数理解析研究所講究録
第 1952 巻 2015 年 100-107

Folding and Skew-Unfolding of One-dimensional Continuous Semimartingales

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Abstract

In this short note first we review folding and unfolding of real-valued continuous semimartingales discussed in ICHIBA & KARATZAS (2014). We fold the continuous semimartingale in two ways : the conventional reflection and the SKOROKHOD reflection, in order to obtain a non-negative, continuous semimartingale. We unfold this folded process in a skew manner by assigning random signs independently of the folded process. Then as a byproduct, the stochastic differential equation is obtained corresponding to skew process. We examine its solvability, some properties of such equation, and then discuss some application of this method to planar processes.

Random perturbation can break symmetry of stochastic systems. It is often important to understand asymmetry or skewness of such system, in order to fill up some gap between the real world phenomena and mathematical theory. Here, given real-valued continuous semimartingale, following PROKAJ (2009), we shall discuss a method of generating continuous, skewed semimartingale.

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ with the so-called "usual conditions" of right continuity and augmentation by null sets, let us consider a real-valued continuous semimartingale $U(\cdot)$ of the form

$$U(t) = M(t) + A(t), \qquad 0 \le t < \infty \tag{1}$$

where $M(\cdot)$ is a continuous local martingale with quadratic variation $\langle M\rangle(\cdot)$, and $A(\cdot)$ is a process of finite first variation on compact intervals. We assume $M(0) = A(0) = 0$, $\langle M\rangle(\infty) = \infty$ for concreteness.

Let us "fold", or reflect, this real-valued semimartingale $U(\cdot)$ about the origin in the following two ways. One is the *conventional reflection*

$$R(t) := |U(t)|, \qquad 0 \le t < \infty; \tag{2}$$

the other is the SKOROKHOD *reflection*

$$S(t) := U(t) + \max_{0\le s\le t}\big(-U(s)\big), \qquad 0 \le t < \infty. \tag{3}$$

The following result, inspired by PROKAJ (2009), shows how the first can be obtained from the second, by suitably unfolding the SKOROKHOD reflection in a possibly "skewed" manner by assigning a random sign (positive with probability α, and negative with probability $1-\alpha$) for each excursion of $S(\cdot)$. It constructs a continuous semimartingale $X(\cdot)$ whose conventional reflection coincides with the SKOROKHOD reflection $S(\cdot)$ of the given semimartingale $U(\cdot)$, and which satisfies the stochastic integral equation, a *skew version* of the celebrated TANAKA equation driven by the continuous semimartingale $U(\cdot)$ with the "skew-unfolding" parameter $\alpha \in (0,1)$.

Proposition 1. [ICHIBA & KARATZAS (2014)] *Let us fix a constant* $\alpha \in (0,1)$ *and take the* SKOROKHOD *reflection* $S(\cdot)$ *in (3). There exists an enlargement* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}(t)\}_{0\le t<\infty}$

of the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ *with a measure-preserving map* $\pi : \Omega \to \widetilde{\Omega}$, *and on this enlarged space a continuous semimartingale* $X(\cdot)$ *that satisfies*

$$|X(\cdot)| = S(\cdot), \qquad L^X(\cdot) = \alpha L^S(\cdot), \quad X(\cdot) = \int_0^{\cdot} \overline{sgn}(X(t))\, \mathrm{d}U(t) + \frac{2\alpha - 1}{\alpha} L^X(\cdot). \quad (4)$$

Here, we use the notation for the *right* and the *symmetric* versions of local time at the origin of a continuous semimartingale as in (1),

$$L^U(\cdot) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\cdot} 1_{\{0\le U(t)<\varepsilon\}}\, \mathrm{d}\langle U\rangle(t), \qquad \widehat{L}^U(\cdot) := \frac{1}{2}\left(L^U(\cdot) + L^{-U}(\cdot)\right), \quad (5)$$

respectively, and the conventions

$$\overline{\mathrm{sgn}}(x) := \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0)}(x), \qquad \mathrm{sgn}(x) := \mathbf{1}_{(0,\infty)}(x) - \mathbf{1}_{(-\infty,0]}(x), \qquad x \in \mathbb{R}$$

