数理解析研究所講究録
第 1950 巻 2015 年 73-92

On Styles of $\lambda 2$-Terms
– Extended Abstract*–

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December 26, 2014

Abstract

Traditionally, two styles of λ-terms with types are well known, i.e., the Church and Curry styles. We still have other styles, e.g., de Bruijn version, domain-free style, and type-free style for polymorphic λ-calculus $\lambda 2$. It is known that some of fundamental properties hold for $\lambda 2$ in any known style, but others depend on styles. In order to capture existing styles in a uniform way, styles of $\lambda 2$-terms are introduced by giving abstract term-trees with indices, and terms in already known styles are obtained as well-typed partially annotated terms following the styles. Next, the notion of partially annotated terms is also defined for 2nd-order existential λ-calculus $\lambda^{\exists}$ We establish a systematic relationship between s-style $\lambda 2$ and s-style $\lambda^{\exists}$ via CPS-translations, which reveals the refined correspondence between type annotations and domains of abstractions. This study makes fundamental properties parametric, and provides new insight and foundations for investigating which annotations cause the differences in fundamental properties.

1 Introduction

Following the founders, we have two styles of λ-terms with types, i.e., the explicit typing (Church style) and the implicit typing (Curry style). Terms in the style of Church [4, 2] are well-typed terms where each variable is attached to a unique type. The use of explicit typing provides the property that the terms enjoy uniqueness of types. On the other hand, terms in the style of Curry [11, 1, 17] are the same as those of untyped λ-calculus, and type inference or checking guarantees that terms are well typed. This style of implicit typing forms a common basis for functional programming. In addition, pseudo-terms *à la* de Bruijn [2] are well known, and this notion can be extended to systems with higher order types and dependent types. Each style has its own advantages depending on the context under which terms are used.

*This work was partly supported by Grants-in-Aid for Scientific Research KAKENHI (C) 25400192, and by the Kayamori Foundation of Informational Science Advancement.

In this paper, we are interested in polymorphic λ-terms where types are defined from type variables denoted by X, Y, Z using constructors $\to$ and $\forall$:

$$A, B ::= X \mid (A \to B) \mid \forall X.A$$

The notation $\mathrm{FV}(A)$ denotes the set of type variables appearing freely in type A. We write $A_1 \equiv A_2$ for the syntactical identity under renaming of bound variables.

In order to define Church-style terms for polymorphic λ-calculus $\lambda 2$, we use the following syntax for raw terms, where each variable is attached to a unique type so that we have $A \equiv B$ for the same variable x such as x^A and x^B, and attached types are included in the syntax of terms.

$$M, N ::= x^A \mid \lambda x^A.M \mid MN \mid \Lambda X.M \mid M[A]$$

The notation $\mathrm{FV}(M)$ denotes the set of term variables appearing freely in term M. Then Church-style terms for $\lambda 2$ are defined inductively as follows:

$$\vdash_{\mathsf{Ch}} x^A : A \qquad \frac{\vdash_{\mathsf{Ch}} M : B}{\vdash_{\mathsf{Ch}} \lambda x^A.M : A \to B} \qquad \frac{\vdash_{\mathsf{Ch}} M : A \to B \quad \vdash_{\mathsf{Ch}} N : A}{\vdash_{\mathsf{Ch}} MN : B}$$

$$\frac{\vdash_{\mathsf{Ch}} M : A}{\vdash_{\mathsf{Ch}} \Lambda X.M : \forall X.A}\ (\mathrm{Ch})^* \qquad \frac{\vdash_{\mathsf{Ch}} M : \forall X.A}{\vdash_{\mathsf{Ch}} M[B] : A[X := B]}$$

where the mark (Ch)* denotes the variable condition that $X \notin \mathrm{FV}(B)$ for each type B such that $x^B \in \mathrm{FV}(M)$.

On the other hand, pseudo-terms *à la* de Bruijn [1, 2] are defined for $\lambda 2$, where free variables do not get ornamented with types, and type assignment rules are defined as usual.

$$M, N ::= x \mid \lambda x{:}A.M \mid MN \mid \Lambda X.M \mid M[A]$$

Finally, the system of type assignment for Curry-style terms [1, 17] of $\lambda 2$ is defined as well, where a context denoted by Γ or Σ is a set of declarations of the form $x : A$ with distinct variables. We write $\Gamma(x) = A$ for $(x{:}A) \in \Gamma$, and $\mathrm{FV}(\Gamma)$ for $\bigcup_{(x:A)\in\Gamma} \mathrm{FV}(A)$.

$$\Gamma, x{:}A \vdash_{\mathsf{Cu}} x : A \qquad \frac{\Gamma \vdash_{\mathsf{Cu}} M : A \to B \quad \Gamma \vdash_{\mathsf{Cu}} N : A}{\Gamma \vdash_{\mathsf{Cu}} MN : B}$$

$$\frac{\Gamma, x{:}A \vdash_{\mathsf{Cu}} M : B}{\Gamma \vdash_{\mathsf{Cu}} \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash_{\mathsf{Cu}} M : A}{\Gamma \vdash_{\mathsf{Cu}} M : \forall X.A}\ (\mathrm{Cu})^* \qquad \frac{\Gamma \vdash_{\mathsf{Cu}} M : \forall X.A}{\Gamma \vdash_{\mathsf{Cu}} M : A[X := B]}$$

where (Cu)* denotes the variable condition $X \notin \mathrm{FV}(\Gamma)$.

In the case of $\lambda 2$, we still have other styles, for example, domain-free style (df) [3, 8], type-free style (tf) [9], and so on. It is well known that some fundamental properties hold for $\lambda 2$ in any known style, but others depend on styles. For instance, inhabitation problems are independent of styles. The subject reduction property with respect to η-reduction holds for Church-style, but not for Curry-style [16]. Moreover, type-related problems are sensitive to styles. The type-checking and type-inference problems are known to be decidable for $\lambda 2$-terms in the Church style or the de Bruijn version, but undecidable for the Curry style by Wells [17].

In order to capture existing styles in a uniform way, we introduce styles of λ-terms by giving abstract term-trees with indices. Then we can obtain λ-terms not only in already known styles but also in new ones as partially annotated terms that are erasures. Now, we can compare terms in different styles in a uniform framework. Next, the notion of pseudo-terms for fully annotated λ-terms is also defined for 2nd-order existential λ-calculus $\lambda^{\exists}$. We establish a systematic relationship between s-style $\lambda 2$ and s-style $\lambda^{\exists}$ via CPS-translations (see Fig. 1), which reveals the refined correspondence between type annotations and domains of abstractions. This study provides new insight and foundations for investigating which annotations cause the differences in decidability of type-related problems which are made parametric with respect to styles.

In this study, annotations play three roles. The first role is that type annotations work as hints or a guide through hard typability. The second is that terms in a certain style are introduced, based on the style, by fully annotated terms, and then fundamental properties can be parametric with respect to well-ordered styles. The third and pivotal role is that annotation information makes it possible to establish natural CPS-translations from pseudo-terms of $\lambda 2$ into those of the 2nd-order existential system $\lambda^{\exists}$, without referring to derivations. In previous work [6], we studied a neat CPS-translation from the Church-style $\lambda 2$ into $\lambda^{\exists}$, where polymorphic functions are interpreted by abstract data types [12], and the translation has been defined by induction on the structure of the derivations. In order to relate type-related problems with each other, however, translations between $\lambda 2$ and $\lambda^{\exists}$ should be defined by pseudo-terms, because definitions of such problems are usually given in terms of raw terms. This idea leads to a framework for reductions from $\lambda 2$ to $\lambda^{\exists}$ families, such that some properties for $\lambda 2$ with a certain style are reduced to those for $\lambda^{\exists}$ with the corresponding style, and in turn that other properties for $\lambda^{\exists}$ parametrized with styles are reduced to those for $\lambda 2$ with the corresponding style.

Fig. 1 shows a brief outline of this idea, where $*^s$ is a CPS-translation from s-style $\lambda 2$ into s-style $\lambda^{\exists}$, and $\sharp^s$ is its inverse translation from a CPS-calculus of $\lambda^{\exists}$ in s-style back to s-style $\lambda 2$. Here, styles s and t range over not only well-known styles such as $\{\mathrm{Ch}, \mathrm{df}, \mathrm{tf}, \mathrm{Cu}\}$ but also intermediate systems between the fully annotated terms and Curry terms. Hereafter, we write $|\ |^t_s$ for an erasure mapping from t-style terms to s-style terms, and in this case we say that style t is greater than style s, denoted by $s < t$. The well-known forgetful map $|\ |^{\mathrm{full}}_s$ is a homomorphism from fully annotated terms to s-style, which erases some information in fully annotated terms, and provides more abstract λ-terms with an intermediate structure in s-style. The erasure map preserves typing. Moreover, the soundness of $*^s$ ($\sharp^t$) guarantees that the composition of the translations $|\ |^t_s$ and $*^s$ ($\sharp^t$ and $|\ |^t_s$) constitutes a homomorphic projection of t-style to s-style. The systematic correspondence presents a bird's-eye view of the whole combination of annotations including new ones, and the relationship between

$$\begin{tikzcd}
\lambda 2: & \text{Fully annotated style} \arrow[r, "|\ |^{\mathrm{full}}_{s}"] \arrow[d, shift right, "*^{\mathrm{full}}"'] & s\text{-style} \arrow[r, "|\ |^{s}_{\mathrm{cu}}"] \arrow[d, shift right, "*^{s}"'] & \text{Curry} \arrow[d, shift right, "*^{\mathrm{cu}}"'] \\
\lambda^{\exists}(\text{CPS}): & \text{Fully annotated style} \arrow[r, "|\ |^{\mathrm{full}}_{s}"] \arrow[u, shift right, "\sharp^{\mathrm{full}}"'] & s\text{-style} \arrow[r, "|\ |^{s}_{\mathrm{cu}}"] \arrow[u, shift right, "\sharp^{s}"'] & \text{Curry} \arrow[u, shift right, "\sharp^{\mathrm{cu}}"']
\end{tikzcd}$$

Figure 1: Systematic relationship between $\lambda 2$ and $\lambda^{\exists}$ with various styles type annotations and decidability of type-related problems for the systems with various styles.

