space X, id_X denotes the identity map of X.

2. Classification of singular fibers

In this section, we recall the classification of singular fibers of proper C^∞ stable maps of 3-dimensional manifolds with boundary into surfaces without boundary.

Let us first recall the following characterization of C^∞ stable maps. In the following, for a 3-manifold M with boundary and a point $p \in \partial M$, we use local coordinates (x, y, z) around p such that $\operatorname{Int} M$ and ∂M correspond to the sets $\{z > 0\}$ and $\{z = 0\}$, respectively.

Proposition 2.1 (Shibata [13], Martins and Nabarro [4]). *Let M be a 3-manifold possibly with boundary and N a surface without boundary. A proper C^∞ map $f\colon M \to N$ is C^∞ stable if and only if it satisfies the following conditions.*

(1) (*Local conditions*)

(1a) *For $p \in \operatorname{Int} M$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y,z) \mapsto \begin{cases} (x,y), & p\text{: regular point,} \\ (x, y^2+z^2), & p\text{: definite fold point,} \\ (x, y^2-z^2), & p\text{: indefinite fold point,} \\ (x, y^3+xy-z^2), & p\text{: cusp point.} \end{cases}$$

(1b) *For $p \in \partial M \setminus S(f)$, the germ of f at p is right-left equivalent to one of the following:*

$$(x,y,z) \mapsto \begin{cases} (x,y), & p\text{: regular point of } f|_{\partial M}, \\ (x, y^2+z), & p\text{: boundary definite fold point,} \\ (x, y^2-z), & p\text{: boundary indefinite fold point,} \\ (x, y^3+xy+z), & p\text{: boundary cusp point.} \end{cases}$$

(1c) *For $p \in \partial M \cap S(f)$, the germ of f at p is right-left equivalent to the map germ*

$$(x,y,z) \mapsto (x, y^2 + xz \pm z^2).$$

(2) (*Global conditions*) *For each $q \in f(S(f)) \cup f(S(f|_{\partial M}))$, the multi-germ*

$$(f|_{S(f)\cup S(f|_{\partial M})}, f^{-1}(q) \cap (S(f) \cup S(f|_{\partial M})))$$

is right-left equivalent to one of the eight multi-germs as depicted in Figure 1, where the ordinary curves correspond to the singular value set $f(S(f))$ and the dotted curves to $f(S(f|_{\partial M}))$: (1) corresponds to a single fold point, (4) corresponds to a single boundary fold point, (3), (6) and (7) represent normal crossings of two immersion germs, each of which corresponds to a fold point or a boundary fold point, (2) corresponds to a cusp point, (5) corresponds to a boundary cusp point, and (8) corresponds to a single point in $\partial M \cap S(f)$.

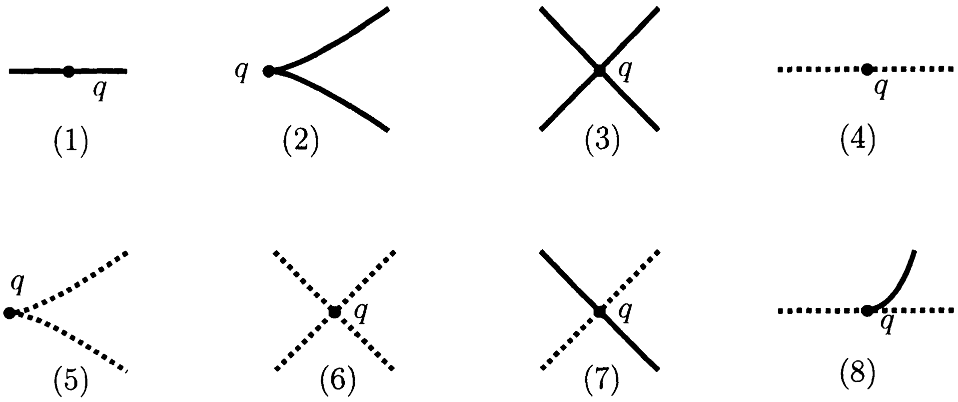

FIGURE 1. Multi-germs of $f|_{S(f)\cup S(f|_{\partial M})}$

Note that if a C^∞ map $f\colon\ M \to N$ is C^∞ stable, then so is $f|_{\partial M} : \partial M \to N$.

In the following, a map germ at a point on the boundary right-left equivalent to the normal form

$$(x,y,z) \mapsto (x, y^2 + xz + z^2) \quad \text{or} \quad (x,y,z) \mapsto (x, y^2 + xz - z^2)$$

is called a *definite* $\Sigma^{2,0}_{1,0}$ *point* or an *indefinite* $\Sigma^{2,0}_{1,0}$ *point*, respectively.

Definition 2.2. Let us consider finitely many fibers of smooth maps with all the dimensions of the sources and the targets being the same. Then, their *disjoint union* is the fiber corresponding to the single map defined on the disjoint union of the sources, where the target spaces are all identified to a single small open disk. This depends on such identifications; however, in the following, we can take "generic identifications" in such a way that the resulting map is C^∞ stable and is unique up to C^∞ equivalence, as long as the identifications are generic.

By using the method developed in [6], the authors [12] have obtained the following classification of singular fibers.

Theorem 2.3. *Let $f\colon\ M \to N$ be a proper C^∞ stable map of a 3-manifold M with boundary into a surface N without boundary. Then, every fiber of f is equivalent to the disjoint union of one of the fibers in the following list, a finite number of copies of a fiber of the trivial circle bundle, and a finite number of copies of a fiber of the trivial I-bundle, where $I = [-1,1]$:*

(1) *fibers as depicted in Figure 2, i.e.* $\widetilde{\mathrm{b0}}^0$, $\widetilde{\mathrm{b0}}^1$, *and* $\widetilde{\mathrm{bI}}^\mu$ *with* $2 \le \mu \le 10$,
(2) *disconnected fibers* $\widetilde{\mathrm{bII}}^{\mu,\nu}$ *with* $2 \le \mu \le \nu \le 10$, *where* $\widetilde{\mathrm{bII}}^{\mu,\nu}$ *means the disjoint union of* $\widetilde{\mathrm{bI}}^\mu$ *and* $\widetilde{\mathrm{bI}}^\nu$,
(3) *the connected fibers as depicted in Figure* 3, *i.e.* $\widetilde{\mathrm{bII}}^\mu$ *with* $11 \le \mu \le 39$, $\widetilde{\mathrm{bII}}^a$, $\widetilde{\mathrm{bII}}^b$, $\widetilde{\mathrm{bII}}^c$, $\widetilde{\mathrm{bII}}^d$, $\widetilde{\mathrm{bII}}^e$ *and* $\widetilde{\mathrm{bII}}^f$.

