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3-dimensional approaches with 3D-GRAPES in high school mathematics

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1 What is 3D-GRAPES?

3D-GRAPES is a free function graphing software made by Katsuhisa TOMODA. GRAPES is an abbreviation for graph presentation and experiment system. 3D-GRAPES is a three-dimensional version of GRAPES. We can get it from the following homepage: `http://www.criced.tsukuba.ac.jp/grapes/`

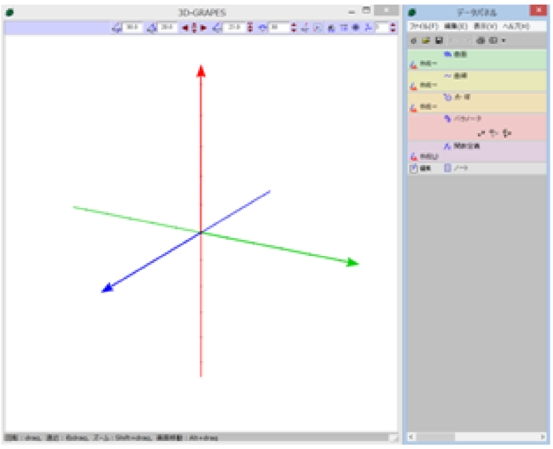

Figure 1. The opening scene of 3D-GRAPES

Figure 1 above is the opening scene of 3D-GRAPES. It has a graphic window and a data panel. We can draw surfaces, lines, points, spheres, and graphs of functions. We made some crystal structures by drawing spheres. (Figure2)

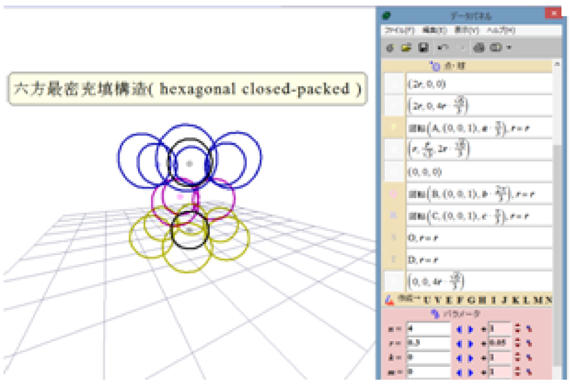

Figure 2. A hexagonal closed-packed structure

2 Visualization of Fibonacci sequence by 3D-GRAPES

Fibonacci sequence $\{F_n\}_{n=1,2,\cdots}$ is a sequence of integers which satisfies the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, \qquad F_1 = F_2 = 1$$

First few numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$. Let φ be the number $\dfrac{1+\sqrt{5}}{2}$. φ satisfies the quadratic equation $\varphi^2 = \varphi + 1$, and it is called the golden ratio. Ratio of consecutive Fibonacci numbers converges to the golden ratio φ. Let 's visualize this by 3D-GRAPES. Let P_n be the point (F_n, F_{n+1}, F_{n+2}). $\displaystyle\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$ implies $\dfrac{F_{n+1}}{F_n} \sim \varphi$ and $\dfrac{F_{n+2}}{F_n+1} \sim \varphi$ for large n. So sequence of points P_n should be on the line $\varphi x = y = \dfrac{z}{\varphi}$ for large n. In fact, points P_n look like on a line for even small n. (Figure3)

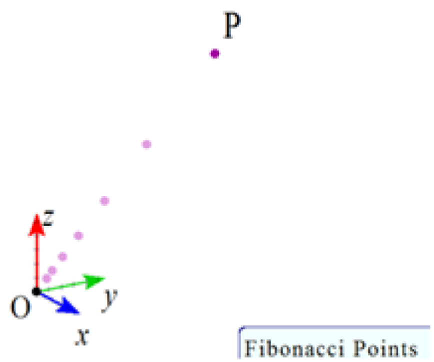

Figure 3. Points P_n

How can we show students the sequence $\dfrac{F_{n+1}}{F_n}$ converges to φ visually? We used the logarithm with base φ. Let Q_n be the point $(\log_\varphi F_n, \log_\varphi F_{n+1}, \log_\varphi F_{n+2})$. $\dfrac{F_{n+1}}{F_n} \sim \varphi$ and $\dfrac{F_{n+2}}{F_{n+1}} \sim \varphi$ hold for large n. This implies $\log_\varphi F_{n+1} - \log_\varphi F_n \sim \log_\varphi \varphi = 1$ and $\log_\varphi F_{n+2} - \log_\varphi F_{n+1} \sim \log_\varphi \varphi = 1$. So points Q_n are nearly on the line $x = y-1 = z-2$ for large n. (Figure4)