The paper is organized as follows. In Section 2, we first introduce fully annotated $\lambda 2$, and styles of terms are introduced in terms of abstract term-trees with indices. Then partially annotated $\lambda 2$ is defined by using erasure based on styles, which makes fundamental properties parametric with respect to well-ordered styles. Secondly we introduce fully annotated $\lambda^{\exists}$ as the counterpart of full $\lambda 2$ in Section 3. In Section 4, we present a framework that connects fundamental properties systematically between $\lambda 2$ and $\lambda^{\exists}$ families by means of CPS-translations. Then we verify fundamental properties preserved under the translations, and show, in a uniform way, decidability results on type-related problems for $\lambda 2$ and $\lambda^{\exists}$ with various styles. In Section 5, we give concluding remarks.

2 Fully annotated and partially annotated $\lambda 2$

2.1 Fully annotated $\lambda 2$

First we introduce $\lambda 2$-terms in fully annotated style (simply called full $\lambda 2$). Pseudo-terms for full $\lambda 2$ and the system of type assignment are defined as follows. A context denoted by Γ is defined as usual, and we write $\mathrm{dom}(\Gamma)$ for $\{x \mid (x : A) \in \Gamma\}$. Let S be a set of term variables, $\Gamma \uparrow S$ denotes the context whose domain $\mathrm{dom}(\Gamma \uparrow S)$ is restricted to S.

$$M, N ::= x \mid \lambda x{:}A.M^B \mid M^A N^B \mid \Lambda X.M^A \mid M^A[B]$$

$$\frac{}{\Gamma, x{:}A \vdash_{\mathsf{full}\lambda 2} x : A}\ (\mathrm{var})$$

$$\frac{\Gamma, x{:}A \vdash_{\mathsf{full}\lambda 2} M : B}{\Gamma \vdash_{\mathsf{full}\lambda 2} \lambda x{:}A.M^B : A \to B}\ (\to I) \qquad \frac{\Gamma \vdash_{\mathsf{full}\lambda 2} M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash_{\mathsf{full}\lambda 2} M^{(A \to B)} N^A : B}\ (\to E)$$

$$\frac{\Gamma \vdash_{\mathsf{full}\lambda 2} M : A}{\Gamma \vdash_{\mathsf{full}\lambda 2} \Lambda X.M^A : \forall X.A}\ (\forall I)^{\star} \qquad \frac{\Gamma \vdash_{\mathsf{full}\lambda 2} M : \forall X.A}{\Gamma \vdash_{\mathsf{full}\lambda 2} M^{\forall X.A}[B] : A[X := B]}\ (\forall E)$$

where $\star$ means that the variable condition $X \notin \mathrm{FV}(\Gamma)$ is imposed on $(\forall I)^{\star}$.

Derivations are uniquely represented by well-typed full $\lambda 2$-terms.

Proposition 1 *Let M be a full $\lambda 2$-term that is not a variable. If $\Gamma_1 \vdash_{\mathsf{full}\lambda 2} M : A_1$ and $\Gamma_2 \vdash_{\mathsf{full}\lambda 2} M : A_2$, then $A_1 \equiv A_2$ and $\Gamma_1 \uparrow \mathrm{FV}(M) = \Gamma_2 \uparrow \mathrm{FV}(M)$.*

Given a well-typed term of full $\lambda 2$, the Church-style term can be defined by $(|M|^{\mathrm{full}}_{\mathrm{ch}})^{\Gamma}$ using the following erasure $|\cdot|^{\mathrm{full}}_{\mathrm{ch}}$ and $\cdot^{\Gamma}$:

Definition 1 ($|\cdot|_{\text{ch}}^{\text{full}}$ **and** $\cdot^{\Gamma}$) *1.* $|x|_{\text{ch}}^{\text{full}} = x \quad x^{\Gamma} = x^{\Gamma(x)}$

2. $|\lambda x\!:\!A.M^B|_{\text{ch}}^{\text{full}} = \lambda x\!:\!A.|M|_{\text{ch}}^{\text{full}}$

3. $|M^A N^B|_{\text{ch}}^{\text{full}} = |M|_{\text{ch}}^{\text{full}}|N|_{\text{ch}}^{\text{full}}$

4. $|\Lambda X.M^A|_{\text{ch}}^{\text{full}} = \Lambda X.|M|_{\text{ch}}^{\text{full}}$

5. $|M^A[B]|_{\text{ch}}^{\text{full}} = |M|_{\text{ch}}^{\text{full}}[B]$

$$\begin{aligned}(\lambda x\!:\!A.M)^{\Gamma} &= \lambda x\!:\!A.M^{\Gamma}\\ (MN)^{\Gamma} &= M^{\Gamma}N^{\Gamma}\\ (\Lambda X.M)^{\Gamma} &= \Lambda X.M^{\Gamma}\\ (M[A])^{\Gamma} &= M^{\Gamma}[A]\end{aligned}$$

Proposition 2 *If* $\Gamma \vdash_{\text{full}\lambda 2} M : A$*, then we have* $\vdash_{\text{Ch}} (|M|_{\text{ch}}^{\text{full}})^{\Gamma} : A$.

Church-style terms are represented by well-typed and partially annotated $\lambda 2$-terms with the erasure and Γ. In this way, $\lambda 2$-terms in well-known style will be obtained from full $\lambda 2$ by erasing, based on styles representing patterns of terms.

2.2 Styles of $\lambda 2$-terms and partially annotated terms

In order to represent styles of terms, we introduce term constructors with indices. General styles of $\lambda 2$-terms will be defined from the set of term trees that are well labelled with indices. A syntax of term trees is defined from term constructors $\mathsf{var}, \lambda, @, \Lambda$, and $@_{\mathsf{T}}$.

$$t \in \mathcal{T}ree ::= \mathsf{var} \mid \lambda.t \mid @.(t_1, t_2) \mid \Lambda.t \mid @_{\mathsf{T}}.t$$

A syntax of styles of $\lambda 2$-terms is defined by term trees together with indices, denoted by n and i that range over the set of natural numbers.

$$s, t \in \mathcal{S}tyle \quad ::= \quad \mathsf{var}(n) \mid \lambda(n,n).s \mid @(n,n).(s,s) \mid \Lambda(i,n).s \mid @_{\mathsf{T}}(n,i).s$$

Here, the indices in indexed constructors informally mean that how many pieces of information are included in terms, and type annotations will be assigned following indices of styles soon.

We define a surjective mapping from $\mathcal{S}tyle$ to $\mathcal{T}ree$, called an erasure mapping.

Definition 2 (Erasure from styles to term-trees)
$|\mathsf{var}(n)| = \mathsf{var}$; $\quad |\lambda(m,n).s| = \lambda.|s|$; $\quad |@(m,n).(s,t)| = @.(|s|,|t|)$;
$|\Lambda(i,n).s| = \Lambda.|s|$; $\quad |@_{\mathsf{T}}(m,i).s| = @_{\mathsf{T}}.|s|$.

Definition 3 (Order on styles) *We define a binary relation* $s \leq t$ *on styles pointwise as follows:*

1. *If* $n_1 \leq n_2$ *then* $\mathsf{var}(n_1) \leq \mathsf{var}(n_2)$.
2. *If* $s_1 \leq s_2$ *with* $|s_1| = |s_2|$, $m_1 \leq m_2$, *and* $n_1 \leq n_2$ *then* $\lambda(m_1,n_1).s_1 \leq \lambda(m_2,n_2).s_2$.
3. *If* $s_1 \leq s_2$ *with* $|s_1| = |s_2|$, $t_1 \leq t_2$ *with* $|t_1| = |t_2|$, $m_1 \leq m_2$, *and* $n_1 \leq n_2$ *then* $@(m_1,n_1).(s_1,t_1) \leq @(m_2,n_2).(s_2,t_2)$.

4. *If* $s_1 \leq s_2$, $i_1 \leq i_2$, *and* $n_1 \leq n_2$ *then* $\Lambda(i_1, n_1).s_1 \leq \Lambda(i_2, n_2).s_2$.

5. *If* $s_1 \leq s_2$, $m_1 \leq m_2$, *and* $i_1 \leq i_2$ *then* $@_{\mathsf{T}}(m_1, i_1).s_1 \leq @_{\mathsf{T}}(m_2, i_2).s_2$.

Note that the relation on styles forms a partial order.

For general styles, we consider subsets of $\mathcal{S}tyle$, which are bijective to $\mathcal{T}ree$.

Definition 4 (General styles) *A subset* $\mathcal{S}t$ *of* $\mathcal{S}tyle$ *is a general style, if for each tree* $t \in \mathcal{T}ree$, *there exists a unique style* $s \in \mathcal{S}t$ *such that* $|s| = t$.

A partial order on general styles can be defined naturally.

Definition 5 (Order on general styles) *Let* $\mathcal{S}t_1, \mathcal{S}t_2$ *be general styles. We define* $\mathcal{S}t_1 \leq \mathcal{S}t_2$ *if* $s_1 \leq s_2$ *for each* $s_1 \in \mathcal{S}t_1$ *and* $s_2 \in \mathcal{S}t_2$ *such that* $|s_1| = |s_2|$.

Note that an erasure mapping $|\ |_{\mathcal{S}t_1}^{\mathcal{S}t_2}$ is induced from general styles $\mathcal{S}t_1 \leq \mathcal{S}t_2$: e.g., if $\lambda(m_1, n_1).t_1 \leq \lambda(m_2, n_2).t_2$, then erase annotations so that the values m_2, n_2 decrease to m_1, n_1, respectively. In turn, a pre-order on styles is in general induced from the identity mapping and the composition of two erasures. Moreover, if one has an erasure that maps $\mathcal{S}t_1$ to $\mathcal{S}t_2$ then an order is naturally induced such that $\mathcal{S}t_2 \leq \mathcal{S}t_1$, since there exists a unique style for each term tree.