In Figures 2 and 3, κ denotes the *codimension* of the set of points in the target N whose corresponding fibers are C^∞ equivalent to the relevant one (see [6] for details). Furthermore, the symbols $\widetilde{\mathrm{b0}}^*$, $\widetilde{\mathrm{bI}}^*$, and $\widetilde{\mathrm{bII}}^*$ mean the names of the corresponding singular fibers. Note that we have named the fibers so that each "connected fiber"

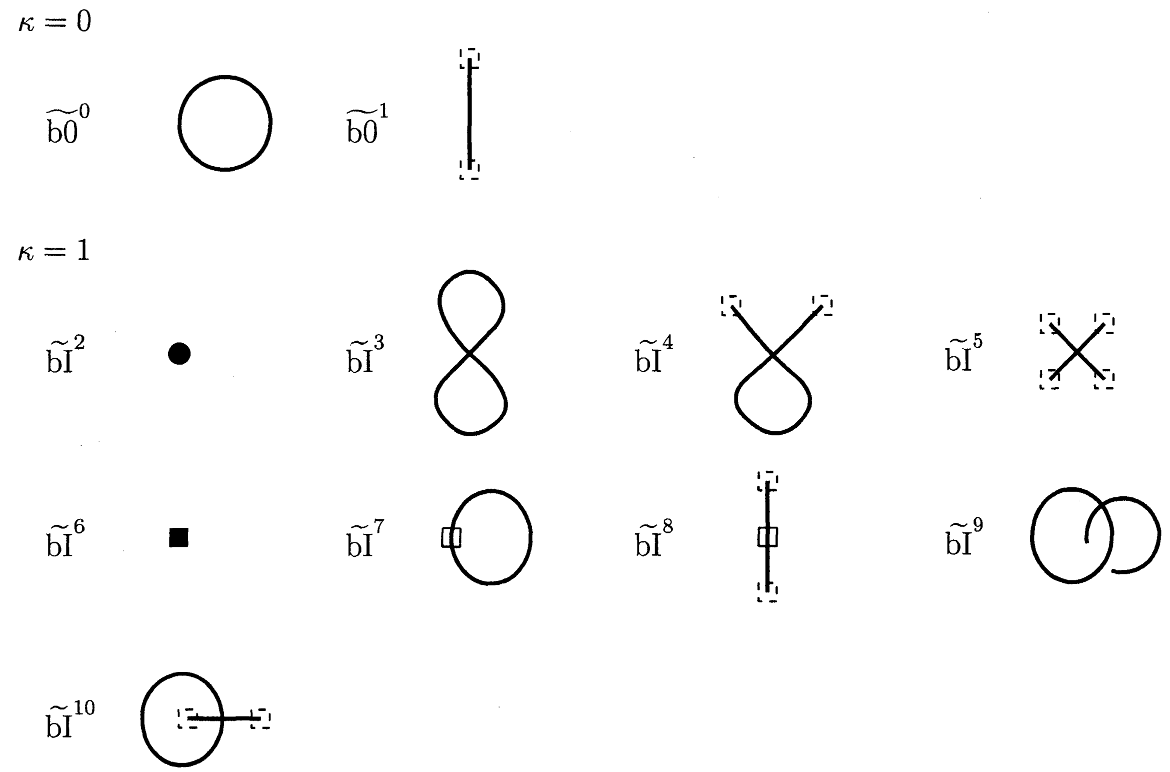

FIGURE 2. List of the fibers of proper C^∞ stable maps of 3-manifolds with boundary into surfaces without boundary; 1

has its own number or letter, and a "disconnected fiber" has the name consisting of the numbers of its "connected components", with the regular fiber components being ignored. Note also that each figure represents a map germ along the corresponding fiber and not just the inverse image of a point.

Remark 2.4. Our classification result of singular fibers of stable maps of compact 3-dimensional manifolds with boundary into surfaces has already been applied in computer science for visual data analysis. More precisely, it helps to visualize characteristic features of certain multi-field data (see Figure 4). For details, see [8, 9].

Remark 2.5. The list of the C^∞ equivalence classes of singular fibers of proper stable Morse functions on surfaces with boundary can be obtained in a similar fashion. The result corresponds to those appearing in Figure 2 with $\kappa = 0, 1$. In fact, it is not difficult to show that the suspensions of the fibers of such functions in the sense of Definition 3.1 coincide with those appearing in the figure. However, in the following, by abuse of notation, we use the symbols in Figure 2 with $\kappa = 0, 1$ for the fibers of stable Morse functions as well.

3. UNIVERSAL COMPLEX

In this section we briefly recall the theory of universal complex of singular fibers. As to the general theory, the reader is referred to [6, Part II].