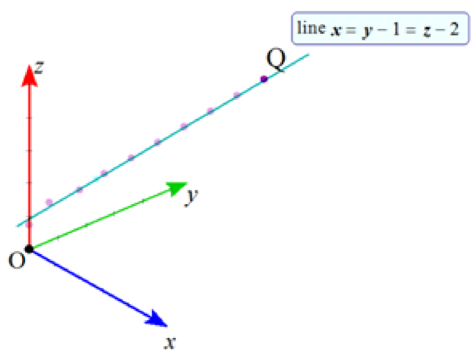

Figure 4. Points Q_n and the line $x = y - 1 = z - 2$

3 Visualization of inverse function by 3D-GRAPES

In pre-calculus we teach students $f^{-1}(f(x)) = x$. In this chapter, we want to show $f^{-1}(f(x)) = x$ visually. We will show it in the case $f(x) = \sqrt{x}$. Let $t = f(x) = \sqrt{x}$. Then $f^{-1}(x) = x^2$. We use t-axis instead of z-axis. (Figure5)

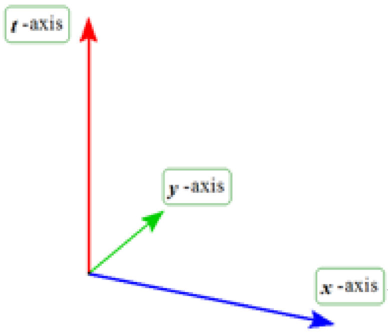

Figure 5. x-axis, y-axis and t-axis

Let A be any point on x-axis. Let B be the point on the graph $t = \sqrt{x}$ such that the x-coordinates of A and B are the same. Let C be the point on t-axis such that the t-coordinates of B and C are the same. (Figure 6,7)

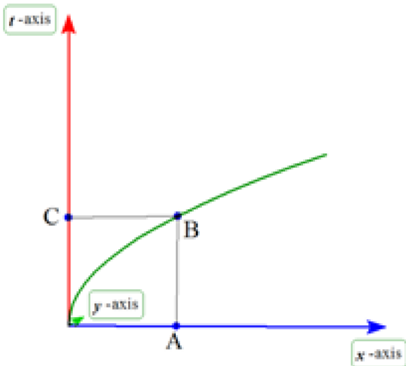

Figure 6. Graph of $t = f(x) = \sqrt{x}$ and points A, B, C

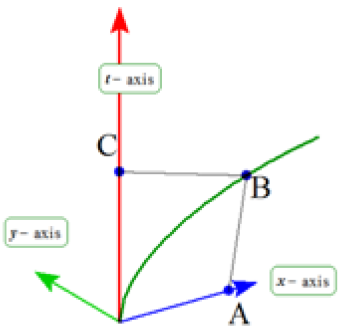

Figure 7. Graph of $t = f(x) = \sqrt{x}$ and points A, B, C (3-dimensional view)

Let $y = f^{-1}(t) = t^2$. Points D, E are chosen as in Figure 8.

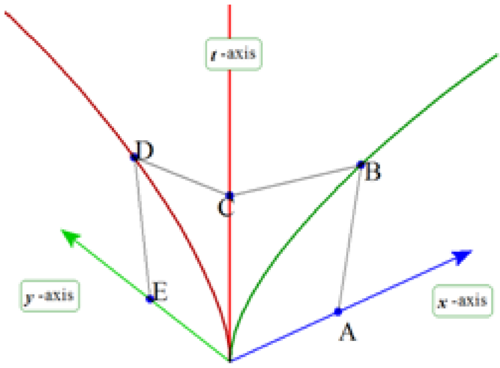

Figure 8. Graphs of $t = f(x) = \sqrt{x}$ and $y = f^{-1}(t) = t^2$

Let F be the point on xy-plane such that x-coordinates of A and F are the same and y-coordinates of E and F are the same.