We define a binary relation on terms and styles such that a term M has a style $s \in \mathcal{S}t$, denoted by $M :: s$.

$$x :: \mathsf{var}(0) \qquad x^A :: \mathsf{var}(1)$$

$$\frac{M :: s}{\lambda x{:}A.M^B :: \lambda(1,1).s} \quad \frac{M :: s}{\lambda x{:}A.M :: \lambda(1,0).s} \quad \frac{M :: s}{\lambda x.M^B :: \lambda(0,1).s} \quad \frac{M :: s}{\lambda x.M :: \lambda(0,0).s}$$

$$\frac{M :: s \quad N :: t}{M^A N^B :: @(1,1).(s,t)} \quad \frac{M :: s \quad N :: t}{M^A N :: @(1,0).(s,t)} \quad \frac{M :: s \quad N :: t}{M N^B :: @(0,1).(s,t)} \quad \frac{M :: s \quad N :: t}{M N :: @(0,0).(s,t)}$$

$$\frac{M :: s}{\Lambda X.M^B :: \Lambda(2,1).s} \quad \frac{M :: s}{\Lambda X.M :: \Lambda(2,0).s} \quad \frac{M :: s}{\Lambda.M^B :: \Lambda(1,1).s} \quad \frac{M :: s}{\Lambda.M :: \Lambda(1,0).s}$$

$$\frac{M :: s}{M^B :: \Lambda(0,1).s} \quad \frac{M :: s}{M :: \Lambda(0,0).s} \quad \frac{M :: s}{M^A[B] :: @_{\mathsf{T}}(1,2).s} \quad \frac{M :: s}{M[B] :: @_{\mathsf{T}}(0,2).s}$$

$$\frac{M :: s}{M^A[] :: @_{\mathsf{T}}(1,1).s} \quad \frac{M :: s}{M[] :: @_{\mathsf{T}}(0,1).s} \quad \frac{M :: s}{M^A :: @_{\mathsf{T}}(1,0).s} \quad \frac{M :: s}{M :: @_{\mathsf{T}}(0,0).s}$$

We concentrate the paper mainly on styles such that each term constructor has fixed indices for simplicity and direct connection to already existing ones. This form of styles can be represented by the following tuples:

Definition 6 (Church, de Bruijn, domain-free, type-free, Curry styles $\lambda 2$)

- *Pseudo-terms for Church style* $\lambda 2$*:* $\langle \mathsf{var}(1), \lambda(1,0), @(0,0), \Lambda(2,0), @_{\mathsf{T}}(0,2) \rangle$
- *Pseudo-terms for de Bruijn style* $\lambda 2$*:* $\langle \mathsf{var}(0), \lambda(1,0), @(0,0), \Lambda(2,0), @_{\mathsf{T}}(0,2) \rangle$
- *Pseudo-terms for domain-free style* $\lambda 2$*:* $\langle \mathsf{var}(0), \lambda(0,0), @(0,0), \Lambda(2,0), @_{\mathsf{T}}(0,2) \rangle$

- *Pseudo-terms for type-free style* $\lambda 2$*:* $\langle \mathsf{var}(0), \lambda(0,0), @(0,0), \Lambda(1,0), @_{\mathsf{T}}(0,1)\rangle$
- *Curry style* $\lambda 2$*:* $\langle \mathsf{var}(0), \lambda(0,0), @(0,0), \Lambda(0,0), @_{\mathsf{T}}(0,0)\rangle$

We define terms of partially annotated $\lambda 2$ (partial $\lambda 2$) by deleting some annotations from full $\lambda 2$ following style s, called s-style $\lambda 2$-terms. In this way, we consider all styles between full and Curry-styles, where Curry-style is the least style, and the style of full $\lambda 2$ is the greatest. Under the order, Definition 1 (Church) is available not only for full $\lambda 2$ but also for any style s greater than or equal to that of de Bruijn.

For any style $s \geq$ Curry, one can naturally define the system of type assignment for s-style terms M under a context Γ as an erasure of the system for full $\lambda 2$, written by $\Gamma \vdash_s M : A$. For any style $s \geq$ type-free, one has a natural form of generation (inversion) lemma called syntax directed, such that from the shape of an s-style term M of $\Gamma \vdash_s M : A$, one can uniquely determine which rule should be applied to derive the judgement.

Proposition 3 (Generation lemma for $s \geq$ type-free**)** *Let* $s \geq$ type-free.

(1) *If* $\Gamma \vdash_s x : A$ *then* $\Gamma(x) = A$.

(2) *If* $\Gamma \vdash_s \lambda x.M : A_1$ *then* $\Gamma, x{:}A_0 \vdash_s M : A_2$ *and* $A_1 = (A_0 \to A_2)$ *for some* A_0, A_2.

(3) *If* $\Gamma \vdash_s M_1 M_2 : A_1$ *then* $\Gamma \vdash_s M_1 : A_0 \to A_1$ *and* $\Gamma \vdash_s M_2 : A_0$ *for some* A_0.

(4) *If* $\Gamma \vdash_s \Lambda.M : A_1$ *then* $\Gamma \vdash_s M : A_2$ *and* $A_1 = \forall X.A_2$ *with* $X \notin \mathrm{FV}(\Gamma)$ *for some* A_2.

(5) *If* $\Gamma \vdash_s M[] : A_1$ *then* $\Gamma \vdash_s M : \forall X.A_2$ *and* $A_1 = A_2[X := A]$ *for some* A, A_2.

Recall that a similar generation lemma holds for Curry-style $\lambda 2$ [1, 17]. For uniqueness of types, we need more annotations than those in the style of type-free.

Proposition 4 (Uniqueness of types for $s \geq$ deBruijn**)** *For any style* $s \geq$ deBruijn, *if* $\Gamma \vdash_s M : A_1$ *and* $\Gamma \vdash_s M : A_2$*, then we have* $A_1 \equiv A_2$.

Proposition 5 (Erasure and lifting for $s \geq$ Curry**)** *Let* s, t *be styles with* Curry $\leq s \leq t$.

(1) *If* $\Gamma \vdash_t M : A$ *then* $\Gamma \vdash_s |M|^t_s : A$.

(2) *If* $\Gamma \vdash_s M : A$ *then there exists a* $\lambda 2$*-term* N *in* t*-style such that* $|N|^t_s = M$ *and* $\Gamma \vdash_t N : A$.

Inhabitation problem by s-style $\lambda 2$-terms (IHP(s)) is defined as follows: Given a type A, determine whether there exists a closed $\lambda 2$-term M in s-style such that $\vdash_s M : A$.

Corollary 1 IHP(s) *is equivalent to each other for any style* $s \geq$ Curry.

3 Existential $\lambda^\exists$ in fully and partially annotated styles

Secondly we define the 2nd-order existential type system $\lambda^\exists$ that is logically a subsystem of minimal logic consisting of falsity, negation, conjunction, and 2nd-order existential quantification. It is known that $\lambda 2$ can be Galois embedded into $\lambda^\exists$ [7], which can be applied to connect fundamental properties with each other between $\lambda 2$ and $\lambda^\exists$. We introduce fully annotated $\lambda^\exists$ (full $\lambda^\exists$)[1] that is the counterpart of the full $\lambda 2$.

1. $\lambda^\exists$-types $\quad A, B ::= X \mid \bot \mid \neg A \mid (A \wedge B) \mid \exists X.A$

2. Pseudo-terms for full $\lambda^\exists$

$$\begin{array}{rcl} M, N & ::= & x \mid \lambda x{:}A.M \mid MN^A \mid \langle M, N\rangle^A \mid (\texttt{let}\ \langle x{:}A, y{:}B\rangle = M\ \texttt{in}\ N) \\ & \mid & \langle A, M\rangle^B \mid (\texttt{let}\ \langle X, y{:}B\rangle = M\ \texttt{in}\ N) \end{array}$$

3. Inference rules for full $\lambda^\exists$-terms

$$\frac{}{\Gamma, x{:}A \vdash_{\mathsf{full}\lambda^\exists} x : A}\ (\text{var}) \qquad \frac{\Gamma, x{:}A \vdash_{\mathsf{full}\lambda^\exists} M : \bot}{\Gamma \vdash_{\mathsf{full}\lambda^\exists} \lambda x{:}A.M : \neg A}\ (\neg I)$$

$$\frac{\Gamma \vdash_{\mathsf{full}\lambda^\exists} M : \neg A \quad \Gamma \vdash_{\mathsf{full}\lambda^\exists} N : A}{\Gamma \vdash_{\mathsf{full}\lambda^\exists} MN^A : \bot}\ (\neg E) \qquad \frac{\Gamma \vdash_{\mathsf{full}\lambda^\exists} M : A \quad \Gamma \vdash_{\mathsf{full}\lambda^\exists} N : B}{\Gamma \vdash_{\mathsf{full}\lambda^\exists} \langle M, N\rangle^{(A\wedge B)} : A \wedge B}\ (\wedge I)$$

$$\frac{\Gamma \vdash M : A_1 \wedge A_2 \quad \Gamma, x{:}A_1, y{:}A_2 \vdash N : B}{\Gamma \vdash_{\mathsf{full}\lambda^\exists} (\texttt{let}\ \langle x{:}A_1, y{:}A_2\rangle = M\ \texttt{in}\ N) : B}\ (\wedge E)$$

$$\frac{\Gamma \vdash_{\mathsf{full}\lambda^\exists} M : A[X := B]}{\Gamma \vdash_{\mathsf{full}\lambda^\exists} \langle B, M\rangle^{\exists X.A} : \exists X.A}\ (\exists I) \qquad \frac{\Gamma \vdash_{\mathsf{full}\lambda^\exists} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\mathsf{full}\lambda^\exists} (\texttt{let}\ \langle X, x{:}A\rangle = M\ \texttt{in}\ N) : B}\ (\exists E)^\star$$

where $\star$ means that the variable condition $X \notin \mathrm{FV}(\Gamma, B)$ is imposed.