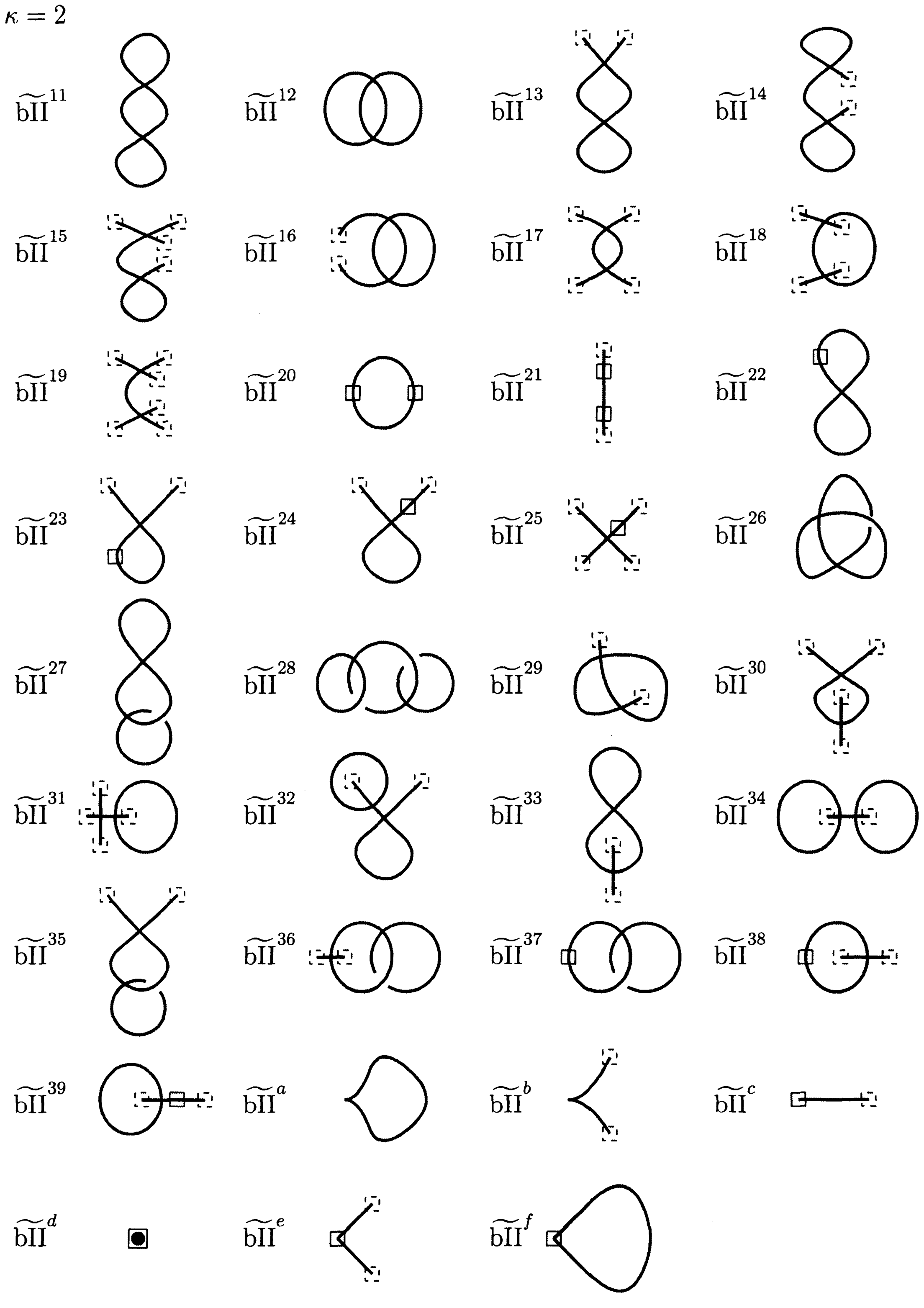

FIGURE 3. List of the fibers of proper C^∞ stable maps of 3-manifolds with boundary into surfaces without boundary; 2

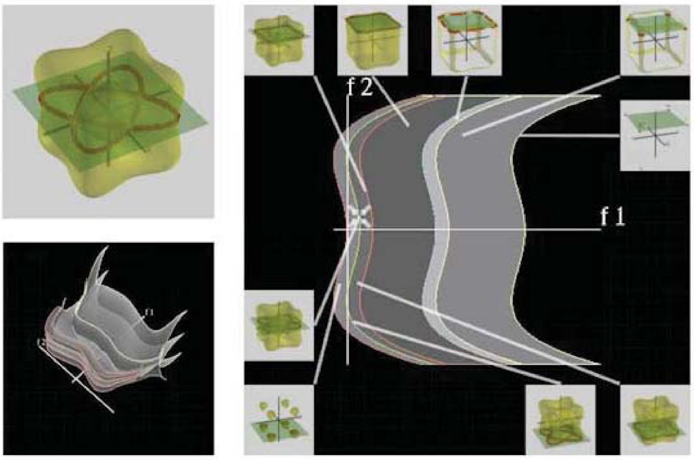

FIGURE 4. User interface for visualizing singular fibers

Throughout this section, M is an m-dimensional manifold which is not necessarily closed, and N is an n-dimensional manifold without boundary. The *codimension* of a smooth map $f\colon M \to N$ is defined to be the difference $\dim N - \dim M \in \mathbb{Z}$. To construct the universal complex of singular fibers of C^∞ maps, we fix an integer $\ell \in \mathbb{Z}$ for the codimension of the maps, and consider the following:

(1) a set τ of fibers of proper Thom maps[1] of codimension ℓ, and
(2) an equivalence relation ρ among the fibers in τ.

We further assume that the set τ and the relation ρ satisfy the following conditions.

(a) The set τ is closed under adjacency relation, i.e. if a fiber is in τ, then so are all nearby fibers.
(b) Each ρ-class is a union of C^0 equivalence classes.
(c) Let $f_i\colon M_i \to N_i$ be proper Thom maps and $q_i \in N_i$, $i = 0, 1$. Suppose that the fibers over q_0 and q_1 are in τ and that they are equivalent with respect to ρ. Then, there exist open neighborhoods U_i of q_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi\colon U_0 \to U_1$ satisfying $\varphi(q_0) = q_1$ and $\varphi(U_0 \cap \mathcal{F}(f_0)) = U_1 \cap \mathcal{F}(f_1)$, for each ρ-class $\mathcal{F}$.

In particular, the above conditions imply that for each proper Thom map $f\colon M \to N$ and each ρ-class $\mathcal{F}$, $\mathcal{F}(f)$ is a C^0 submanifold of constant codimension unless it is not empty, where

$$\mathcal{F}(f) = \{q \in N \,|\, \text{the fiber over } q \text{ belongs to the class } \mathcal{F}\}.$$

The *codimension* of $\mathcal{F}$ is defined to be that of $\mathcal{F}(f)$ in N, and is denoted by $\kappa(\mathcal{F})$.

We call a proper Thom map $f\colon M \to N$ a *τ-map* if all of its fibers are in τ.

For each $\kappa \in \mathbb{Z}$, let $C^\kappa(\tau, \rho)$ be the formal $\mathbb{Z}_2$-vector space spanned by the ρ-classes of codimension κ in τ. If there are no such fibers, then we set $C^\kappa(\tau, \rho) = 0$.

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[1] A *Thom map* is a C^∞ stratified map with respect to Whitney regular stratifications such that it is a submersion on each stratum and satisfies certain regularity conditions. See, for example, [2] for more details.

We can naturally define a $\mathbb{Z}_2$-linear map $\delta_\kappa\colon C^\kappa(\tau,\rho) \to C^{\kappa+1}(\tau,\rho)$ by using adjacencies to obtain the cochain complex

$$\mathcal{C}(\tau,\rho) = (C^\kappa(\tau,\rho), \delta_\kappa)_\kappa,$$

which is called the *universal complex of singular fibers for τ-maps with respect to the equivalence relation ρ*, and denote its cohomology group of dimension κ by $H^\kappa(\tau,\rho)$.

In order to formulate cobordisms and their invariants associated with cohomology classes of the universal complex we need the following notion of suspension of a Thom map.

Definition 3.1. For a proper Thom map $f\colon M \to N$, let us consider the product map

$$f \times \mathrm{id}_{\mathbb{R}}\colon M \times \mathbb{R} \to N \times \mathbb{R}.$$

We call $f \times \mathrm{id}_{\mathbb{R}}$ and the fiber of $f \times \mathrm{id}_{\mathbb{R}}$ over a point $(q,0) \in N \times \mathbb{R}$ the *suspension* of f and the *suspension* of the fiber of f over $q \in N$, respectively.

Let τ be a set of fibers for proper Thom maps of codimension ℓ as above. For a dimension pair (m,n) with $n - m = \ell$, let $\tau(m,n)$ denote the set of fibers in τ for proper Thom maps of manifolds of dimension m into those of dimension n. The equivalence relation on $\tau(m,n)$ induced by ρ is denoted by $\rho_{m,n}$.

In addition to conditions (a)–(c) above, we assume the following two additional conditions.

- (d) The suspension of each fiber in $\tau(m,n)$ belongs also to $\tau(m+1,n+1)$.
- (e) If two fibers in $\tau(m,n)$ are equivalent with respect to $\rho_{m,n}$, then their suspensions are also equivalent with respect to $\rho_{m+1,n+1}$.