for the symmetric and the left-continuous versions, respectively, of the signum function. We also denote by $\mathbb{F}^U = \{\mathcal{F}^U(t)\}_{0\le t<\infty}$ the "natural filtration" of $U(\cdot)$, that is, the smallest filtration that satisfies the usual conditions and with respect to which $U(\cdot)$ is adapted; we set $\mathcal{F}^U(\infty) := \sigma\big(\bigcup_{0\le t<\infty} \mathcal{F}^U(t)\big)$. Equalities between stochastic processes, such as in (4), are to be understood throughout in the almost sure sense.

When there is no skewness, i.e., $\alpha = 1/2$, it reduces the TANAKA equation (see PROKAJ (2009))

$$X(\cdot) = X(0) + \int_0^{\cdot} \overline{\mathrm{sgn}}(X(t))\, \mathrm{d}U(t)$$

We can find the skew unfolding for the conventional reflection.

Proposition 2. [ICHIBA & KARATZAS (2014)] *Fix a constant* $\alpha \in (0,1)$. *There exists an enlargement* $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, $\widehat{\mathbb{F}} = \{\widehat{\mathcal{F}}(t)\}_{0\le t<\infty}$ *of* $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$, *with a measure-preserving map* $\pi : \Omega \to \widehat{\Omega}$, *and on this enlarged space a continuous semimartingale* $\widehat{X}(\cdot)$ *that satisfies*

$$|\widehat{X}(\cdot)| = |U(\cdot)|, \qquad L^{\widehat{X}}(\cdot) = \alpha L^{|U|}(\cdot),$$

$$\widehat{X}(\cdot) = \widehat{X}(0) + \int_0^{\cdot} \overline{sgn}(\widehat{X}(t))\, \mathrm{d}\widehat{U}(t) + \frac{2\alpha - 1}{\alpha} L^{\widehat{X}}(\cdot);$$

Here

$$\widehat{U}(\cdot) := \int_0^{\cdot} \overline{sgn}(U(t))\, \mathrm{d}U(t)$$

is the LÉVY *transform of the semimartingale* $U(\cdot)$, *and the classical reflection* $R(\cdot) = |U(\cdot)|$ *of* $U(\cdot)$ *coincides with the* SKOROKHOD *reflection of the process* $\widehat{U}(\cdot)$, *namely*

$$\widehat{S}(t) := \widehat{U}(t) + \max_{0\le s\le t}\left(-\widehat{U}(s)\right), \qquad 0 \le t < \infty.$$

Example 1 (From One Skew Brownian Motion to Another). Suppose that $U(\cdot)$ is a skew BM with parameter $\gamma \in (0,1)$, i.e., (ITÔ & MCKEAN (1963), WALSH (1978), HARRISON & SHEPP (1981))

$$U(\cdot) = B(\cdot) + \frac{2\gamma - 1}{\gamma} L^U(\cdot)$$

for some standard, real-valued BM $B(\cdot)$. We have in this case $\int_0^\infty \mathbf{1}_{\{U(t)=0\}}\,\mathrm{d}t = 0$ as well as the local time property

$$2\,L^U(\cdot) - L^{|U|}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{U(t)=0\}}\,\mathrm{d}U(t) = \frac{2\gamma-1}{\gamma}\,L^U(\cdot)\,,$$

thus $L^U(\cdot) = \gamma L^{|U|}(\cdot)$ and therefore $R(\cdot) = |U(\cdot)| = \int_0^{\cdot} \overline{\mathrm{sgn}}(U(t))\,\mathrm{d}U(t) + L^{|U|}(\cdot) = W(\cdot) + L^{|U|}(\cdot)$.