Following the idea of partial $\lambda 2$, partially annotated $\lambda^\exists$ (partial $\lambda^\exists$) is defined as well, and for this, styles are also defined for $\lambda^\exists$ where n, i range over the set of natural numbers:

$$\begin{array}{rcl} s, t \in \mathit{Style} & ::= & \mathsf{var}(n) \mid \lambda(n).s \mid @(n).(s,t) \mid \mathsf{pair}(n).(s,t) \mid \mathsf{let}(n,n).(s,t) \\ & \mid & \mathsf{pair}_{\mathsf{T}}(i,n).s \mid \mathsf{let}_{\mathsf{T}}(i,n).(s,t) \end{array}$$

A binary relation between terms and styles is partly listed together with inference rules:

$$\frac{\Gamma \vdash_{\lambda^\exists} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\lambda^\exists} (\texttt{let}\ \langle X, x{:}A\rangle = M\ \texttt{in}\ N) : B}\ \mathsf{let}_{\mathsf{T}}(2,1) \qquad \frac{\Gamma \vdash_{\lambda^\exists} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\lambda^\exists} (\texttt{let}\ \langle X, x\rangle = M\ \texttt{in}\ N) : B}\ \mathsf{let}_{\mathsf{T}}(2,0)$$

$$\frac{\Gamma \vdash_{\lambda^\exists} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\lambda^\exists} (\texttt{let}\ \langle x{:}A\rangle = M\ \texttt{in}\ N) : B}\ \mathsf{let}_{\mathsf{T}}(1,1) \qquad \frac{\Gamma \vdash_{\lambda^\exists} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\lambda^\exists} (\texttt{let}\ \langle x\rangle = M\ \texttt{in}\ N) : B}\ \mathsf{let}_{\mathsf{T}}(1,0)$$

[1] Full $\lambda^\exists$ will be denoted by the tuple $\langle \mathsf{var}(0), \lambda(1), @(1), \mathsf{pair}(1), \mathsf{let}(1,1), \mathsf{pair}_{\mathsf{T}}(2,1), \mathsf{let}_{\mathsf{T}}(2,1)\rangle$.

$$\frac{\Gamma \vdash_{\lambda^{\exists}} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\lambda^{\exists}} N[x := M^A] : B}\ \mathsf{let_T}(0,1) \qquad \frac{\Gamma \vdash_{\lambda^{\exists}} M : \exists X.A \quad \Gamma, x{:}A \vdash N : B}{\Gamma \vdash_{\lambda^{\exists}} N[x := M] : B}\ \mathsf{let_T}(0,0)$$

An order on styles is naturally defined as well. Note that some of partial $\lambda^{\exists}$ already appeared in the literature, e.g., $\langle \mathsf{pair_T}(0,0), \mathsf{let_T}(0,0)\rangle$ in Sørensen and Urzyczyn [16], where

$$\frac{\Gamma \vdash M : A[X := B]}{\Gamma \vdash \langle B, M\rangle : \exists X.A}\ \mathsf{pair_T}(2,0) \qquad \frac{\Gamma \vdash M : A[X := B]}{\Gamma \vdash \langle M\rangle : \exists X.A}\ \mathsf{pair_T}(1,0) \qquad \frac{\Gamma \vdash M : A[X := B]}{\Gamma \vdash M : \exists X.A}\ \mathsf{pair_T}(0,0)$$

4 Systematic relationship between s-$\lambda 2$ and s-$\lambda^{\exists}$

Next we introduce translations $*, \sharp$ between the full $\lambda 2$ and the full $\lambda^{\exists}$, and moreover, by using the erasure map, the translations can be modified systematically for partial $\lambda 2$ and $\lambda^{\exists}$ in s-style, including domain-free, type-free, and Curry $\lambda 2$ and $\lambda^{\exists}$ respectively, denoted by $*^s, \sharp^s$. Note that the following definition of CPS-translation reveals an interesting correspondence between type annotations of full $\lambda 2$ and domains of abstractions of $\lambda^{\exists}$.

4.1 CPS-translation from full $\lambda 2$ into full $\lambda^{\exists}$

Definition 7 (**CPS-translation** $*$ **from** $\lambda 2$ **into** $\lambda^{\exists}$) *In the following, $*^{\text{full}}$ may be written simply by $*$, and a is a fresh variable.*

$X^* = X$, $(A \to B)^* = (\neg A^* \wedge B^*)$, $(\forall X.A)^* = \exists X.A^*$.

1. $(x^A)^* = (x^{\neg A^*} a)$,
2. $(\lambda x{:}A.M^B)^* = (\mathtt{let}\ \langle x{:}\neg A^*, a{:}B^*\rangle = a\ \mathtt{in}\ M^*)$,
3. $(M^A N^B)^* = M^*[a := \langle \lambda a{:}B^*.N^*, a\rangle^{A^*}]$,
4. $(\Lambda X.M^A)^* = (\mathtt{let}\ \langle X, a{:}A^*\rangle = a\ \mathtt{in}\ M^*)$,
5. $(M^A[B])^* = M^*[a := \langle B^*, a\rangle^{A^*}]$.

Proposition 6 (Soundness of full $\lambda 2$)
If $\Gamma \vdash_{\text{full}\lambda 2} M : A$, then we have $\neg\Gamma^* \vdash_{\text{full}\lambda^{\exists}} \lambda a{:}A^*.M^* : \neg A^*$.

Proof. By induction on the derivation. □

Note that the target calculus is essentially the full $\lambda^{\exists}$, although the variable a in (xa) has no annotations; substituted instances of a with pair(1), $\mathsf{pair_T}(1,1)$ always have an annotation.

For an inverse translation, we define a subcalculus that properly includes CPS-images of $\lambda 2$-terms and types, called CPS-terms and CPS-types respectively.

CPS-types:

$A, B ::= X \mid (\neg A \wedge B) \mid \exists X.A$

The calculus consists of two categories; denotations P and continuations C, for

$\lambda 2$ in s-style	$\lambda(n_1, n_2)$	$@(n_3, n_4)$	$\Lambda(i_1, n_5)$	$@_{\mathsf{T}}(n_6, i_2)$
$\lambda^{\exists}$ in s^*-style	$\mathrm{let}(n_1, n_2)$	$\mathrm{pair}(n_3), \lambda(n_4)$	$\mathrm{let}_{\mathsf{T}}(i_1, n_5)$	$\mathrm{pair}_{\mathsf{T}}(i_2, n_6)$

Table 1: Refined correspondence between s-style $\lambda 2$ and s^*-style $\lambda^{\exists}$

which two kinds of variables, denoted by x, y and a respectively, are used, and CPS-types are denoted by A^*, B^*.

CPS-terms:

$P ::= (x^{\neg A^*}C) \mid (\lambda a\!:\!A^*.P)C \mid (\mathtt{let}\ \langle x\!:\!\neg A^*, a\!:\!B^*\rangle = C\ \mathtt{in}\ P)$
$\quad \mid (\mathtt{let}\ \langle X, a\!:\!B^*\rangle = C\ \mathtt{in}\ P)$

$C ::= a \mid \langle x^{\neg A^*}, C\rangle^{B^*} \mid \langle \lambda a\!:\!A^*.P, C\rangle^{B^*} \mid \langle A^*, C\rangle^{B^*}$

Here, a restriction on occurrences of the continuation variable a is imposed, such that P and C involve exactly one free occurrence of a, namely, a linear variable. The categories are closed under substitutions such as $P[x := \lambda a\!:\!A^*.P'], P[a := C], C[x := \lambda a\!:\!A^*.P'], C[a := C']$. An inverse $\sharp^{\mathrm{full}}$ is defined for CPS-types and CPS-terms, where a continuation C is inverse translated to a term-context $C^{\sharp}$ with a hole $[\ ]$, which is defined as usual.

Definition 8 (Inverse translation $\sharp^{\mathrm{full}}$)
$X^{\sharp} = X,\ (\neg A \wedge B)^{\sharp} = (A^{\sharp} \to B^{\sharp}),\ (\exists X.A)^{\sharp} = \forall X.A^{\sharp}.$

(1) (a) $(x^{\neg A}C)^{\sharp} = C^{\sharp}[x^{A^{\sharp}}]$, (b) $((\lambda a\!:\!A.P)C)^{\sharp} = C^{\sharp}[P^{\sharp A^{\sharp}}]$,

(c) $(\mathtt{let}\ \langle x\!:\!\neg A, a\!:\!B\rangle = C\ \mathtt{in}\ P)^{\sharp} = C^{\sharp}[\lambda x\!:\!A^{\sharp}.P^{\sharp B^{\sharp}}]$,

(d) $(\mathtt{let}\ \langle X, a\!:\!A\rangle = C\ \mathtt{in}\ P)^{\sharp} = C\sharp[\Lambda X.P^{\sharp A^{\sharp}}]$;

(2) (a) $a^{\sharp} = [\ ]$, (b) $((\langle\lambda a\!:\!A.P, C\rangle^{B})^{\sharp} = C^{\sharp}[[\ ]^{B^{\sharp}}P^{\sharp A^{\sharp}}]$,

(c) $((\langle A, C\rangle^{B})^{\sharp} = C^{\sharp}[[\ ]^{B^{\sharp}}A^{\sharp}]$.

Next section, we show the completeness of the full $\lambda 2$ with respect to the full $\lambda^{\exists}$ such that $\Gamma \vdash_{\mathsf{full}\lambda 2} M : A \iff \neg\Gamma^* \vdash_{\mathsf{full}\lambda^{\exists}} \lambda a\!:\!A^*.M^* : \neg A^*$, based on which the completeness of s-$\lambda 2$ with respect to s-$\lambda^{\exists}$ will be obtained.

4.2 Correspondence between s-style $\lambda 2$ and s^*-style $\lambda^{\exists}$

Although we have observed the dual correspondence between $\lambda 2$ and $\lambda^{\exists}$ [6, 7], the introduction of full annotations establishes much detailed and informative correspondence between partial $\lambda 2$ and partial $\lambda^{\exists}$. Let s be a style of partial $\lambda 2$ with

$s = \langle \lambda(n_1, n_2), @(n_3, n_4), \Lambda(i_1, n_5),\ @_{\mathsf{T}}(n_6, i_2)\rangle$.

From Definition 7 for full $\lambda 2$ with an erasure mapping, one has a CPS-translation $*^s$ from s-style $\lambda 2$ into t-style $\lambda^{\exists}$ such that

$t = \langle \lambda(n_4), @(0), \mathrm{pair}(n_3), \mathrm{let}(n_1, n_2), \mathrm{pair}_{\mathsf{T}}(i_2, n_6), \mathrm{let}_{\mathsf{T}}(i_1, n_5)\rangle$.