For each $\kappa \in \mathbb{Z}$, the suspension induces the $\mathbb{Z}_2$-linear map

$$s_\kappa\colon C^\kappa(\tau(m+1,n+1),\rho_{m+1,n+1}) \to C^\kappa(\tau(m,n),\rho_{m,n}),$$

where for a $\rho_{m+1,n+1}$-class $\mathcal{F}$, $s_\kappa(\mathcal{F})$ is the sum of all $\rho_{m,n}$-classes of codimension κ whose suspensions are in $\mathcal{F}$. Note that s_κ is well-defined. We can show that the system of $\mathbb{Z}_2$-linear maps $\{s_\kappa\}$ defines a cochain map

$$\{s_\kappa\}\colon \mathcal{C}(\tau(m+1,n+1),\rho_{m+1,n+1}) \to \mathcal{C}(\tau(m,n),\rho_{m,n}).$$

Definition 3.2. Let

$$c = \sum_{\kappa(\mathcal{F})=\kappa} n_{\mathcal{F}}\mathcal{F}$$

be a κ-dimensional cochain of $\mathcal{C}(\tau,\rho)$ with $n_{\mathcal{F}} \in \mathbb{Z}_2$. For a τ-map $f\colon M \to N$, $c(f)$ denotes the set of points $q \in N$ such that the fiber over q is in $\mathcal{F}$ with $n_{\mathcal{F}} \neq 0$. If c is a cocycle, then we can show that $c(f)$ is a $\mathbb{Z}_2$-cycle of closed support of codimension κ in N.

It is known that if two cocycles c and c' are are cohomologous, then the $\mathbb{Z}_2$-cycles $c(f)$ and $c'(f)$ are $\mathbb{Z}_2$-homologous in N for each τ-map $f\colon M \to N$.

Definition 3.3. Let $[c]$ be a κ-dimensional cohomology class of $\mathcal{C}(\tau,\rho)$ represented by a cochain c. For a τ-map $f\colon M \to N$, define $[c(f)] \in H^c_{n-\kappa}(N;\mathbb{Z}_2)$ to be the $\mathbb{Z}_2$-homology class represented by the $\mathbb{Z}_2$-cycle $c(f)$ of closed support. This is well-defined by virtue of the above remark.

Furthermore, define the $\mathbb{Z}_2$-linear map $\varphi_f \colon H^\kappa(\tau, \rho) \to H^\kappa(N; \mathbb{Z}_2)$ by $\varphi_f([c]) = [c(f)]^*$, where $[c(f)]^* \in H^\kappa(N; \mathbb{Z}_2)$ is the Poincaré dual of $[c(f)] \in H^c_{n-\kappa}(N; \mathbb{Z}_2)$.

Let us introduce a geometric equivalence relation for τ-maps.

Definition 3.4. Two τ-maps $f_i \colon M_i \to N$, $i = 0, 1$, of compact manifolds with boundary into a manifold without boundary are *τ-cobordant* if there exist a compact manifold X with corners and a τ-map $F \colon X \to N \times [0, 1]$ that satisfy the following conditions:

(1) $\partial X = M_0 \cup Q \cup M_1$, where M_0, M_1 and Q are codimension 0 smooth submanifolds of ∂X, $M_0 \cap M_1 = \emptyset$, and $\partial Q = (M_0 \cap Q) \cup (M_1 \cap Q)$,
(2) X has corners along ∂Q,
(3) $F|_{M_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$ and $F|_{M_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$, where $M_0 \times [0, \varepsilon)$ and $M_1 \times (1 - \varepsilon, 1]$ denote the collar neighborhoods (with corners) of M_0 and M_1 in X, respectively.

In this case, we call the map F a *τ-cobordism* between f_0 and f_1.

Note that the τ-cobordism relation is an equivalence relation among the τ-maps into a fixed manifold N. For a manifold N, we denote by $Cob_\tau(N)$ the set of all equivalence classes of τ-maps of compact manifolds into N with respect to the τ-cobordism.

It is known that, for each cohomology class $[c] \in H^\kappa(\tau(m + 1, n + 1), \rho_{m+1,n+1})$ and an n-dimensional manifold N without boundary, we obtain the map

$$I_{[c]} \colon Cob_\tau(N) \to H^\kappa(N; \mathbb{Z}_2)$$

defined by $I_{[c]}(f) = \varphi_f([s_{\kappa*}c])$, which does not depend on the choice of a representative f of a given τ-cobordism class, where $s_{\kappa*} \colon H^\kappa(\tau(m + 1, n + 1), \rho_{m+1,n+1}) \to H^\kappa(\tau(m, n), \rho_{m,n})$ is the homomorphism induced by the suspension. In other words, each element in

$$H^\kappa(\tau(m + 1, n + 1), \rho_{m+1,n+1})$$

induces a τ-cobordism invariant for τ-maps into an n-dimensional manifold N through suspension.

4. Co-orientable fibers

In this section, we consider fibers that are (strongly) co-orientable in the sense of [6, Definition 10.5]. In the following, τ is a certain set of fibers and ρ is an equivalence relation for fibers in τ as in the previous section.

Definition 4.1. A ρ-equivalence class $\widetilde{\mathcal{F}}$ of fibers of τ-maps is *strongly co-orientable* if for a τ-map $M \to N$ and a point $q \in N$ whose fiber belongs to $\widetilde{\mathcal{F}}$, every local homeomorphism around $q \in N$ preserving the adjacent equivalence classes necessarily preserves the orientation of the normal direction to the submanifold corresponding to $\widetilde{\mathcal{F}}$. For a ρ-class of co-orientable fibers, it is *co-oriented* if the orientation of the above normal direction is given.

For each $\kappa \in \mathbb{Z}$, let $CO^\kappa(\tau, \rho)$ be the formal free $\mathbb{Z}$-module spanned by the ρ-classes of co-oriented fibers of codimension κ in τ. Here, a ρ-class with the reversed co-orientation is identified with the (-1)-times the original class. If there are no such fibers, then we set $CO^\kappa(\tau, \rho) = 0$.

We can naturally define a $\mathbb{Z}$-module homomorphism $\delta_\kappa\colon CO^\kappa(\tau,\rho) \to CO^{\kappa+1}(\tau,\rho)$ by using adjacencies and co-orientations to obtain the cochain complex

$$\mathcal{CO}(\tau,\rho) = (CO^\kappa(\tau,\rho), \delta_\kappa)_\kappa,$$

which is called the *universal complex of co-orientable fibers for τ-maps with respect to the equivalence relation ρ*, and we denote its cohomology group of dimension κ by $H^\kappa(\mathcal{CO}(\tau,\rho);\mathbb{Z})$.

As in the previous section, for each $\kappa \in \mathbb{Z}$, the suspension induces the $\mathbb{Z}$-module homomorphism

$$s_\kappa\colon CO^\kappa(\tau(m+1,n+1),\rho_{m+1,n+1}) \to CO^\kappa(\tau(m,n),\rho_{m,n}).$$

We can also show that the system of homomorphisms $\{s_\kappa\}$ defines a cochain map

$$\{s_\kappa\}\colon \mathcal{CO}(\tau(m+1,n+1),\rho_{m+1,n+1}) \to \mathcal{CO}(\tau(m,n),\rho_{m,n}).$$

5. Universal complex for stable maps of n-dimensional manifolds with boundary into $(n-1)$-dimensional manifolds

In order to discuss more specific cases, for a positive integer n, let $b\mathcal{S}_{\mathrm{pr}}(n,n-1)$ be the set of fibers for proper C^0 stable Thom maps of n-dimensional manifolds with boundary into $(n-1)$-dimensional manifolds without boundary. We put

$$b\mathcal{S}_{\mathrm{pr}} = \bigcup_{n=1}^{\infty} b\mathcal{S}_{\mathrm{pr}}(n,n-1).$$

Remark 5.1. If the dimension pair $(n,n-1)$ is in the nice range, then C^0 stable maps are C^∞ stable (for example, see [1]), and consequently they are Thom maps. For example, this is the case if $n \le 8$.