Here we have denoted the LÉVY transform as

$$W(\cdot) := \widehat{U}(\cdot) = \int_0^{\cdot} \overline{\mathrm{sgn}}(U(t))\left(\mathrm{d}B(t) + \frac{2\gamma-1}{\gamma}\,\mathrm{d}L^U(t)\right) = \int_0^{\cdot} \mathrm{sgn}(U(t))\,\mathrm{d}B(t)\,,$$

and observed that it is another standard BM. Thus, the stochastic integral equation becomes

$$\widehat{X}(\cdot) = \int_0^{\cdot} \mathrm{sgn}(\widehat{X}(t))\,\mathrm{d}W(t) + \frac{2\alpha-1}{\alpha}\,L^{\widehat{X}}(\cdot) = \widehat{W}(\cdot) + \frac{2\alpha-1}{\alpha}\,L^{\widehat{X}}(\cdot)$$

with $\widehat{W}(\cdot) = \int_0^{\cdot} \mathrm{sgn}(\widehat{X}(t))\,\mathrm{d}W(t)$ yet another standard BM.

The HARRISON-SHEPP (1981) theory characterizes now $\widehat{X}(\cdot)$ as skew Brownian motion with skewness parameter α.

$$\mathcal{F}^{\widehat{U}}(\cdot) = \mathcal{F}^{W}(\cdot) = \mathcal{F}^{|U|}(\cdot) \subsetneq \mathcal{F}^{\widehat{X}}(\cdot) = \mathcal{F}^{\widehat{W}}(\cdot)\,.$$

Example 2 (Skew Bessel Processes). Suppose that $U^2(\cdot)$ is a squared BESSEL process with dimension $\delta \in (1,2)$, i.e., $U^2(\cdot)$ is the unique strong solution of the equation

$$U^2(t) = \delta t + 2\int_0^t \sqrt{U^2(t)}\,\mathrm{d}B(t)\,, \quad 0 \le t < \infty$$

for some standard, real-valued Brownian motion $B(\cdot)$. It is well known that when $\delta \in (1,2)$, the square root $R(\cdot) := |U(\cdot)| \ge 0$ of this process is a semimartingale that keeps visiting the origin almost surely, and can be decomposed as

$$R(\cdot) = \int_0^{\cdot} \frac{\delta-1}{2\,R(t)} \cdot \mathbf{1}_{\{R(t)\neq 0\}}\mathrm{d}t + B(\cdot)\,,$$

with $L^R(\cdot) \equiv 0$, $\int_0^{\cdot} \mathbf{1}_{\{R(t)=0\}}\mathrm{d}t \equiv 0$. Now let us apply Proposition 2. Given $\alpha \in (0,1)$, we unfold the nonnegative BESSEL process $R(\cdot)$ to obtain

$$\widehat{X}(\cdot) = Z(\cdot)R(\cdot) = \int_0^{\cdot} Z(t)\mathrm{d}R(t) + (2\alpha-1)L^R(\cdot) = \int_0^{\cdot} \frac{\delta-1}{2\,\widehat{X}(t)} \cdot \mathbf{1}_{\{\widehat{X}(t)\neq 0\}}\mathrm{d}t + \widehat{\beta}(\cdot)\,,$$

with $Z(\cdot) = \mathrm{sgn}(\widehat{X}(\cdot))$ and with $\widehat{\beta}(\cdot) := \int_0^{\cdot} Z(t)\mathrm{d}B(t)$ being another standard Brownian motion on an extended probability space.

With the appropriate scalings $g(x) := |x|^{2-\delta}/(2-\delta)$ and $G(x) := \mathrm{sgn}(x)\cdot g(x)$ for $x \in \mathbb{R}$ one can show that

$$G(\widehat{X}(T)) = \mathrm{sgn}(\widehat{X}(T))g(\widehat{X}(T)) = \int_0^T \mathrm{sgn}(\widehat{X}(t))\mathrm{d}(g(\widehat{X}(t))) + (2\alpha-1)\,L^{g(\widehat{X})}(T)$$

as well as

$$L^{G(\widehat{X})}(\cdot) - L^{-G(\widehat{X})}(\cdot) = (2\alpha - 1)\big(L^{G(\widehat{X})}(\cdot) + L^{-G(\widehat{X})}(\cdot)\big)$$

and

$$(1-\alpha)\, L^{G(\widehat{X})}(\cdot) = \alpha\, L^{-G(\widehat{X})}(\cdot)\,, \quad L^{g(\widehat{X})}(\cdot) = \frac{1}{2}\big(L^{G(\widehat{X})}(\cdot) + L^{-G(\widehat{X})}(\cdot)\big)\,.$$

