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Dirichlet form approach to infinite interacting Lévy processes

Syota Esaki
Faculty of Science,
Chiba university

1 Introduction

Let $\Phi:\mathbb{R}^{d}\rightarrow \mathbb{R}\cup\{\infty\}$ be a self potential, $\Psi:\mathbb{R}^{d}\times \mathbb{R}^{d}\rightarrow \mathbb{R}\cup\{\infty\}$ be a interaction pair potential with $\Psi(x, y)=\Psi(y, x)$. We then consider ISDE (infinite dimensional stochastic differential equation)

$$dX_{j}(t)=dB_{j}(t)-\frac{1}{2}\nabla\Phi(X_{j}(t))-\frac{1}{2}\sum_{\substack{k:k\in \mathbb{Z}\\ k\neq j}}\nabla\Psi(X_{j}(t), X_{k}(t))dt, \quad j\in \mathbb{N}. \tag{1}$$

The existence and uniqueness of solutions of ISDE (1) has been studied in many researches. The stochastice process $(X_{j}(t))_{j\in \mathbb{N}}$ describes a system of interacting Brownian motions (IBM). On the other hand, IBM is constructed by using Dirichlet form technique [2, 5, 6]. For a local function f on S the symmetric function $\widetilde{f}$ such that

$$f(\sum_{j\in \mathbb{N}}\delta_{s_{j}})=\widetilde{f}((s_{j})_{j\in \mathbb{N}})$$

is associated. We call a local function f is smooth if the associated function $\widetilde{f}$ is smooth. We denote by $\mathscr{D}_{\circ}$ the set of all local smooth functions on S. We introduce a square field on $\mathscr{D}_{\circ}$ given by

$$\mathbb{D}_{\mathrm{BM}}[f, g](\mathsf{s})=\frac{1}{2}\sum_{i=1}^{\infty}\nabla_{i}\widetilde{f}(s)\cdot\nabla_{i}\widetilde{g}(s), \quad f, g\in \mathscr{D}_{\circ},$$

where $\nabla_{i}=(\frac{\partial}{\partial s_{i1}}, \ldots, \frac{\partial}{\partial s_{id}})$,

$$s=(s_{j})_{j\in \mathbb{N}}=(s_{j}^{1}, s_{j}^{2}, \ldots, s_{j}^{d})_{j\in \mathbb{N}},$$

and a bilinear form $(\mathscr{E}_{\mathrm{BM}}, \mathscr{D}_{\infty,\mathrm{BM}})$ defined by

$$\mathscr{E}_{\mathrm{BM}}(f, g)=\int_{\mathsf{S}}\mathbb{D}_{\mathrm{BM}}[f, g](\mathsf{s})d\mu, \quad f, g\in \mathscr{D}_{\infty,\mathrm{BM}},$$
$$\mathscr{D}_{\infty,\mathrm{BM}}=\{f\in \mathscr{D}_{\circ};\mathscr{E}_{\mathrm{BM}}(f, f)<\infty, f\in L^{2}(\mathsf{S}, \mu)\}.$$

Under some assumptions, $(\mathscr{E}_{\mathrm{BM}}, \mathscr{D}_{\infty,\mathrm{BM}})$ is closable and its closure, denoted by $(\mathscr{E}_{\mathrm{BM}}, \mathscr{D}_{\mathrm{BM}})$, is a local, quasi-regular Dirichlet form. Therefore there is a diffusion $(\mathsf{X}_{\mathrm{BM}}, \mathbb{P}_{\mathsf{s},\mathrm{BM}})$ associated with $(\mathscr{E}_{\mathrm{BM}}, \mathscr{D}_{\mathrm{BM}})$. If μ is (Φ,Ψ)-quasi-Gibbs measure with smooth functions Φ and Ψ then its L^2-generator L_{BM} is given by

$$L_{\mathrm{BM}} f(\mathsf{s}) = \frac{1}{2}\sum_{i=1}^{\infty}\left\{\Delta_i \widetilde{f} - \left\{(\nabla\Phi)(s_i) + \sum_{j=1, j\neq i}^{\infty}(\nabla\Psi)(s_i, s_j)\right\}\nabla_i \widetilde{f}\right\},$$

where $\Delta_i = (\frac{\partial^2}{\partial s_{i1}^2}, \ldots, \frac{\partial^2}{\partial s_{id}})$. In addition the ISDE associated with $(\mathscr{E}_{\mathrm{BM}}, \mathscr{D}_{\mathrm{BM}})$ is described by (1). We remark that although the logarithmic interaction potentials $\Psi(x,y) = -\beta\log|x-y|$ are unbounded at infinity, there exists quasi-Gibbs states associated with them for $\beta = 1,2,4$ and related IBMs can be constructed by the Dirichlet form approach [5, 6].