Furthermore, let $\rho_{n,n-1}(2)$ be the *C^0 equivalence relation modulo two regular fibers* for fibers in $b\mathcal{S}_{\mathrm{pr}}(n,n-1)$: i.e., two fibers in $b\mathcal{S}_{\mathrm{pr}}(n,n-1)$ are $\rho_{n,n-1}(2)$-equivalent if they become C^0 equivalent after we add some regular fibers to each of them with the numbers of added components having the same parity. Note that, under this equivalence, for $n = 2,3$, we do not distinguish the fibers of types $\widetilde{\mathrm{b0}}^0$ with $\widetilde{\mathrm{b0}}^1$. Therefore, in the following, we denote both of them by $\widetilde{\mathrm{b0}}$.

We denote by $\rho(2)$ the equivalence relation on $b\mathcal{S}_{\mathrm{pr}}$ which is induced by $\rho_{n,n-1}(2)$, $n \ge 1$. Note that the set $b\mathcal{S}_{\mathrm{pr}}$ and the equivalence relation $\rho(2)$ satisfy conditions (a)–(e) described above.

For a C^0 equivalence class $\widetilde{\mathcal{F}}$ of singular fibers, denote by $\widetilde{\mathcal{F}}_o$ (or $\widetilde{\mathcal{F}}_e$) the equivalence class with respect to $\rho_{n,n-1}(2)$ which consists of singular fibers of type $\widetilde{\mathcal{F}}$ with an odd number (resp. even number) of regular fiber components. For $n = 2,3$, we denote by $\widetilde{\mathrm{b0}}_o$ and $\widetilde{\mathrm{b0}}_e$ the equivalence class with respect to $\rho_{n,n-1}(2)$ which consist exclusively of an odd (resp. even) number of regular fiber components.

We will also consider a certain restricted class of stable maps. For a positive integer n, let $\mathcal{AS}_{\mathrm{pr}}(n,n-1)$ be the set of fibers for proper admissible C^0 stable Thom maps of n-dimensional manifolds with boundary into $(n-1)$-dimensional manifolds without boundary, where a C^0 stable map $f\colon M \to N$ of a manifold with boundary into a manifold without boundary is *admissible* if it is a submersion on a neighborhood of ∂M. In particular, a stable map $f\colon M \to N$ of a 3-dimensional manifold with boundary into a surface without boundary is admissible if and only if it has no definite $\Sigma^{2,0}_{1,0}$ points nor indefinite $\Sigma^{2,0}_{1,0}$ points.

Note that stable Morse functions on compact surfaces and their suspensions are always admissible.

Furthermore, set

$$\mathcal{AS}_{\mathrm{pr}} = \bigcup_{n=1}^{\infty} \mathcal{AS}_{\mathrm{pr}}(n, n-1).$$

Note that the above set together with the equivalence relation induced by $\rho(2)$, which we still denote by $\rho(2)$ by abuse of notation, satisfy conditions (a)–(e) mentioned before.

6. Universal complex of co-orientable singular fibers of stable maps

By analyzing the adjacencies of fibers, we easily get the following.

Lemma 6.1. *Those equivalence classes with respect to $\rho_{3,2}(2)$ which are strongly co-orientable are $\widetilde{\mathrm{b0}}_*$, $\widetilde{\mathrm{bI}}^2_*$, $\widetilde{\mathrm{bI}}^3_*$, $\widetilde{\mathrm{bI}}^4_*$, $\widetilde{\mathrm{bI}}^6_*$, $\widetilde{\mathrm{bI}}^8_*$, $\widetilde{\mathrm{bII}}^{2,3}_*$, $\widetilde{\mathrm{bII}}^{2,4}_*$, $\widetilde{\mathrm{bII}}^{2,6}_*$, $\widetilde{\mathrm{bII}}^{2,8}_*$, $\widetilde{\mathrm{bII}}^{3,4}_*$, $\widetilde{\mathrm{bII}}^{3,6}_*$, $\widetilde{\mathrm{bII}}^{3,8}_*$, $\widetilde{\mathrm{bII}}^{4,6}_*$, $\widetilde{\mathrm{bII}}^{4,8}_*$, $\widetilde{\mathrm{bII}}^{6,8}_*$, $\widetilde{\mathrm{bII}}^{13}_*$, $\widetilde{\mathrm{bII}}^{22}_*$, $\widetilde{\mathrm{bII}}^{23}_*$, $\widetilde{\mathrm{bII}}^{24}_*$, $\widetilde{\mathrm{bII}}^{a}_*$, $\widetilde{\mathrm{bII}}^{b}_*$, $\widetilde{\mathrm{bII}}^{c}_*$, $\widetilde{\mathrm{bII}}^{d}_*$, $\widetilde{\mathrm{bII}}^{e}_*$, $\widetilde{\mathrm{bII}}^{f}_*$, where $*$ denotes o or e. The other equivalence classes are not strongly co-orientable.*

Let us fix a co-orientation for each co-orientable equivalence class of codimension one in such a way that the co-orientation points from $\widetilde{\mathrm{b0}}_e$ to $\widetilde{\mathrm{b0}}_o$. For each co-orientable equivalence class of codimension two, we fix a co-orientation as in Figures 5, 6 and 7. (For those equivalence classes which do not appear in the figures, we fix their co-orientations in a similar fashion.)

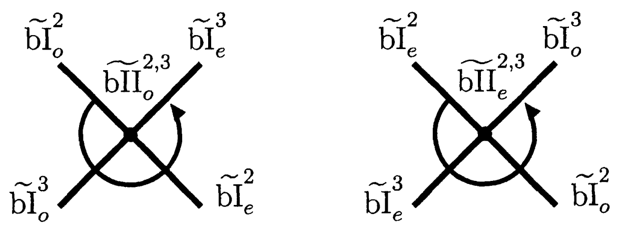

FIGURE 5. Co-orientations for $\widetilde{\mathrm{bII}}^{2,3}_*$

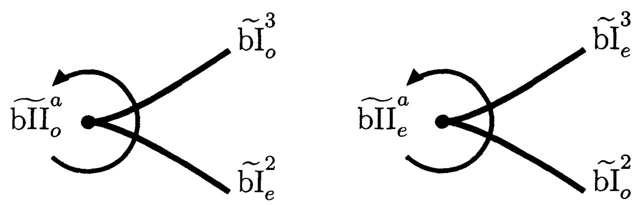

FIGURE 6. Co-orientations for $\widetilde{\mathrm{bII}}^a_*$

Then, for the universal complex $\mathcal{CO}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ of co-orientable fibers, the coboundary homomorphism is given by the following formulas. Recall that the cochain