Here there exists a (nonnegative) one-dimensional BESSEL process $\rho(\cdot)$ such that $\rho(0) = (2-\delta)^{\delta-1} g(\widehat{X}(0))$ and

$$g(\widehat{X}(t)) = \frac{1}{2-\delta}\,\big|\widehat{X}(t)\big|^{2-\delta} = \frac{1}{(2-\delta)^{\delta-1}}\,\rho(\Lambda(t))\,, \qquad 0 \le t < \infty\,,$$

(by Proposition XI.1.11 of REVUZ & YOR ('99)) where

$$\Lambda(t) := \inf\{s \ge 0 : K(s) \ge t\}\,, \quad K(s) := \int_0^s \big(\rho(u)\big)^{\frac{2\delta-2}{2-\delta}} \mathrm{d}u\,,$$

that is, $g(\widehat{X}(\cdot))$ is a time-changed, conventionally reflected Brownian motion with the stochastic clock $\Lambda(\cdot)$. Thus the constructed process is the $\delta-$dimensional skew BESSEL process with skewness parameter $\alpha \in (0,1)$ studied in BLEI (2012). □

Let us discuss here two questions on the solvability of (4). A first question that arises regarding the stochastic integral equation in (4), is whether it can be written in the more conventional form

$$X(\cdot) = \int_0^{\cdot} \operatorname{sgn}(X(t))\, \mathrm{d}U(t) + \frac{2\alpha - 1}{\alpha}\, L^X(\cdot)\,, \tag{6}$$

in terms of the asymmetric (left-continuous) version of the signum function. For this, it is necessary and sufficient to have

$$\int_0^{\cdot} \mathbf{1}_{\{X(t)=0\}}\, \mathrm{d}U(t) \equiv 0\,, \qquad \text{or equivalently} \qquad \int_0^{\cdot} \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}U(t) \equiv 0 \tag{7}$$

in the context of Proposition 1. Now from (1), (3) it is clear that $M(\cdot)$ is the local martingale part of the continuous semimartingale $S(\cdot)$, so we have $\langle S\rangle(\cdot) = \langle U\rangle(\cdot) = \langle M\rangle(\cdot)$ and

$$\int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}\langle M\rangle(t) = 0 \tag{8}$$

(e.g., KARATZAS & SHREVE (1991) Exercise 3.7.10). This gives $\int_0^{\cdot} \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}M(t) \equiv 0$, so (7) will follow if and only if

$$\int_0^{\cdot} \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}A(t) \equiv 0 \tag{9}$$

holds; and on the strength of (8), a sufficient condition for (9) is that $A(\cdot)$ be absolutely continuous with respect to the quadratic variation process $\langle M\rangle(\cdot)$. We have the following result.

Proposition 3. [ICHIBA & KARATZAS (2014)] *For a given continuous semimartingale $U(\cdot)$ of the form (1) the stochastic integral equation of (4) can be cast equivalently in the form (6), if and only if (9) holds; and in this case we have the identification $L^S(t) = \max_{0\le s\le t}\big(-U(s)\big)$ and the filtration comparisons*

$$\mathcal{F}^{|X|}(t) = \mathcal{F}^U(t) \subseteq \mathcal{F}^X(t)\,, \quad 0 \le t < \infty\,. \tag{10}$$