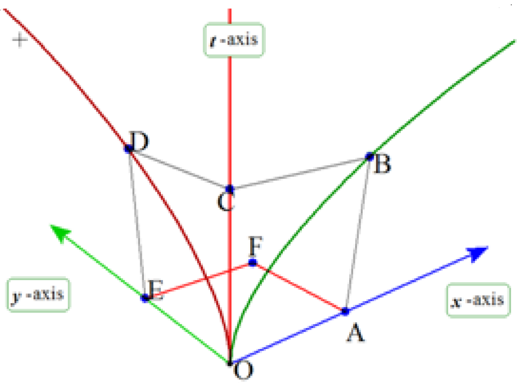

Figure 9. Point F on xy-plane

As figure OBA and ODE are congruent, so OA=OE. It means F is on the line $y = x$. Thus $f^{-1}(f(x)) = x$ holds for $f(x) = \sqrt{x}$. (Figure 10)

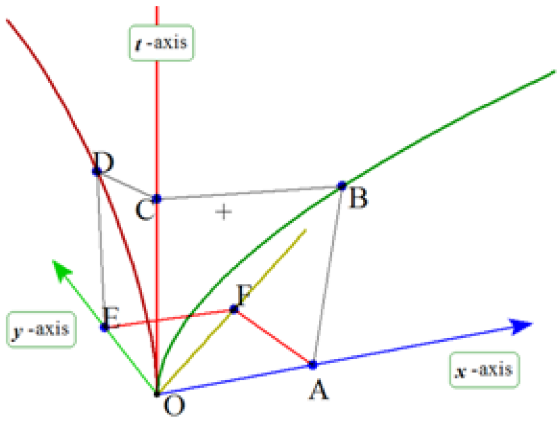

Figure 10. $f^{-1}(f(x)) = x$ holds for $f(x) = \sqrt{x}$

4 Visualization of limits of trigonometric functions by 3D-GRAPES

In this chapter, we visualize $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$. To prove this, we usually use the inequality $\sin\theta<\theta<\tan\theta$. But there is another way. Suppose that A be the point $(1,0,0)$, B be a point on xy-plane such that OB= 1. A and B are on the unit circle on xy-plane. Let $\angle$AOB= θ (rad). Then the coordinate of B is $(\cos\theta,\sin\theta,0)$ and $\hat{AB}=\theta$. Let C be the point $(\cos\theta,\sin\theta,\sin\theta)$, then C is right above B and on the plane $y=z$.(Figure 11)

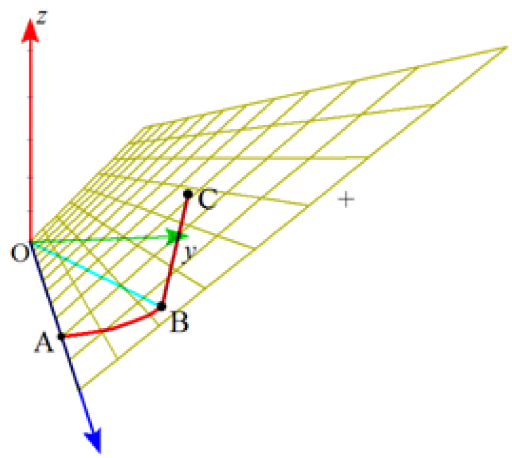

Figure 11. Plane $y=z$ and points A, B, C

If θ is very small, ABC looks like an isosceles triangle. So, if $\theta\sim 0$, then $\sin\theta\sim\theta$. (Figure 12)

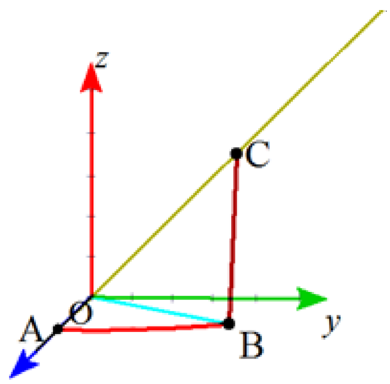

Figure 12. ABC looks like an isosceles triangle

5 Ellipse and hyperbola on xy-plane

In this chapter, we consider ellipse and hyperbola on xy-plane. Before drawing curves, let ' s consider the following problem.

Problem : Find the Maximum and minimum value of y such that

$$y = \left(2 - \frac{\sqrt{2}}{2}\right)\sin^2\theta + \sqrt{2}\sin\theta\cos\theta + \left(2 + \frac{\sqrt{2}}{2}\right)\cos^2\theta \quad (0 \leqq \theta < 2\pi)$$

The solution is as follows.