From now on, we may write simply s^* for such a target style t, and $*$ for $*^s$. Moreover, with the help of erasure, the inverse $\sharp$ for full $\lambda^{\exists}$ is available as well to each instance of s^*-style $\lambda^{\exists}$. The refined correspondence between partial $\lambda 2$ and $\lambda^{\exists}$ is summarized in Table 1. We show instances of partial $\lambda 2$ and $\lambda^{\exists}$. See also Section 3 for $\mathrm{let}_{\mathsf{T}}$ and $\mathrm{let}_{\mathsf{T}}$.

- De Bruijn $\lambda^{\exists} = \langle \mathsf{var}(0), \lambda(0), @(0), \mathrm{pair}(0), \mathrm{let}(1,0), \mathrm{pair}_{\mathsf{T}}(2,0), \mathrm{let}_{\mathsf{T}}(2,0)\rangle$
- Domain-free $\lambda^{\exists} = \langle \mathsf{var}(0), \lambda(0), @(0), \mathrm{pair}(0), \mathrm{let}(0,0), \mathrm{pair}_{\mathsf{T}}(2,0), \mathrm{let}_{\mathsf{T}}(2,0)\rangle$ [13]
- Hole-application $\lambda^{\exists} = \langle \mathsf{var}(0), \lambda(0), @(0), \mathrm{pair}(0), \mathrm{let}(1,0), \mathrm{pair}_{\mathsf{T}}(1,0), \mathrm{let}_{\mathsf{T}}(2,0)\rangle$
- Type-free $\lambda^{\exists} = \langle \mathsf{var}(0), \lambda(0), @(0), \mathrm{pair}(0), \mathrm{let}(1,0), \mathrm{pair}_{\mathsf{T}}(1,0), \mathrm{let}_{\mathsf{T}}(1,0)\rangle$
- Curry$^{+\Lambda(1,0)}$ $\lambda 2 = \langle \mathsf{var}(0), \lambda(0,0), @(0,0), \Lambda(1,0), @_{\mathsf{T}}(0,0)\rangle$
  Curry$^{+\mathsf{let}_{\mathsf{T}}(1,0)}$ $\lambda^{\exists} = \langle \mathsf{var}(0), \lambda(0), @(0), \mathrm{pair}(0), \mathrm{let}(0,0), \mathrm{pair}_{\mathsf{T}}(0,0), \mathrm{let}_{\mathsf{T}}(1,0)\rangle$
- Curry$^{+\Lambda(0,1)}$ $\lambda 2 = \langle \mathsf{var}(0), \lambda(0,0), @(0,0), \Lambda(0,1), @_{\mathsf{T}}(0,0)\rangle$
  Curry$^{+\mathsf{let}_{\mathsf{T}}(0,1)}$ $\lambda^{\exists} = \langle \mathsf{var}(0), \lambda(0), @(0), \mathrm{pair}(0), \mathrm{let}(0,0), \mathrm{pair}_{\mathsf{T}}(0,0), \mathrm{let}_{\mathsf{T}}(0,1)\rangle$

Note that the systems of Curry^{+} $\lambda 2$ and $\lambda^{\exists}$ seem to be not found in the literature up to our knowledge. In particular, $\lambda^{\exists}$ systems in s-style with $s \geq$ Curry^{+} play an important role here.

4.3 A systematic reduction from partial $\lambda 2$ into partial $\lambda^{\exists}$

We introduce a framework that can relate systematically corresponding systems between $\lambda 2$ and $\lambda^{\exists}$. In the following, we show commutativity of the translations $*, \sharp$ and erasure $|\ |$; lifting of CPS-terms in s-style up to those in t-style with $s \leq t$; and a back translation $\sharp$ from full $\lambda^{\exists}$ to full $\lambda 2$.

Proposition 7 (Commutativity of translations $*, \sharp$ and erasure $|\ |$) (1) *Let M be a $\lambda 2$-term M in* t-style *and $s \leq t$. Then we have $(M^{*^t})^{\sharp^t} = M$ and $(|M|^t_s)^{*^s} = |M^{*^t}|^t_s$.*

(2) *Let P, C be CPS-terms of* t-style *$\lambda^{\exists}$ with $s \leq t$. Then $(|P|^t_s)^{\sharp^s} = |P^{\sharp^t}|^t_s$ and $(|C|^t_s)^{\sharp^s} = |C^{\sharp^t}|^t_s$.*

Proof. By induction on the structures. We show some of the cases in Fig. 2. □
$\lambda 2$ has the lemma of lifting as in [1], i.e., Proposition 5 ($s \geq$ Curry): If $\Gamma \vdash_{s\text{-}\lambda 2} N : A$ then there exists a term M in the full $\lambda 2$ such that $|M|^{\mathrm{full}}_s = N$ and $\Gamma \vdash_{\mathrm{full}\lambda 2} M : A$. Here, CPS-terms in style $s \geq$ Curry$^{+\mathsf{let}_{\mathsf{T}}(1,0)}$ or $s =$ Curry$^{+\mathsf{let}_{\mathsf{T}}(0,1)}$ have the following lemma that plays an important role. We write CUR for the set $\{s \mid s \geq \text{Curry}^{+\Lambda(1,0)}\} \cup \{\text{Curry}^{+\Lambda(0,1)}\}$, and CUR^* for $\{s \mid s \geq \text{Curry}^{+\mathsf{let}_{\mathsf{T}}(1,0)}\} \cup \{\text{Curry}^{+\mathsf{let}_{\mathsf{T}}(0,1)}\}$,

Proposition 8 (Key proposition: lifting CPS-terms and types for $s \in \mathsf{CUR}^*$) *Let $s \in \mathsf{CUR}^*$.*

(1) *If $\Gamma, a : A \vdash_{s\text{-}\lambda^{\exists}} P : \bot$ in s-style $\lambda^{\exists}$, then there exist a* CPS-*term Q in the full $\lambda^{\exists}$, a CPS-type A', and Γ' consisting of CPS-types such that $|Q|^{\mathrm{full}}_s = P$ and $\neg\Gamma', a{:}A' \vdash_{\mathrm{full}\lambda^{\exists}} Q : \bot$.*

1. Case of $(\lambda x{:}A.M^B)::\lambda(1,1)$:

$$\begin{tikzcd}
\lambda x{:}A.M^B \arrow[r, "|\ |^{\mathrm{full}}_{\mathrm{db}}"] \arrow[d, "*^{\mathrm{full}}"'] & \lambda x{:}A.|M|^{\mathrm{full}}_{\mathrm{db}} \arrow[r, "|\ |^{\mathrm{db}}_{\mathrm{cu}}"] \arrow[d, "*^{\mathrm{db}}"'] & \lambda x.|M|^{\mathrm{full}}_{\mathrm{cu}} \arrow[d, "*^{\mathrm{cu}}"'] \\
\mathtt{let}\ \langle x{:}\neg A^*, a{:}B^*\rangle = a\ \mathtt{in}\ M^* \arrow[r, "|\ |^{\mathrm{full}}_{\mathrm{db}}"] & \mathtt{let}\ \langle x{:}\neg A^*, a\rangle = a\ \mathtt{in}\ |M^*|^{\mathrm{full}}_{\mathrm{db}} \arrow[r, "|\ |^{\mathrm{db}}_{\mathrm{cu}}"] & \mathtt{let}\ \langle x, a\rangle = a\ \mathtt{in}\ |M^*|^{\mathrm{full}}_{\mathrm{cu}}
\end{tikzcd}$$

2. Case of $(M^A N^B)::@(1,1)$:

$$\begin{tikzcd}
M^A N^B \arrow[r, "|\ |^{\mathrm{full}}"] \arrow[d, "*^{\mathrm{full}}"'] & |M|^{\mathrm{full}}(|N|^{\mathrm{full}})^B \arrow[r, "|\ |_{\mathrm{cu}}"] \arrow[d, "*"'] & |M|^{\mathrm{full}}_{\mathrm{cu}}|N|^{\mathrm{full}}_{\mathrm{cu}} \arrow[d, "*^{\mathrm{cu}}"'] \\
M^*[a := \langle\lambda a{:}B^*.N^*, a\rangle^{A^*}] \arrow[r, "|\ |^{\mathrm{full}}"] & |M^*|^{\mathrm{full}}[a := \langle\lambda a{:}B^*.|N^*|^{\mathrm{full}}, a\rangle] \arrow[r, "|\ |_{\mathrm{cu}}"] & |M^*|^{\mathrm{full}}_{\mathrm{cu}}[a := \langle\lambda a.|N^*|^{\mathrm{full}}_{\mathrm{cu}}, a\rangle]
\end{tikzcd}$$

3. Case of $(M^A[B])::@_{\mathsf{T}}(1,2)$:

$$\begin{tikzcd}
M^A[B] \arrow[r, "|\ |^{\mathrm{full}}_{\mathrm{db}}"] \arrow[d, "*^{\mathrm{full}}"'] & |M|^{\mathrm{full}}_{\mathrm{db}}[B] \arrow[r, "|\ |^{\mathrm{db}}_{\mathrm{tf}}"] \arrow[d, "*^{\mathrm{db}}"'] & |M|^{\mathrm{full}}_{\mathrm{tf}}[] \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] \arrow[d, "*^{\mathrm{tf}}"'] & |M|^{\mathrm{full}}_{\mathrm{cu}} \arrow[d, "*^{\mathrm{cu}}"'] \\
M^*[a := \langle B^*, a\rangle^{A^*}] \arrow[r, "|\ |^{\mathrm{full}}_{\mathrm{db}}"] & |M^*|^{\mathrm{full}}_{\mathrm{db}}[a := \langle B^*, a\rangle] \arrow[r, "|\ |^{\mathrm{db}}_{\mathrm{tf}}"] & |M^*|^{\mathrm{full}}_{\mathrm{tf}}[a := \langle a\rangle] \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] & |M^*|^{\mathrm{full}}_{\mathrm{cu}}
\end{tikzcd}$$