Whereas, a sufficient condition for (9) to hold, is that there exist an $\mathbb{F}$−progressively measurable process $p(\cdot)$, locally integrable with respect to $\langle M\rangle(\cdot)$ and such that

$$A(\cdot) = \int_0^{\cdot} p(t)\,\mathrm{d}\langle M\rangle(t)\,. \tag{11}$$

A second question that arises regarding the skew-TANAKA equation of (4), is whether it can be solved uniquely. It is well-known that we cannot expect pathwise uniqueness or strength to hold for this equation. Such strong existence and uniqueness fail already with $\alpha = 1/2$ and $U(\cdot)$ a standard Brownian motion, in which case we have in (10) also the strict inclusion $\mathcal{F}^U(t) \subsetneqq \mathcal{F}^X(t)$ for all $t \in (0,\infty)$ (e.g., KARATZAS & SHREVE (1991), Example 5.3.5). The SKOROKHOD reflection of $U(\cdot)$ can then be "unfolded" into a Brownian motion $X(\cdot)$, whose filtration is strictly finer than that of the original Brownian motion $U(\cdot)$: the unfolding cannot be accomplished without the help of some additional randomness.

The issue, therefore, is whether *uniqueness in distribution* holds for the skew-TANAKA equation of (4), under appropriate conditions. We shall address this question in the case of a continuous local martingale $U(\cdot)$ with $U(0) = 0$ and $\langle U\rangle(\infty) = \infty$. Let us recall a few notions and facts about such a process, starting with its DAMBIS-DUBINS-SCHWARZ representation

$$U(t) = B\big(\langle U\rangle(t)\big)\,, \qquad 0 \le t < \infty \tag{12}$$

(cf. KARATZAS & SHREVE (1991) Theorem 3.4.6); here $B(\theta) = U(Q(\theta)),\ 0 \le \theta < \infty$ is standard Brownian motion, and $Q(\cdot)$ the right-continuous inverse of the continuous, increasing process $\langle U\rangle(\cdot)$.

We say that this $U(\cdot)$ is *pure,* if each $\langle U\rangle(t)$ is $\mathcal{F}^B(\infty)$−measurable; we say that it is an OCONE *martingale,* if the processes $B(\cdot)$ and $\langle U\rangle(\cdot)$ are independent (cf. OCONE (1993) and Appendix of DUBINS ET AL. (1993)). As discussed in VOSTRIKOVA & YOR (2000) , a pure OCONE martingale is a Gaussian process.

Proposition 4. [ICHIBA & KARATZAS (2014)] *Suppose that $U(\cdot)$ is a continuous local martingale with $U(0) = 0$ and $\langle U\rangle(\infty) = \infty$. Then uniqueness in distribution holds for the skew-TANAKA equation of (4), or equivalently of (6), provided that either*

(i) $U(\cdot)$ is pure; or that (ii) the quadratic variation process $\langle U\rangle(\cdot)$ is adapted to a Brownian motion $\Gamma(\cdot) := (\Gamma_1(\cdot), \cdots, \Gamma_n(\cdot))'$, with values in some Euclidean space $\mathbb{R}^n$ and independent of the real-valued Brownian motion $B(\cdot)$ in the representation (12).

There is a counterexample that fails to be unique in distribution for general OCONE martingale: for $u > 0,\ v > 0$ with $u \ne v$ and a BM $\beta(\cdot)$ let us define $B(\cdot) = \int_0^{\cdot} \mathrm{sgn}(\beta(t))\mathrm{d}\beta(t)$, $U(\cdot) = B(\langle U\rangle(\cdot))$ with

$$\langle U\rangle(t) := t\cdot \mathbf{1}_{\{t\le 1\}} + \{1 + (u\cdot \mathbf{1}_{\{\beta(1)>0\}} + v\cdot \mathbf{1}_{\{\beta(1)\le 0\}})(t-1)\}\cdot \mathbf{1}_{\{t>1\}}\,.$$

Both $X(\cdot) := \beta(\langle U\rangle(\cdot))$ and $\Xi(\cdot) := -X(\cdot)$ solve the skew TANAKA equation but $\mathbb{E}[(X(2))^3] \ne 0$, and hence there is no uniqueness in distribution.

What rules out the strong solution for (4)? The next proposition tells us about the case when the additive (Brownian) noise helps to have the pathwise uniqueness and the strong solution.