In our research we discretize this interacting particle systems. Especially, in this paper we consider infinite particle systems in which each particle undergoes a jump type Lévy process with long range interaction.

Let $\mathbb{D}[\cdot,\cdot]$ be the square field on $\mathscr{D}_\circ$ defined by

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{j=1}^{\infty}\int_{\mathbb{R}^d} \nabla_j^y \widetilde{f}(s)\cdot\nabla_j^y\widetilde{g}(s)\, p(|y-s_j|)dy, \quad f,g\in\mathscr{D}_\circ,$$

where

$$\nabla_j^y \widetilde{f}(s) = \widetilde{f}(s_1,\ldots,s_{j-1},y,s_{j+1},\ldots) - \widetilde{f}(s).$$

Here $p : [0,\infty)\to[0,\infty)$ is a density of a (finite or infinite) measure such that

$$\int_{\mathbb{R}^d}(1\wedge|y|^2)\,p(|y|)dy < \infty. \tag{2}$$

Then we introduce the bilinear form $(\mathscr{E}, \mathscr{D}_\infty)$ defined by

$$\mathscr{E}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g](\mathsf{s})d\mu, \quad f,g\in\mathscr{D}_\infty,$$

$$\mathscr{D}_\infty = \{f\in\mathscr{D}_\circ; \mathscr{E}(f,f)<\infty, f\in L^2(\mathsf{S},\mu)\}.$$

We show that under assumptions (B.1)–(B.4) in addition to (A.0)–(A.2) in section 2, $(\mathscr{E},\mathscr{D}_\infty)$ is closable and its closure, denoted by $(\mathscr{E},\mathscr{D})$, is a quasi-regular Dirichlet form. Therefore there is a special standard process $(\mathsf{X},\mathbb{P}_{\mathsf{s}})$ associated with $(\mathscr{E},\mathscr{D})$. These assumptions are quite mild and a system of interacting α-stable processes ($\alpha\in(0,2)$) satisfies them if α is greater than κ, the growth order of the density (the 1-correlation function) of μ. Since we consider the case that a jump rate density do not have the expectation (e.g. the Cauchy process) we need to consider the influence of the number of particles coming from far points and the long range interaction. In addition in the case that the density goes infinity at the infinity point, the parameter α is restricted. However in the case even if particles move independently, infinite particles can concentrate at a point. Suppose that μ is a (Φ,Ψ)-quasi Gibbs measure. Let μ_x be the reduced Palm measure defined by $\mu_x = \mu(\cdot - \delta_x |\, \mathsf{s}(\{x\})\geq 1)$ for $x\in\mathbb{R}^d$, $\rho^1(x)$ be the 1-correlation function of μ defined by $\int_A \rho^1(x)dx = \int_{\mathsf{S}} \mathsf{s}(A)d\mu$ for any bounded measurable subset $A\subset\mathbb{R}^d$ and

$$c_{\mathsf{s}\setminus x}(x,y) = 1 + \frac{\rho^1(y)}{\rho^1(x)}\frac{d\mu_y}{d\mu_x}(\mathsf{s}\setminus x), \tag{3}$$

for $x \in \mathsf{s}$, where $\mathsf{s} \setminus x = \mathsf{s} - \delta_x$ and $d\mu_y/d\mu_x$ denote the Radon-Nikodym density of μ_y for μ_x. Then under the some assumptions its L^2-generator L is given by

$$Lf(\mathsf{s}) = \frac{1}{2}\sum_{j=1}^{\infty}\int_S \nabla_j^y \widetilde{f}(s)\ c_{\mathsf{s}\setminus s_j}(s_j, y)p(|y - s_j|)dy.$$

According to the arguments in [3, 4] we can show that the associated labeled process solves the following ISDE:

$$X_j(t) = X_j(0) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_0^{c_{\mathsf{X}(s-)\setminus X_j(s-)}(X_j(s-),X_j(s-)+u)} uN_p(dsdudr),$$

for all $i \in \mathbb{N}$, where $\mathsf{X}(t) = \sum_i \delta_{X_i(t)}$ and $N_p(dsdudr)$ is the Poisson point process on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}$ with intensity $dsp(|u|)dudr$. In forthcoming paper we construct the ISDE associated by the present infinite particle systems and discuss the uniqueness of the solution of the ISDE. Our result is more interesting for a quasi-Gibbs state which is not a Gibbs state. For example consider the Ginibre random point process μ_{gin}, which is a probability measure on the configuration space on $\mathbb{R}^2$ with self potential $\Phi(x) = 0$ and interaction potential $\Psi(x, y) = -2\log|x - y|$. From Theorem 1.3 in Osada and Shirai [7] we see that $c_{\mathsf{s}}(x, y)$ in (3) is written by

$$c_{\mathsf{s}\setminus x}(x,y) = 1 + \lim_{r\to\infty}\prod_{|s_i|<r}\frac{|y - s_i|^2}{|x - s_i|^2}.$$