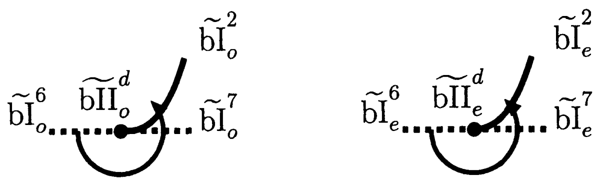

FIGURE 7. Co-orientations for $\widetilde{\mathrm{bII}}^{d}_{*}$

complex is defined over $\mathbb{Z}$.

$$
\begin{aligned}
\delta_0(\widetilde{\mathrm{b0}}_o) &= \widetilde{\mathrm{bI}}^{2}_{o} + \widetilde{\mathrm{bI}}^{2}_{e} + \widetilde{\mathrm{bI}}^{3}_{o} + \widetilde{\mathrm{bI}}^{3}_{e} + \widetilde{\mathrm{bI}}^{4}_{o} + \widetilde{\mathrm{bI}}^{4}_{e} + \widetilde{\mathrm{bI}}^{6}_{o} + \widetilde{\mathrm{bI}}^{6}_{e} + \widetilde{\mathrm{bI}}^{8}_{o} + \widetilde{\mathrm{bI}}^{8}_{e},\\
\delta_0(\widetilde{\mathrm{b0}}_e) &= -\widetilde{\mathrm{bI}}^{2}_{o} - \widetilde{\mathrm{bI}}^{2}_{e} - \widetilde{\mathrm{bI}}^{3}_{o} - \widetilde{\mathrm{bI}}^{3}_{e} - \widetilde{\mathrm{bI}}^{4}_{o} - \widetilde{\mathrm{bI}}^{4}_{e} - \widetilde{\mathrm{bI}}^{6}_{o} - \widetilde{\mathrm{bI}}^{6}_{e} - \widetilde{\mathrm{bI}}^{8}_{o} - \widetilde{\mathrm{bI}}^{8}_{e},\\
\delta_1(\widetilde{\mathrm{bI}}^{2}_{o}) &= \widetilde{\mathrm{bII}}^{2,3}_{o} - \widetilde{\mathrm{bII}}^{2,3}_{e} + \widetilde{\mathrm{bII}}^{2,4}_{o} - \widetilde{\mathrm{bII}}^{2,4}_{e} + \widetilde{\mathrm{bII}}^{2,6}_{o} - \widetilde{\mathrm{bII}}^{2,6}_{e} + \widetilde{\mathrm{bII}}^{2,8}_{o}\\
&\quad - \widetilde{\mathrm{bII}}^{2,8}_{e} - \widetilde{\mathrm{bII}}^{a}_{e} - \widetilde{\mathrm{bII}}^{b}_{e} + \widetilde{\mathrm{bII}}^{d}_{o},\\
\delta_1(\widetilde{\mathrm{bI}}^{2}_{e}) &= \widetilde{\mathrm{bII}}^{2,3}_{o} - \widetilde{\mathrm{bII}}^{2,3}_{e} + \widetilde{\mathrm{bII}}^{2,4}_{o} - \widetilde{\mathrm{bII}}^{2,4}_{e} + \widetilde{\mathrm{bII}}^{2,6}_{o} - \widetilde{\mathrm{bII}}^{2,6}_{e} + \widetilde{\mathrm{bII}}^{2,8}_{o}\\
&\quad - \widetilde{\mathrm{bII}}^{2,8}_{e} + \widetilde{\mathrm{bII}}^{a}_{o} + \widetilde{\mathrm{bII}}^{b}_{o} - \widetilde{\mathrm{bII}}^{d}_{e},\\
\delta_1(\widetilde{\mathrm{bI}}^{3}_{o}) &= -\widetilde{\mathrm{bII}}^{2,3}_{o} + \widetilde{\mathrm{bII}}^{2,3}_{e} + \widetilde{\mathrm{bII}}^{3,4}_{o} - \widetilde{\mathrm{bII}}^{3,4}_{e} + \widetilde{\mathrm{bII}}^{3,6}_{o} - \widetilde{\mathrm{bII}}^{3,6}_{e} + \widetilde{\mathrm{bII}}^{3,8}_{o}\\
&\quad - \widetilde{\mathrm{bII}}^{3,8}_{e} + \widetilde{\mathrm{bII}}^{13}_{e} + \widetilde{\mathrm{bII}}^{22}_{o} - \widetilde{\mathrm{bII}}^{a}_{o},\\
\delta_1(\widetilde{\mathrm{bI}}^{3}_{e}) &= -\widetilde{\mathrm{bII}}^{2,3}_{o} + \widetilde{\mathrm{bII}}^{2,3}_{e} + \widetilde{\mathrm{bII}}^{3,4}_{o} - \widetilde{\mathrm{bII}}^{3,4}_{e} + \widetilde{\mathrm{bII}}^{3,6}_{o} - \widetilde{\mathrm{bII}}^{3,6}_{e} + \widetilde{\mathrm{bII}}^{3,8}_{o}\\
&\quad - \widetilde{\mathrm{bII}}^{3,8}_{e} - \widetilde{\mathrm{bII}}^{13}_{o} - \widetilde{\mathrm{bII}}^{22}_{e} + \widetilde{\mathrm{bII}}^{a}_{e},\\
\delta_1(\widetilde{\mathrm{bI}}^{4}_{o}) &= -\widetilde{\mathrm{bII}}^{2,4}_{o} + \widetilde{\mathrm{bII}}^{2,4}_{e} - \widetilde{\mathrm{bII}}^{3,4}_{o} + \widetilde{\mathrm{bII}}^{3,4}_{e} + \widetilde{\mathrm{bII}}^{4,6}_{o} - \widetilde{\mathrm{bII}}^{4,6}_{e} + \widetilde{\mathrm{bII}}^{4,8}_{o} - \widetilde{\mathrm{bII}}^{4,8}_{e}\\
&\quad - \widetilde{\mathrm{bII}}^{13}_{e} - \widetilde{\mathrm{bII}}^{22}_{o} + \widetilde{\mathrm{bII}}^{23}_{o} + \widetilde{\mathrm{bII}}^{24}_{o} - \widetilde{\mathrm{bII}}^{24}_{e} - \widetilde{\mathrm{bII}}^{b}_{o} - \widetilde{\mathrm{bII}}^{f}_{o}\\
\delta_1(\widetilde{\mathrm{bI}}^{4}_{e}) &= -\widetilde{\mathrm{bII}}^{2,4}_{o} + \widetilde{\mathrm{bII}}^{2,4}_{e} - \widetilde{\mathrm{bII}}^{3,4}_{o} + \widetilde{\mathrm{bII}}^{3,4}_{e} + \widetilde{\mathrm{bII}}^{4,6}_{o} - \widetilde{\mathrm{bII}}^{4,6}_{e} + \widetilde{\mathrm{bII}}^{4,8}_{o} - \widetilde{\mathrm{bII}}^{4,8}_{e}\\
&\quad + \widetilde{\mathrm{bII}}^{13}_{o} + \widetilde{\mathrm{bII}}^{22}_{e} - \widetilde{\mathrm{bII}}^{23}_{e} + \widetilde{\mathrm{bII}}^{24}_{o} - \widetilde{\mathrm{bII}}^{24}_{e} + \widetilde{\mathrm{bII}}^{b}_{e} + \widetilde{\mathrm{bII}}^{f}_{e},\\
\delta_1(\widetilde{\mathrm{bI}}^{6}_{o}) &= -\widetilde{\mathrm{bII}}^{2,6}_{o} + \widetilde{\mathrm{bII}}^{2,6}_{e} - \widetilde{\mathrm{bII}}^{3,6}_{o} + \widetilde{\mathrm{bII}}^{3,6}_{e} - \widetilde{\mathrm{bII}}^{4,6}_{o} + \widetilde{\mathrm{bII}}^{4,6}_{e} + \widetilde{\mathrm{bII}}^{6,8}_{o}\\
&\quad - \widetilde{\mathrm{bII}}^{6,8}_{e} - \widetilde{\mathrm{bII}}^{c}_{e} - \widetilde{\mathrm{bII}}^{d}_{o} - \widetilde{\mathrm{bII}}^{e}_{e} - \widetilde{\mathrm{bII}}^{f}_{e},\\
\delta_1(\widetilde{\mathrm{bI}}^{6}_{e}) &= -\widetilde{\mathrm{bII}}^{2,6}_{o} + \widetilde{\mathrm{bII}}^{2,6}_{e} - \widetilde{\mathrm{bII}}^{3,6}_{o} + \widetilde{\mathrm{bII}}^{3,6}_{e} - \widetilde{\mathrm{bII}}^{4,6}_{o} + \widetilde{\mathrm{bII}}^{4,6}_{e} + \widetilde{\mathrm{bII}}^{6,8}_{o}\\
&\quad - \widetilde{\mathrm{bII}}^{6,8}_{e} + \widetilde{\mathrm{bII}}^{c}_{o} + \widetilde{\mathrm{bII}}^{d}_{e} + \widetilde{\mathrm{bII}}^{e}_{o} + \widetilde{\mathrm{bII}}^{f}_{o},\\
\delta_1(\widetilde{\mathrm{bI}}^{8}_{o}) &= -\widetilde{\mathrm{bII}}^{2,8}_{o} + \widetilde{\mathrm{bII}}^{2,8}_{e} - \widetilde{\mathrm{bII}}^{3,8}_{o} + \widetilde{\mathrm{bII}}^{3,8}_{e} - \widetilde{\mathrm{bII}}^{4,8}_{o} + \widetilde{\mathrm{bII}}^{4,8}_{e} - \widetilde{\mathrm{bII}}^{6,8}_{o}\\
&\quad + \widetilde{\mathrm{bII}}^{6,8}_{e} - \widetilde{\mathrm{bII}}^{23}_{o} - \widetilde{\mathrm{bII}}^{24}_{o} + \widetilde{\mathrm{bII}}^{24}_{e} - \widetilde{\mathrm{bII}}^{c}_{o} - \widetilde{\mathrm{bII}}^{e}_{o},\\
\delta_1(\widetilde{\mathrm{bI}}^{8}_{e}) &= -\widetilde{\mathrm{bII}}^{2,8}_{o} + \widetilde{\mathrm{bII}}^{2,8}_{e} - \widetilde{\mathrm{bII}}^{3,8}_{o} + \widetilde{\mathrm{bII}}^{3,8}_{e} - \widetilde{\mathrm{bII}}^{4,8}_{o} + \widetilde{\mathrm{bII}}^{4,8}_{e} - \widetilde{\mathrm{bII}}^{6,8}_{o}\\
&\quad + \widetilde{\mathrm{bII}}^{6,8}_{e} + \widetilde{\mathrm{bII}}^{23}_{e} - \widetilde{\mathrm{bII}}^{24}_{o} + \widetilde{\mathrm{bII}}^{24}_{e} + \widetilde{\mathrm{bII}}^{c}_{e} + \widetilde{\mathrm{bII}}^{e}_{e}.
\end{aligned}
$$