Proposition 5. [ICHIBA & KARATZAS (2014)] *Assume that the continuous semimartingale $U(\cdot) = U(0) + M(\cdot) + A(\cdot)$ satisfies the conditions of Proposition 1; and that*

$$V(\cdot) = N(\cdot) + \Delta(\cdot)$$

is another continuous semimartingale, with continuous local martingale part $N(\cdot)$ *and finite variation part* $\Delta(\cdot)$ *which satisfy* $N(0) = \Delta(0) = 0$ *and*

$$\langle M, N\rangle(\cdot) \equiv 0\,, \qquad \langle M\rangle(\cdot) = \int_0^{\cdot} q(t)\,\mathrm{d}\langle N\rangle(t)$$

for some $\mathbb{F}$*−progressively measurable process* $q(\cdot)$ *with values in a compact interval* $[0, b]$. *Then pathwise uniqueness holds for the perturbed skew-*TANAKA *equation*

$$X(\cdot) = \int_0^{\cdot} sgn\big(X(t)\big)\,\mathrm{d}U(t) + V(\cdot) + \frac{2\alpha - 1}{\alpha}\,L^X(\cdot)\,, \tag{13}$$

provided that either
(i) $\alpha = 1/2$, *or that*
(ii) $U(\cdot)$ *and* $V(\cdot)$ *are independent, standard Brownian motions. In this case a weak solution exists, and is thus strong by the* YAMADA-WATANABE *theory.*

Proof uses the ideas of LE GALL (1983) (cf. NAKAO (1972)); see also FERNHOLZ, ICHIBA, KARATZAS & PROKAJ (2013) and PROKAJ (2013). For this perturbed skew-Tanaka equation (13) there are some related results which can be found in KARATZAS, SHIRYAEV & SHKOLNIKOV (2011), FERNHOLZ, ICHIBA & KARATZAS (2013), ICHIBA, KARATZAS & PROKAJ (2013) and ICHIBA, KARATZAS & SHKOLNIKOV (2013).

Finally, let us discuss very briefly this unfolding procedure in the planar domain with the following example. Let us define two dimensional analogue of the symmetric signum function via

$$\mathfrak{f}_1(x) = \cos(\theta)\,, \quad \mathfrak{f}_2(x) = \sin(\theta)$$

for $x = (r, \theta) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ in the polar coordinates, and $\mathfrak{f}_1(\mathbf{0}) = \mathfrak{f}_2(\mathbf{0}) = 0$.

Proposition 6. [ICHIBA ET AL. (2015)] *Given a probability measure* $\boldsymbol{\nu}$ *on* $[0, 2\pi)$ *and the* SKOROKHOD *reflection* $S(\cdot)$ *in (3), there exists an enlargement* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}(t)\}_{0\le t<\infty}$ *of* $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0\le t<\infty}$ *with a measure-preserving map* $\pi : \widetilde{\Omega} \to \Omega$, *and on this enlarged space a continuous semimartingale* $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ *which solves the system of stochastic integral equations*

$$X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}S(t) + \gamma_i L^S(T)\,, \qquad 0 \le T < \infty \tag{14}$$

for $i = 1, 2$; *and* $\boldsymbol{\gamma} := (\gamma_1, \gamma_2)'$ *is the real constant vector with*

$$\gamma_1 := \int_0^{2\pi} \cos(\theta)\boldsymbol{\nu}(\mathrm{d}z)\,, \quad \gamma_2 := \int_0^{2\pi} \cos(\theta)\boldsymbol{\nu}(\mathrm{d}z)$$

Here we fix a vector $\mathrm{x} := (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2$ *with* $\mathrm{x}_i = \mathfrak{f}_i(\mathrm{x})\,S(0)$, $i = 1, 2$.

When $U(\cdot)$ is a Brownian motion, the resulting process $X(\cdot)$ coincides with the WALSH Brownian motion introduced by WALSH (1978). We may introduce a change-of-variable formula for the WALSH semimartingale $X(\cdot)$ in (14) driven by the SKOROKHOD reflection $S(\cdot)$ and also characterize WALSH diffusions by the corresponding Martingale Problem. The details can be found in ICHIBA ET AL. (2015).