Solution: As

$$\begin{aligned} y &= \left(2 - \frac{\sqrt{2}}{2}\right)\sin^2\theta + \sqrt{2}\sin\theta\cos\theta + \left(2 + \frac{\sqrt{2}}{2}\right)\cos^2\theta \quad (0 \leqq \theta < 2\pi) \\ &= \left(2 - \frac{\sqrt{2}}{2}\right)\frac{1-\cos 2\theta}{2} + \frac{\sqrt{2}}{2}\sin 2\theta + \left(2 + \frac{\sqrt{2}}{2}\right)\frac{1+\cos 2\theta}{2} \\ &= 2 + \frac{\sqrt{2}}{2}\sin 2\theta + \frac{\sqrt{2}}{2}\cos 2\theta = 2 + \sin\left(2\theta + \frac{\pi}{4}\right) \end{aligned}$$

so maximum is 3 and minimum is 1.

As you see in the solution of the problem, if $x^2 + y^2 = 1$, the quadratic form can be written as

$$ax^2 + bxy + cy^2 = k + m\sin(2\theta + \alpha)$$

for some k, m, θ, α. We consider a curve in space such that

$$x = \cos\theta, \quad y = \sin\theta, \quad z = k + m\sin(2\theta + \alpha)$$

What does this curve look like? For example, curve

$$x = \cos\theta, y = \sin\theta, z = 2 + 2\sin\left(2\theta + \frac{\pi}{4}\right)$$

is shown in Figure 13.

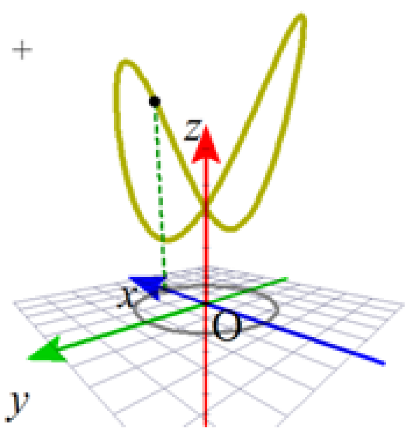

Figure 13. The curve $x = \cos\theta, y = \sin\theta, z = 2 + 2\sin\left(2\theta + \frac{\pi}{4}\right)$

And curve $x = \cos\theta, y = \sin\theta, z = 2 + \sin 2\theta$ is shown in Figure 14. By comparing two graphs, we can see if α changes, the curve rotates around the z-axis.

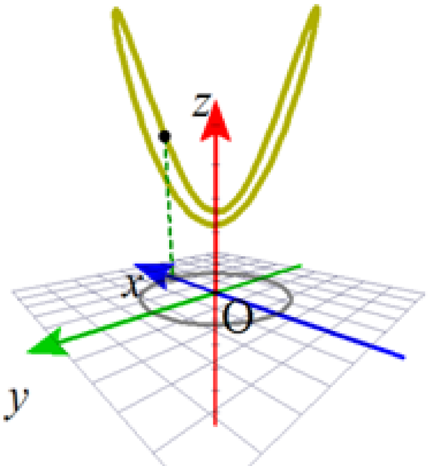

Figure 14. The curve $x = \cos\theta, y = \sin\theta, z = 2 + \sin 2\theta$

Furthermore curve $x = \cos\theta, y = \sin\theta, z = 1.6 + 2\sin 2\theta$ is shown in Figure 15. By comparing these graphs, we can see if k changes, the curve goes up or goes down.

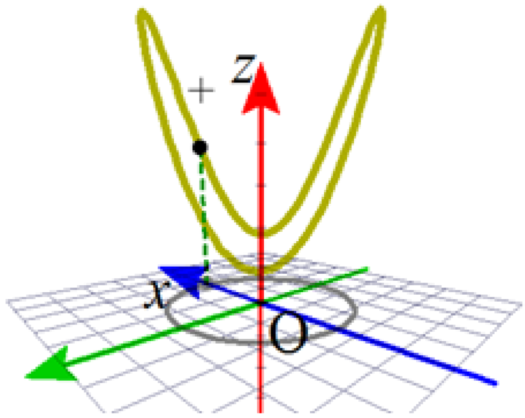

Figure 15. The curve $x = \cos\theta, y = \sin\theta, z = 1.6 + 2\sin 2\theta$

Let ' s draw the curve $ax^2 + bxy + cy^2 = 1$ on xy-plane. By substituting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$x = \frac{\cos\theta}{\sqrt{k + m\sin(2\theta + \alpha)}}, \quad y = \frac{\sin\theta}{\sqrt{k + m\sin(2\theta + \alpha)}}, \quad z = 0$$

For example, curve $x = \cos\theta, y = \sin\theta, z = 2 + \sin 2\theta$ and ellipse are shown in Figure 16.