4. Case of $\langle B, C\rangle^A::\mathrm{pair}_{\mathsf{T}}(2,1)$:

$$\begin{tikzcd}
C^\sharp[[\ ]^{A^\sharp} B^\sharp] \arrow[r, "|\ |^{\mathrm{full}}_{\mathrm{db}}"] & |C^\sharp|^{\mathrm{full}}_{\mathrm{db}}[[\ ]B^\sharp] \arrow[r, "|\ |^{\mathrm{db}}_{\mathrm{tf}}"] & |C^\sharp|^{\mathrm{full}}_{\mathrm{tf}}[[\ ][]] \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] & |C^\sharp|^{\mathrm{full}}_{\mathrm{cu}} \\
\langle B, C\rangle^A \arrow[r, "|\ |^{\mathrm{full}}_{\mathrm{db}}"] \arrow[u, "\sharp^{\mathrm{full}}"'] & \langle B, |C|^{\mathrm{full}}_{\mathrm{db}}\rangle \arrow[r, "|\ |^{\mathrm{db}}_{\mathrm{tf}}"] \arrow[u, "\sharp^{\mathrm{db}}"'] & \langle |C|^{\mathrm{full}}_{\mathrm{tf}}\rangle \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] \arrow[u, "\sharp^{\mathrm{tf}}"'] & |C|^{\mathrm{full}}_{\mathrm{cu}} \arrow[u, "\sharp^{\mathrm{cu}}"']
\end{tikzcd}$$

Figure 2: Proposition 7: Commutativity of the translations $*, \sharp$ and erasure $|\ |$

$$\begin{array}{ccc}
\Gamma \vdash_{t\text{-}\lambda 2} N : A \text{ for } \exists\ t\text{-style } N \text{ s.t. } |N| = M & \xleftarrow{\text{lifting } |\ |^{-1}} & \Gamma \vdash_{s\text{-}\lambda 2} M : A \\
\text{CPS-trans. } *^t \big\downarrow & & \big\downarrow (a) \\
\neg\Gamma^*, a{:}A^* \vdash_{t\text{-}\lambda^\exists} N^* : \bot & \xrightarrow{\text{erasing } |\ |} & \neg\Gamma^*, a{:}A^* \vdash_{s\text{-}\lambda^\exists} |N^*| : \bot
\end{array}$$

Figure 3: Theorem 1(1): Soundness of s-style $\lambda 2$ via t-style together with lifting, $*^t$, erasing, and the commutativity. On the arrow (a), we have $M^{*^s} = (|N|^t_s)^{*^s} = |N^{*^t}|^t_s$.

$$\begin{array}{ccc}
(\Sigma_1^f)^\sharp \vdash_{t\text{-}\lambda 2} Q^\sharp : (A^f)^\sharp & \xrightarrow{|\ |} & (\Sigma_1^f)^\sharp \vdash_{s\text{-}\lambda 2} |Q^\sharp| : (A^f)^\sharp \\
\text{Inverse } \sharp^t \big\uparrow & & \big\uparrow (b) \\
\neg\Sigma_1^f, a{:}A^f \vdash_{t\text{-}\lambda^\exists} Q : \bot \text{ for } \exists\ t\text{-style } Q \text{ s.t. } |Q| = P & \xleftarrow{|\ |^{-1}} & \Sigma_1, a{:}A \vdash_{s\text{-}\lambda^\exists} P : \bot
\end{array}$$

Figure 4: Theorem 1(4): Completeness of s-style $\lambda 2$ via t-style together with lifting by CPS-term Q, $\sharp^t$, erasing, and the commutativity; see also Appendix (A) for the forcing function f. The arrow (b) has $P^{\sharp^s} = (|Q|^t_s)^{\sharp^s} = |Q^{\sharp^t}|^t_s$.

(2) *If* $\Gamma, a : A \vdash_{s\text{-}\lambda^\exists} C : B$ *in s-style* $\lambda^\exists$, *then there exist a* CPS-*term* D *in the full style, CPS-types* A', B', *and* Γ' *consisting of CPS-types such that* $|D|^{\text{full}}_s = C$ *and* $\neg\Gamma', a{:}A' \vdash_{\text{full}\lambda^\exists} D : B'$.

Proof. By induction on the derivations. See also Appendix (A) for the details. □

Finally, the inverse translation $\sharp$ works only for CPS-types denoted by A^*, B^*, and Γ^*.

Proposition 9 (Translation $\sharp$ from full $\lambda^\exists$ back to full $\lambda 2$)

(1) *If* $\neg\Gamma^*, a{:}A^* \vdash_{\text{full}\lambda^\exists} P : \bot$ *then* $(\Gamma^*)^\sharp \vdash_{\text{full}\lambda 2} P^\sharp : (A^*)^\sharp$.

(2) *If* $\neg\Gamma^*, a{:}A^* \vdash_{\text{full}\lambda^\exists} C : B^*$ *then* $(\Gamma^*)^\sharp, x{:}(B^*)^\sharp \vdash_{\text{full}\lambda 2} C^\sharp[x] : (A^*)^\sharp$, *where* x *is fresh.*

Proof. By induction on the derivations. See Appendix (B). □

Note that an inverse of erasure $|\ |^t_s$ is called lifting, denoted by $(|\ |^t_s)^{-1}$; and erasing $|\ |^t_s$ and lifting $(|\ |^t_s)^{-1}$ provide homomorphisms from t-style to s-style, and vice versa. The composition of lifting and erasing (*not* erasing and lifting) constitutes the isomorphism. Now, under the framework, see Fig. 3 and Fig. 4, the soundness and completeness of s-style $\lambda 2$ are established by lifting, the soundness and completeness of the full style, erasing, and the commutativity. This idea is applied to connect type-related problems parametrized with styles by the following theorem.

Theorem 1 (1) *For any* $s \geq$ Curry, *if* $\Gamma \vdash_{s\text{-}\lambda 2} M : A$ *then* $\neg\Gamma^*, a{:}A^* \vdash_{s^*\text{-}\lambda^\exists} M^* : \bot$.

(2) *For any style* $s^* \in \mathsf{CUR}^*$, *if* $\neg\Gamma^*, a : A^* \vdash_{s^*\text{-}\lambda^\exists} M^* : \bot$ *then we have* $\Gamma \vdash_{s\text{-}\lambda 2} M : A$.

(3) *For any style* $s \in \mathsf{CUR}$ *with style* $@(n, 0)^2$, *we have* $\Gamma \vdash_{s\text{-}\lambda 2} M : A$ *for some type* A *if and only if* $\neg\Gamma^* \vdash_{s^*\text{-}\lambda^\exists} \lambda a.M^* : B$ *for some type* B.

(4) *For any style* $s \in \mathsf{CUR}$, *we have* $\Gamma \vdash_{s\text{-}\lambda 2} M : A$ *for some context* Γ *and type* A *if and only if* $\Sigma \vdash_{s^*\text{-}\lambda^\exists} M^* : B$ *for some context* Σ *and type* B.

Proof. (1) Suppose that $\Gamma \vdash M : A$ in s-style $\lambda 2$. Then, by Proposition 5 (lifting), there exists a full $\lambda 2$-term N, such that $|N|_s^{\text{full}} = M$ and $\Gamma \vdash N : A$ in full $\lambda 2$. Thus, from Proposition 6, $\neg\Gamma^*, a\!:\!A^* \vdash N^{*^{\text{full}}} : \bot$ in full $\lambda^\exists$. Hence, by erasing, $\neg\Gamma^*, a\!:\!A^* \vdash |N^{*^{\text{full}}}|_s^{\text{full}} : \bot$ in the s-style $\lambda^\exists$, where $|N^{*^{\text{full}}}|_s^{\text{full}} = (|N|_s^{\text{full}})^{*^s} = M^{*^s}$ by Proposition 7, see also Fig. 3.
(2) Suppose that $\neg\Gamma^*, a : A^* \vdash M^{*^s} : \bot$ in s-style $\lambda^\exists$. Then, from Proposition 8, there exists a CPS-term, to say Q in full $\lambda^\exists$, such that $|Q|_s^{\text{full}} = M^{*^s}$ and $\neg(\neg\Gamma^*)^f, a\!:\!(A^*)^f \vdash Q : \bot$ in full $\lambda^\exists$, where $\neg(\neg\Gamma^*)^f = \neg\Gamma^*$ and $(A^*)^f = A^*$. See also Appendix (A) for the function f forcing non CPS-types into CPS-types. Hence, from Proposition 9, we have $(\Gamma^*)^\sharp \vdash Q^{\sharp^{\text{full}}} : (A^*)^\sharp$ in full $\lambda 2$, where $(A^*)^\sharp = A$ and $(\Gamma^*)^\sharp = \Gamma$. Therefore, by erasing, we obtain $\Gamma \vdash |Q^{\sharp^{\text{full}}}|_s^{\text{full}} : A$ in s-style $\lambda 2$, where $|Q^{\sharp^{\text{full}}}|_s^{\text{full}} = (|Q|_s^{\text{full}})^{\sharp^s} = (M^{*^s})^{\sharp^s} = M$ by Proposition 7.
(3) Similarly to the above. For domain-free abstraction $\lambda a.M^*$ with style $\lambda(0)$, it is enough to have style $@(n, 0)$ from Table 1.
(4) ($\Rightarrow$) is the same as done in (1) above.
($\Leftarrow$): Suppose that there exist Σ and B such that $\Sigma \vdash M^* : B$ in s-style $\lambda^\exists$. From the definition of M^*, we should have $B = \bot$ and $\Sigma = \Sigma_1, a\!:\!A$ for some Σ_1 and A^3, such that $\Sigma_1, a\!:\!A \vdash M^* : \bot$ in s-style $\lambda^\exists$. Thus the same method used in (2) above proves this part, as shown in Fig. 4. □

Corollary 2 *The CPS-translation is an order-embedding with respect to the order on styles* $s \in \mathsf{CUR}$.