By appropriate scale function for the WALSH Brownian motion, we may construct the WALSH Brownian motion $Y(\cdot) := (Y_1(\cdot), Y_2(\cdot))'$ with ray-dependent polar drifts:

$$Y(T) = \mathrm{y} + \int_0^T \mathfrak{f}(Y(t))\big(- \boldsymbol{\lambda}(\arg(Y(t))\mathrm{d}t + \mathrm{d}W(t)\big) + \gamma L^{\|Y\|}(T) \tag{15}$$

with the spinning measure $\boldsymbol{\nu}(\mathrm{d}\theta)$ and some measurable function $\boldsymbol{\lambda} : \sup(\boldsymbol{\nu}) \to (0, \infty)$. Assume that $\min_{0\le\theta\le 2\pi} \boldsymbol{\lambda}(\theta) > 0$. The resulting process is positive recurrent. Its stationary distribution is expressed in polar coördinates as

$$\mathbb{P}^e(\|Y(t)\| \in \mathrm{d}r\,,\, \arg(Y(t)) \in \mathrm{d}\theta) = \Big(\int_0^{2\pi} \frac{1}{2\,[\boldsymbol{\lambda}(v)]^2}\,\boldsymbol{\nu}(\mathrm{d}v)\Big)^{-1} \cdot \frac{e^{-2\,\boldsymbol{\lambda}(\theta)\,r}}{[\boldsymbol{\lambda}(\theta)]}\,\mathrm{d}r\,\boldsymbol{\nu}(\mathrm{d}\theta) \tag{16}$$

for every $t > 0$; and in particular, if $\boldsymbol{\lambda}(\theta) \equiv \lambda$ (a positive constant), then the stationary distribution becomes $2\,\lambda\, e^{-2\,\lambda r}\mathrm{d}r\,\boldsymbol{\nu}(\mathrm{d}\theta)$ for $r > 0$, $\theta \in \mathrm{supp}(\boldsymbol{\nu})$. Here $\mathbb{P}^e$ stands for the stationary distribution.

In a similar manner, again by appropriate scale function, we may construct the WALSH diffusion driven by the ray dependent ORNSTEIN-UHLENBECK process via

$$Y(T) = \mathrm{y} + \int_0^T \mathfrak{f}(Y(t))\big(- \boldsymbol{\lambda}(\arg(Y(t)))\|Y(t)\|\mathrm{d}t + \mathrm{d}W(t)\big) + \gamma L^{\|Y\|}(T)$$

For each $\theta \in \mathrm{supp}(\boldsymbol{\nu})$ the driving ORNSTEIN-UHLENBECK process has the scale function $s(x) = \int_0^x \exp(\boldsymbol{\lambda}(\theta)y^2)\mathrm{d}y$, the speed measure $m(\mathrm{d}x) = 2\exp(-\boldsymbol{\lambda}(\theta)x^2)$, and the LÉVY measure

$$\mathcal{V}(\mathrm{d}u) = \frac{1}{\sqrt{2\pi}}\Big(\frac{2\boldsymbol{\lambda}(\theta)}{1 - e^{-2\boldsymbol{\lambda}(\theta)u}}\Big)^{3/2}$$

of the inverse local time. Then the stationary distribution of $Y(\cdot)$ can be expressed in polar coördinates as for every $t > 0$

$$\mathbb{P}^e(\|Y(t)\| \in \mathrm{d}r\,,\, \arg(Y(t)) \in \mathrm{d}\theta) = \Big(\int_0^{2\pi} \frac{\sqrt{\pi}}{2\,\boldsymbol{\lambda}(v)}\,\boldsymbol{\nu}(\mathrm{d}v)\Big)^{-1} \cdot \frac{e^{-\,\boldsymbol{\lambda}(\theta)\,r^2}}{[\boldsymbol{\lambda}(\theta)]^{1/2}}\,\mathrm{d}r\,\boldsymbol{\nu}(\mathrm{d}\theta)\,. \tag{17}$$

We expect the folding/unfolding procedure may be applicable in higher dimensional processes described by stochastic differential equations with degeneracy.

Part of research was supported by NSF-DMS-13-13373.

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