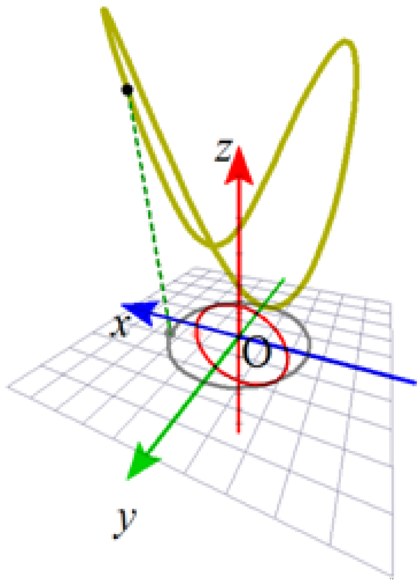

Figure 16. Curve and ellipse

When the z-coordinate of the point on the curve is minimum, the foot is on the major axis of the ellipse. And when the z-coordinate of the point on the curve is maximum, the foot is on the minor axis of the ellipse. What will happen if we decrease k?

As k decrease, the ellipse on the plane becomes larger. When the curve touched down, ellipse is no longer ellipse but two parallel lines.(Figure 17)

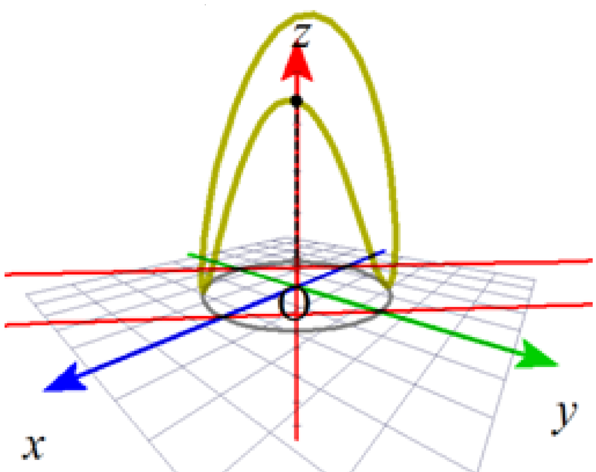

Figure 17. When the curve touched down

Furthermore, if there is a point on the curve under xy-plane, then $ax^2+bxy+cy^2=1$ is a hyperbola.(Figure 18)

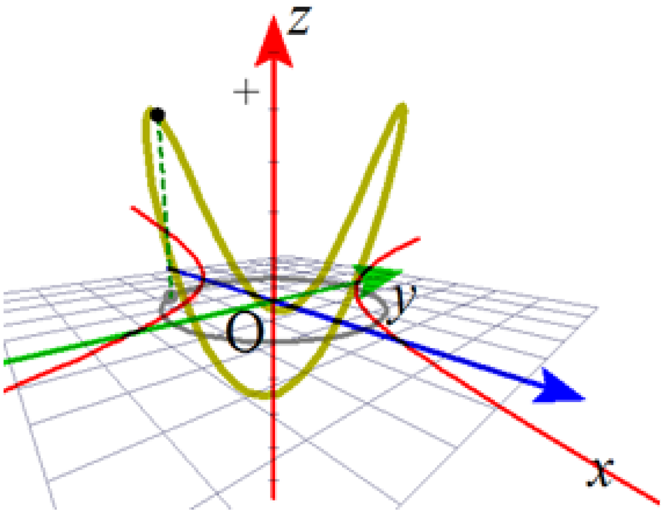

Figure 18. $ax^2 + bxy + cy^2 = 1$ is a hyperbola

Every time we demonstrate these changes of shape, they amaze audience. We are going to continue making 3D visualization of mathematical themes.

Reference

[1] Abraham Arcavi, The Educational Potential of Friendly Graphing Software: The case of GRAPES
`http://www.criced.tsukuba.ac.jp/grapes/doc/arcavi_en.pdf`

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