Note that in a particular case, similar formulas have been obtained in [7, §6]. Then, by a straightforward computation, we get the following.

Proposition 6.2. *The cohomology groups of the universal complex*

$$\mathcal{CO}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$$

of co-orientable fibers for proper stable maps of 3*-manifolds with boundary into surfaces without boundary, are described as follows:*

(1) $H^0(\mathcal{CO}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,
(2) $H^1(\mathcal{CO}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by*

$$\gamma_1 = [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = -[\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].$$

Note that the ranks of $CO^i(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$, $i = 0, 1, 2$, are equal to 2, 10 and 40, respectively.

Suppose that we have a stable Morse function $f \colon V \to W$ of a compact surface V with boundary into a 1-dimensional manifold W without boundary. Furthermore, we assume that W is oriented. Using the orientation of W, we can co-orient each co-orientable singular fiber (of codimension 1) of f. In this way, we can define a $b\mathcal{S}_{\mathrm{pr}}$-cobordism invariant for such stable Morse functions as above for each cohomology class of the universal complex.

Unfortunately, it turns out that the $b\mathcal{S}_{\mathrm{pr}}$-cobordism invariant $s_{1*}\gamma_1$ is trivial, which can be proved by using the same argument as in [6, Lemma 14.1]. This reflects the fact that any two stable Morse functions on compact surfaces are $b\mathcal{S}_{\mathrm{pr}}$-cobordant.

For admissible maps, we get the following.

Proposition 6.3. *The cohomology groups of the universal complex*

$$\mathcal{CO}(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$$

of co-orientable fibers for proper admissible stable maps of 3*-manifolds with boundary into surfaces without boundary, are described as follows:*

(1) $H^0(\mathcal{CO}(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,
(2) $H^1(\mathcal{CO}(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by*

$$\gamma_2 = [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = -[\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o]$$

Note that the ranks of $CO^i(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$, $i = 0, 1, 2$, are equal to 2, 10 and 34, respectively.

Unfortunately, it turns out again that the $\mathcal{AS}_{\mathrm{pr}}$-cobordism invariant $s_{1*}\gamma_2$ is trivial, which is shown by using the same argument as in [6, Lemma 14.1].

This supports the following conjecture.

Conjecture 6.4. *The* $\mathcal{AS}_{\mathrm{pr}}$*-cobordism group of stable Morse functions on compact surfaces with boundary with values in* $\mathbb{R}$ *is isomorphic to* $\mathbb{Z}_2$.

Note that the $\mathcal{AS}_{\mathrm{pr}}$-cobordism classes as above form an abelian group, where the addition is given by the disjoint union, and the inverse element of a given map f is given by $-f$. Then, for each cohomology class of the universal complex of co-orientable fibers with coefficients in $\mathbb{Z}$, the associated cobordism invariant gives a homomorphism into $\mathbb{Z}$. Therefore, if the answer to the above conjecture is affirmative, then such a homomorphism must necessarily vanish.