Proof. Let $s \in \mathsf{CUR}$. From Theorem 1 (1,2) and Table 1, an s-style $\lambda 2$ is embedded into s^*-style $\lambda^\exists$ such that $\Gamma \vdash_{s\text{-}\lambda 2} M : A$ iff $\neg\Gamma^*, a\!:\!A^* \vdash_{s^*\text{-}\lambda^\exists} M^* : \bot$. Let $s \leq t$. Then t-style $\lambda 2$ is embedded into t^*-style $\lambda^\exists$ as well with $s^* \leq t^*$. □

4.4 Application to fundamental properties preserved under the translations: decidability correspondence between problems

As a by-product, decidability of the following type-related problems between s-style $\lambda 2$ and s-style $\lambda^\exists$ is preserved by Theorem 1.

Definition 9 (TCP(s), TIP(s), TPP(s))

[2] See Table 1, where style $@(n, 0)$ corresponds to domain-free style $\lambda(0)$.

[3] In general, Σ_1, A may not consist only of CPS-types, but this is overcome by the forcing function.

(1) *Type checking problem of s-style terms* TCP(s) : *Given an s-style* λ*-term* M*, a type* A*, and a context* Γ*, determine whether* $\Gamma \vdash_s M : A$.

(2) *Type inference problem of s-style* λ*-terms* (TIP(s)) : *Given an s-style* λ*-term* M *and a context* Γ*, determine whether* $\Gamma \vdash_s M : A$ *for some type* A.

(3) *Typability problem of s-style terms* (TPP(s)): *Given an s-style* λ*-term* M*, determine whether* $\Gamma \vdash_s M : A$ *for some context* Γ *and type* A.

For any style $s \in \mathsf{CUR}$, TCP(s) follows Theorem 1(1)(2); for style s with $@(n,0)$, TIP(s) follows Theorem 1(3); and for style $s \in \mathsf{CUR}$, TPP(s) follows Theorem 1(4). Therefore, the decidability relationship between type-related problems are summarized as follows.

Proposition 10 (Decidability correspondence between $\lambda 2$ and $\lambda^{\exists}$) *1. For any style* $s \in \mathsf{CUR}$*, the undecidable results of* TCP(s) *for s-style* $\lambda 2$ *imply those for* s^**-style* $\lambda^{\exists}$*. In turn, the decidable results of* TCP(s^*) *for* s^**-*$\lambda^{\exists}$ *imply those for s-*$\lambda 2$*.*

2. For any style $s \in \mathsf{CUR}$ *with* $@(n,0)$*, the undecidable results of* TIP(s) *for s-style* $\lambda 2$ *imply those for* s^**-style* $\lambda^{\exists}$*.*

3. For any style $s \in \mathsf{CUR}$*, the undecidable results of* TPP(s) *for s-style* $\lambda 2$ *imply those for* s^**-style* $\lambda^{\exists}$*.*

Not only already known examples but also new ones follow Proposition 10. For example, the undecidable results of TCP(df-$\lambda 2$) in Fujita and Schubert [8] and TCP(tf-$\lambda 2$) [9] can be applied to show the corresponding results of TCP(df-$\lambda^{\exists}$) and TCP(tf-$\lambda^{\exists}$) respectively. Undecidability of TIP(df-$\lambda^{\exists}$) and TIP(tf-$\lambda^{\exists}$) are derived from that of the corresponding TIP(df-$\lambda 2$) [8] and TIP(tf-$\lambda 2$) [9]. The undecidable results of TPP(ha-$\lambda 2$), TPP(df-$\lambda 2$) [8], and TPP(tf-$\lambda 2$) [9] imply those of TPP(ha-$\lambda^{\exists}$), TPP(df-$\lambda^{\exists}$) in Nakazawa *et al.* [13], and TPP(tf-$\lambda^{\exists}$) respectively.

5 Concluding remarks

Fundamental properties are dependent on styles or representations of terms, and many formulations of terms are introduced and studied under various contexts. There have been a number of noteworthy investigations including, e.g., partial type-reconstruction by Pfenning [14]; explicit type scheme annotations by Odersky and Läufer [10]; bidirectional type-checking of predicative System F by Dunfield and Krishnaswami [5] and references therein.

In order to capture existing styles in a uniform way, we introduced styles of terms by giving abstract term-trees with indices, which present a bird's-eye view of not only existing systems but also new ones such as Curry^{+}-styles in Section 4.2. We note that TCP for a variant of Curry$^{+\Lambda(1,0)}$-$\lambda 2$ becomes undecidable by reducing the semi-unification problem following Wells [17], see Appendix (C) for the details.

As an application to decidability of type-related problems, it is worthwhile to investigate intermediate structures between decidable and undecidable systems. For this principal objective, the notion of fully annotated and partially annotated $\lambda 2$-terms based on styles is useful, and moreover, we introduced the counterpart systems partial $\lambda^{\exists}$ (2nd-order existential type systems) and the framework that handles both $\lambda 2$ and $\lambda^{\exists}$ families systematically by means of translations. The CPS-translation provides a natural interpretation from $\lambda 2$ into $\lambda^{\exists}$, such that type annotations of $\lambda 2$ correspond to domains of abstractions of $\lambda^{\exists}$. At the current stage, Theorem 1(2,3,4) excludes the Curry style, since the key proposition Proposition 8 (Lifting CPS-terms) would become the most involved for the Curry-style $\lambda^{\exists}$ with the style $\mathsf{let}_{\mathsf{T}}(0,0)$. Further studies are needed for this case.

As further work, the notion of styles should be extended to systems with deponent types, and the promising is the application to reduction properties, e.g., the Church-Rosser property is challenging.

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Appendix

A Proposition 8 (Lifting CPS-terms and types for style $t \in \mathsf{CUR}^*$)

(1) If $\Gamma, a{:}A \vdash_{t\text{-}\lambda^{\exists}} P : \bot$ in t-style $\lambda^{\exists}$, then there exist a CPS-term Q in the full style, a CPS-type A', and a context Γ' consisting of CPS-types such that $|Q|_t^{\text{full}} = P$ and $\neg\Gamma', a{:}A' \vdash_{\text{full}\lambda^{\exists}} Q : \bot$.

(2) If $\Gamma, a{:}A \vdash_{t\text{-}\lambda^{\exists}} C : B$ in t-style $\lambda^{\exists}$, then there exist a CPS-term D in the full style, a CPS-type B', and a context Γ' consisting of CPS-types such that $|D|_t^{\text{full}} = C$ and $\neg\Gamma', a{:}A' \vdash_{\text{full}\lambda^{\exists}} D : B'$.

Proof. First we define a function that forces non CPS-types into CPS-types, as follows:

1. $X^f = X$;

2. $(\neg A)^f = A^f$;
3. $(\exists X.A)^f = \exists X.A^f$;
4. $(A \wedge B)^f = \neg A^f \wedge B^f$;
5. $\perp^f = Z$ where Z is a fixed and fresh type variable.

Note that we have $A^f = A$ for any CPS-type A. Now we prove the following statement by induction on the derivations.

(1) If $\Gamma, a\!:\!A \vdash_{t\text{-}\lambda^\exists} P : \perp$ in t-style $\lambda^\exists$, then there exists a CPS-term Q in the full style such that $|Q|_t^{\mathsf{full}} = P$ and $\neg\Gamma^f, a\!:\!A^f \vdash_{\mathsf{full}\lambda^\exists} Q : \perp$.

(2) If $\Gamma, a\!:\!A \vdash_{t\text{-}\lambda^\exists} C : B$ in t-style $\lambda^\exists$, then there exists a CPS-term D in the full style such that $|D|_t^{\mathsf{full}} = C$ and $\neg\Gamma^f, a\!:\!A^f \vdash_{\mathsf{full}\lambda^\exists} D : B^f$.

Note that for style $t \geq$ tf, the system t-$\lambda^\exists$ is so-called syntax directed, for instance see Proposition 3 (Generation lemma), so that the lifting lemma holds naturally. Moreover, the lemma holds for systems with style $t \geq \mathrm{Curry}^{+\mathsf{let}_\mathsf{T}(1,0)}$ as well. In addition, $\mathrm{Curry}^{+\mathsf{let}_\mathsf{T}(0,1)}$-$\lambda^\exists$ also enjoys the property. We show some of the cases here.

1. Case P of (xC); $\mathsf{var}(0)$ and $@(0)$:

$$\frac{\Gamma, x\!:\!\neg B \vdash x : \neg B \quad \Gamma, x\!:\!\neg B, a\!:\!A \vdash C : B}{\Gamma, x\!:\!\neg B, a\!:\!A \vdash xC : \perp}\ (\neg E)$$

Note that $\neg(\neg B)^f = \neg B^f$ for any $\lambda^\exists$-type B. From the induction hypothesis, we have a CPS-term D in the full style such that $|D| = C^f$ and $\neg\Gamma^f, x : \neg B^f, a : A^f \vdash D : B^f$. Hence, $\neg\Gamma^f, x : \neg B^f, a : A^f \vdash xD : \perp$ by $(\neg E)$, where $|xD| = xC$.

2. Case of ($\mathtt{let}\ \langle x, a\rangle = C\ \mathtt{in}\ P$)::$\mathsf{let}(0,0)$:

$$\frac{\Gamma, a\!:\!A \vdash C : A_1 \wedge B \quad \Gamma, x\!:\!A_1, a\!:\!B \vdash P : \perp}{\Gamma, a\!:\!A \vdash \mathtt{let}\ \langle x, a\rangle = C\ \mathtt{in}\ P : \perp}\ (\wedge E)$$

From the induction hypotheses, we have CPS-terms D and Q in the full style such that $\neg\Gamma^f, a\!:\!A^f \vdash D : (A_1 \wedge B)^f$ and $\neg\Gamma^f, x\!:\!\neg A_1^f, a\!:\!B^f \vdash Q : \perp$, together with $|D| = C$ and $|Q| = P$, where $(A_1 \wedge B)^f = \neg A_1^f \wedge B^f$. Thus, $\neg\Gamma^f, a : A^f \vdash \mathtt{let}\ \langle x\!:\!\neg A_1^f, a\!:\!B^f\rangle = D\ \mathtt{in}\ Q : \perp$ by $(\wedge E)$, where $|\mathtt{let}\ \langle x\!:\!\neg A_1^f, a\!:\!B^f\rangle = D\ \mathtt{in}\ Q| = (\mathtt{let}\ \langle x, a\rangle = C\ \mathtt{in}\ P)$.