7. Subcomplexes corresponding to orientable manifolds

We can naturally define the subcomplex $\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$ of the universal complex $\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ with $\mathbb{Z}_2$-coefficients which is generated by the classes of fibers of maps of orientable manifolds. In this case, its cohomology classes give $b\mathcal{S}_{\mathrm{pr}}$-cobordism invariants of maps of orientable manifolds with respect to orientable cobordisms.

Remark 7.1. If the source 3-manifold is orientable, then the singular fibers of types $\widetilde{\mathrm{bI}}^9$, $\widetilde{\mathrm{bI}}^{10}$, $\widetilde{\mathrm{bII}}^{26}$, $\widetilde{\mathrm{bII}}^{27}$, $\widetilde{\mathrm{bII}}^{28}$, $\widetilde{\mathrm{bII}}^{29}$, $\widetilde{\mathrm{bII}}^{30}$, $\widetilde{\mathrm{bII}}^{31}$, $\widetilde{\mathrm{bII}}^{32}$, $\widetilde{\mathrm{bII}}^{33}$, $\widetilde{\mathrm{bII}}^{34}$, $\widetilde{\mathrm{bII}}^{35}$, $\widetilde{\mathrm{bII}}^{36}$, $\widetilde{\mathrm{bII}}^{37}$, $\widetilde{\mathrm{bII}}^{38}$, and $\widetilde{\mathrm{bII}}^{39}$ never appear. Thus, the above-mentioned complex is obtained by just ignoring these fibers.

Then, using results of [12], we get the following.

Proposition 7.2. *The cohomology groups of $\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$ are described as follows:*

(1) $H^0(\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))) \cong \mathbb{Z}_2$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,

(2) $H^1(\mathcal{C}(b\mathcal{S}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *generated by*

$$\begin{aligned}
\beta &= [\widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8] = [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^7], \\
\gamma &= [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].
\end{aligned}$$

By the same reason as before, these cohomology classes give rise to trivial cobordism invariants.

For admissible maps, we can also consider the subcomplex $\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$ of the universal complex $\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ with $\mathbb{Z}_2$-coefficients which is generated by the classes of fibers of maps of orientable manifolds. In this case, its cohomology classes give $\mathcal{AS}_{\mathrm{pr}}$-cobordism invariants of maps of orientable manifolds with respect to orientable cobordisms.

Then, using results of [12], we get the following.

Proposition 7.3. *The cohomology groups of the universal complex*

$$\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))$$

for admissible stable maps of orientable 3-manifolds with boundary to surfaces without boundary with respect to the C^0 equivalence modulo two regular fiber components are described as follows:

(1) $H^0(\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))) \cong \mathbb{Z}_2$, *generated by* $[\widetilde{\mathrm{b0}}_o + \widetilde{\mathrm{b0}}_e]$,

(2) $H^1(\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho_{3,2}(2))) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *generated by*

$$\begin{aligned}
\alpha &= [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^5], \\
\beta &= [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^7] = [\widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8], \\
\gamma &= [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].
\end{aligned}$$

The non-trivial example given in [12, Corollary 4.9] is a stable Morse function on D^2, which is orientable. Therefore, we see that $s_{1*}\alpha$ again induces a non-trivial $\mathbb{Z}_2$-invariant of the orientable $\mathcal{AS}_{\mathrm{pr}}$-cobordisms of stable Morse functions on compact orientable surfaces.

We can also consider the subcomplex $\mathcal{CO}(b\mathcal{S}_{\rm pr}(3,2)^{\rm ori}, \rho_{3,2}(2))$ of the universal complex $\mathcal{CO}(b\mathcal{S}_{\rm pr}(3,2), \rho_{3,2}(2))$ with coefficients in $\mathbb{Z}$ which is generated by the classes of fibers of maps of orientable manifolds. In this case as well, its cohomology classes give $b\mathcal{S}_{\rm pr}$-cobordism invariants of maps of orientable manifolds with respect to orientable cobordisms.

Then, using the coboundary formulas in §6, we get the following.

Proposition 7.4. *The cohomology groups of the universal complex*

$$\mathcal{CO}(b\mathcal{S}_{\rm pr}(3,2)^{\rm ori}, \rho_{3,2}(2))$$

of co-orientable fibers for proper stable maps of 3*-manifolds with boundary into surfaces without boundary, are described as follows:*

(1) $H^0(\mathcal{CO}(b\mathcal{S}_{\rm pr}(3,2), \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by* $[\widetilde{\rm b0_{\it o} + b0_{\it e}}]$,
(2) $H^1(\mathcal{CO}(b\mathcal{S}_{\rm pr}(3,2), \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by*

$$\gamma_1 = [\widetilde{\rm bI}^2_o + \widetilde{\rm bI}^3_e + \widetilde{\rm bI}^4_e + \widetilde{\rm bI}^6_o + \widetilde{\rm bI}^8_e] = -[\widetilde{\rm bI}^2_e + \widetilde{\rm bI}^3_o + \widetilde{\rm bI}^4_o + \widetilde{\rm bI}^6_e + \widetilde{\rm bI}^8_o].$$

Finally, for the subcomplex

$$\mathcal{CO}(\mathcal{AS}_{\rm pr}(3,2)^{\rm ori}, \rho_{3,2}(2))$$

of the universal complex $\mathcal{CO}(\mathcal{AS}_{\rm pr}(3,2), \rho_{3,2}(2))$ with coefficients in $\mathbb{Z}$ which is generated by the classes of fibers of maps of orientable manifolds, we have the following.

Proposition 7.5. *The cohomology groups of the universal complex*

$$\mathcal{CO}(\mathcal{AS}_{\rm pr}(3,2)^{\rm ori}, \rho_{3,2}(2))$$

of co-orientable fibers for proper admissible stable maps of orientable 3*-manifolds with boundary into surfaces without boundary, are described as follows:*

(1) $H^0(\mathcal{CO}(\mathcal{AS}_{\rm pr}(3,2)^{\rm ori}, \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by* $[\widetilde{\rm b0_{\it o} + b0_{\it e}}]$,
(2) $H^1(\mathcal{CO}(\mathcal{AS}_{\rm pr}(3,2)^{\rm ori}, \rho_{3,2}(2)); \mathbb{Z}) \cong \mathbb{Z}$, *generated by*

$$\gamma_2 = [\widetilde{\rm bI}^2_o + \widetilde{\rm bI}^3_e + \widetilde{\rm bI}^4_e + \widetilde{\rm bI}^6_o + \widetilde{\rm bI}^8_e] = -[\widetilde{\rm bI}^2_e + \widetilde{\rm bI}^3_o + \widetilde{\rm bI}^4_o + \widetilde{\rm bI}^6_e + \widetilde{\rm bI}^8_o]$$

We can show again that all these cohomology classes in Propositions 7.4 and 7.5 give trivial cobordism invariants.

These results support the following conjecture.

Conjecture 7.6. *The orientable* $\mathcal{AS}_{\rm pr}$*-cobordism group of stable Morse functions on compact orientable surfaces with boundary with values in* $\mathbb{R}$ *is isomorphic to* $\mathbb{Z}_2$.

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