3. Case of ($\mathtt{let}\ \langle a\rangle = C\ \mathtt{in}\ P$)::$\mathsf{let}_\mathsf{T}(1,0)$:

$$\frac{\Gamma, a\!:\!A \vdash C : \exists X.B \quad \Gamma, a\!:\!B \vdash P : \perp}{\Gamma, a\!:\!A \vdash \mathtt{let}\ \langle a\rangle = C\ \mathtt{in}\ P : \perp}\ (\exists E)$$

From induction hypotheses, we have CPS-terms D and Q in the full style such that $\neg\Gamma^f, a\!:\!A^f \vdash D : \exists X.B^f$ and $\neg\Gamma^f, a\!:\!B^f \vdash Q : \perp$, together with $|D| = C$ and $|Q| = P$. Hence, $\neg\Gamma^f, a\!:\!A^f \vdash \mathtt{let}\ \langle X, a\!:\!B^f\rangle = D\ \mathtt{in}\ Q : \perp$ by $(\exists E)$, where $|\mathtt{let}\ \langle X, a\!:\!B^f\rangle = D\ \mathtt{in}\ Q| = (\mathtt{let}\ \langle a\rangle = C\ \mathtt{in}\ P)$.

4. Case of (`let` $\langle X, a\rangle = C$ `in` P)::$\mathsf{let_T}(2,0)$ follows the same pattern as the above.

5. Case C of a::$\mathsf{var}(0)$ is verified by $\neg\Gamma^f, a{:}A^f \vdash a : A^f$ by using (var).

6. Case of $\langle\lambda a.P, C\rangle$::$\mathsf{pair}(0)$:

$$\dfrac{\dfrac{\Gamma, a{:}A_1 \vdash P : \bot}{\Gamma \vdash \lambda a.P : \neg A_1}(\neg I) \quad \Gamma, a{:}A \vdash C : B}{\Gamma, a{:}A \vdash \langle\lambda a.P, C\rangle : \neg A_1 \wedge B}(\wedge I)$$

From the induction hypotheses, we have CPS-terms D and Q in the full style such that $\neg\Gamma^f, a{:}A_1^f \vdash Q : \bot$ and $\neg\Gamma^f, a{:}A^f \vdash D : B^f$, together with $|D| = C$ and $|Q| = P$. Thus, $\neg\Gamma^f, a{:}A^f \vdash \langle\lambda a{:}A_1^f.Q, D\rangle : (\neg A_1 \wedge B)^f$ by $(\neg I)$ and $(\wedge I)$, where $(\neg A_1 \wedge B)^f = \neg A_1^f \wedge B^f$, and $|\langle\lambda a{:}A_1^f.Q, D\rangle| = \langle\lambda a.P, C\rangle$.

7. Case of C::$\mathsf{pair_T}(0,0)$:

$$\dfrac{\Gamma, a{:}A \vdash C : B[X := A_1]}{\Gamma, a{:}A \vdash C : \exists X.B}(\exists I)$$

From the induction hypothesis, we have a CPS-term D in the full style such that $\neg\Gamma^f, a{:}A^f \vdash D : (B[X := A_1])^f$, together with $|D| = C$, where $(B[X := A_1])^f = B^f[X := A_1^f]$, provided that the variable X is different from the distinguished variable Z. Hence, $\neg\Gamma^f, a{:}A^f \vdash \langle A_1^f, D\rangle^{\exists X.B^f} : \exists X.B^f$ by $(\exists E)$, where $|\langle A_1^f, D\rangle^{\exists X.B^f}| = |D| = C$.

8. Cases of $\langle C\rangle$::$\mathsf{pair_T}(1,0)$ and $\langle A_1, C\rangle$::$\mathsf{pair_T}(2,0)$:

9. Case of $\mathrm{Curry}^{+\mathsf{let_T}(0,1)}$-$\lambda^{\exists}$, in particular, $(\exists E)$ with style $\mathsf{let_T}(0,1)$:

   Since variable a is a linear variable, into which a CPS-term attached with a type is substituted, one can decompose the judgement $\Gamma, a{:}A \vdash P[a := C^B] : \bot$ into $\Gamma, a{:}A \vdash C : \exists X.B$ and $\Gamma, a{:}B \vdash P : \bot$; and the judgement $\Gamma, a{:}A_1 \vdash C_2[a := C_1^{A_2}] : B$ into $\Gamma, a{:}A_1 \vdash C_1 : \exists X.A_2$ and $\Gamma, a{:}A_2 \vdash C_2 : B$. Then follow the same pattern as the above[4]. □

B Proposition 9 (Translation $\sharp$ from full $\lambda^{\exists}$ back to full $\lambda 2$)

Let A^*, B^* be CPS-types and Γ^* be a context consisting of CPS-types.

(1) If $\neg\Gamma^*, a{:}A^* \vdash_{\mathsf{full}\lambda^{\exists}} P : \bot$ then $(\Gamma^*)^\sharp \vdash_{\mathsf{full}\lambda 2} P^\sharp : (A^*)^\sharp$.

[4]Although the decomposition may not be unique, e.g., take $P[a := C_1^{B_1}[a := C_2^{B_2}]]$, each decomposition can be related by the so-called permuted conversion or structural reduction [15].

(2) If $\neg\Gamma^*, a\!:\!A^* \vdash_{\mathsf{full}\lambda^\exists} C : B^*$ then $(\Gamma^*)^\sharp, x\!:\!(B^*)^\sharp \vdash_{\mathsf{full}\lambda 2} C^\sharp[x] : (A^*)^\sharp$, where x is fresh.

Proof. By induction on the derivations. We show some of the cases here.

1. Case of ($\mathtt{let}\ \langle x\!:\!\neg A_1^*, a\!:\!A_2^*\rangle = C\ \mathtt{in}\ P$):

$$\frac{\neg\Gamma^*, a\!:\!A^* \vdash C : \neg A_1^* \wedge A_2^* \quad \neg\Gamma^*, x\!:\!\neg A_1^*, a\!:\!A_2^* \vdash P : \bot}{\neg\Gamma^*, a\!:\!A^* \vdash \mathtt{let}\ \langle x\!:\!\neg A_1^*, a\!:\!A_2^*\rangle = C\ \mathtt{in}\ P : \bot}\ (\wedge E)$$

   From the induction hypotheses, we have $\Gamma^{*\sharp}, x\!:\!(A_1^{*\sharp} \to A_2^{*\sharp}) \vdash C^\sharp[x] : A^{*\sharp}$ and $\Gamma^{*\sharp}, x\!:\!A_1^{*\sharp} \vdash P^\sharp : A_2^{*\sharp}$. Then $\Gamma^{*\sharp} \vdash \lambda x\!:\!A_1^{*\sharp}.P^\sharp A_2^{*\sharp} : A_1^{*\sharp} \to A_2^{*\sharp}$, and $\Gamma^{*\sharp} \vdash C^\sharp[\lambda x\!:\!A_1^{*\sharp}.P^\sharp A_2^{*\sharp}] : A^{*\sharp}$.

2. Case of $\langle A_2^*, C\rangle^{\exists X.A_1^*}$:

$$\frac{\neg\Gamma^*, a\!:\!A^* \vdash C : A_1^*[X := A_2^*]}{\neg\Gamma^*, a\!:\!A^* \vdash \langle A_2^*, C\rangle^{\exists X.A_1^*} : \exists X.A_1^*}\ (\exists I)$$

   From the induction hypothesis, we have $\Gamma^{*\sharp}, x\!:\!(A_1^*[X := A_2^*])^\sharp \vdash C^\sharp[x] : A^\sharp$. And we also have $x : \forall X.A_1^{*\sharp} \vdash x^{\forall X.A_1^{*\sharp}} A_2^{*\sharp} : A_1^{*\sharp}[X := A_2^{*\sharp}]$, where $A_1^{*\sharp}[X := A_2^{*\sharp}] = (A_1^*[X := A_2^*])^\sharp$.

Hence, $\Gamma^{*\sharp}, x\!:\!\forall X.A_1^{*\sharp} \vdash C^\sharp[x^{\forall X.A_1^{*\sharp}} A_2^{*\sharp}] : A^{*\sharp}$ with $C^\sharp[x^{\forall X.A_1^{*\sharp}} A_2^{*\sharp}] = (\langle A_2^*, C\rangle^{\exists X.A_1^*})^\sharp[x]$.
□

C TCP for a variant of Curry$^{+\Lambda(1,0)}$-$\lambda 2$ is undecidable

We write $\vec{\forall}.A$ for the universal closure of A, i.e., $\vec{\forall}.A = \forall X_1 \ldots X_n.A$ for $\mathrm{FV}(A) = \{X_1, \ldots, X_n\}$, and accordingly put the following rule $(\vec{\forall} I)$ with style $\Lambda(1,0)$ to $\lambda 2$:

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \vec{\Lambda}.M : \vec{\forall}.A}\ (\vec{\forall} I) :: \Lambda(1,0)$$

where $X \notin \mathrm{FV}(\Gamma)$ for each $X \in \mathrm{FV}(A)$. Let A_1, A_2, B_1, B_2 be $\lambda 2$-types, and X, X_1, X_2, Y be fresh type variables. Then, as done in Wells [17], the semi-unification problem (SUP) is reduced to TCP of the variant of Curry$^{+\Lambda(1,0)}$-$\lambda 2$.

Proposition 11 *An instance of SUP $\{A_1 \le B_1, A_2 \le B_2\}$ has a solution if and only if $b : \forall X.(X \to X) \to Y, c : \vec{\forall}.(B_1 \to X_1) \to (X_2 \to B_2) \to (X_1 \to X_2) \vdash b(\lambda x.\vec{\Lambda}.cxx) : Y$ in Curry$^{+\Lambda(1,0)}$-$\lambda 2$.*

Proof. Following the proof of Theorem 4.1 in Wells [17]. □

This proposition implies that TCP for the variant of Curry$^{+\Lambda(1,0)}$-$\lambda 2$ is undecidable, and that of the corresponding Curry$^{+\mathsf{let}_\mathsf{T}(1,0)}$-$\lambda^\exists$ also becomes undecidable from Proposition 10.