In addition we remark that we can not consider an Glauber dynamics by the same way on the present paper. Of course if we take an invariant measure μ from Gibbs measures we can consider an equilibrium Glauber dynamics. Indeed to consider the dynamics we use the absolute continuity of the Palm measure with respect to the Gibbs measure. In this case L^2-generator L_{Gla} of the equilibrium Glauber dynamics is given by

$$L_{\mathrm{Gla}}f(\mathsf{s}) = \int_S (f(\mathsf{s}\cdot x) - f(\mathsf{s}))\rho(x)\frac{d\mu_x}{d\mu}(\mathsf{s})dx + \sum_{x\in \mathrm{supp}\mathsf{s}}(f(\mathsf{s}\setminus x) - f(\mathsf{s})).$$

Here we set $\mathsf{s}\cdot x = \mathsf{s} + \delta_x$ for $\mathsf{s} \in \mathsf{S}$ and $x \in \mathbb{R}^d$. However for an quasi-Gibbs measure in general the Palm measure is not absolute continuous with respect to the quasi-Gibbs measure (e.g. Ginibre random point field [7]). Hence in these case an equilibrium Glauber dynamics for a quasi-Gibbs measure is not well-defined.

2 Setup and main result

Let S be a closed set in $\mathbb{R}^d$ such that $0 \in S$ and $\overline{S^{\mathrm{int}}} = S$, where S^{int} denote the interia of S. Let $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}; \mathsf{s}(K) < \infty$ for all compact sets $K \subset S\}$, where δ_a stands for the delta measure at a. We endow S with the vague topology. Then S is a Polish space. We call S the configuration space over S. We denote by $\mathscr{D}_\circ$ the set of all local smooth functions on S. For $f, g \in \mathscr{D}_\circ$ we set $\mathbb{D}[f, g] : \mathsf{S} \to \mathbb{R}$ by

$$\mathbb{D}[f,g](\mathsf{s}) = \frac{1}{2}\sum_{j=1}^{\infty}\int_S \nabla_j^y \widetilde{f}(s)\cdot \nabla_j^y \widetilde{g}(s)\ p(|y - s_j|)dy,$$

where $p : [0, \infty) \to [0, \infty)$ is a density of a (finite or infinite) measure satisfying the condition (2). We set

$$\mathscr{E}(f, g) = \int_{\mathsf{S}} \mathbb{D}[f, g](\mathsf{s}) d\mu,$$
$$\mathscr{D}_\infty = \{f \in \mathscr{D}_\circ \cap L^2(\mathsf{S}, \mu); \mathscr{E}(f, f) < \infty\}.$$

Let $S_r = \{x \in S; |x| \le r\}$. We introduce some assumptions as the following.

$$\begin{gathered}\text{There exist } k\text{-density function of } \mu \text{ on } S_r, \text{ denoted by } \sigma_r^k, \\ \text{and } k\text{-correlation function, denoted by } \rho^k, \text{ for all } k, r \in \mathbb{N}.\end{gathered} \tag{A.0}$$

$$(\mathscr{E}, \mathscr{D}_\infty) \text{ is closable on } L^2(\mathsf{S}, \mu). \tag{A.1}$$

$$\sigma_r^k \in L^p(S_r^k, dx) \text{ for all } k, r \in \mathbb{N} \text{ with some } 1 < p \le \infty. \tag{A.2}$$

By (A.1) we denote by $(\mathscr{E}, \mathscr{D})$ the closure of $((\mathscr{E}, \mathscr{D}_\infty), L^2(\mathsf{S}, \mu))$. In addition we introduce conditions (B.1)–(B.4):

$$\rho^1(x) = O\left(|x|^\kappa\right) \text{ as } |x| \to \infty \text{ for some } \kappa \ge 0. \tag{B.1}$$

$$p(r) = O(r^{-(d+\alpha)}) \text{ as } r \to \infty \text{ for some } \alpha > \kappa. \tag{B.2}$$

$$p(r) = O(r^{-(d+\beta)}) \text{ as } r \to +0 \text{ for some } 0 < \beta < 2. \tag{B.3}$$

$$\frac{\mathrm{Var}\left[\mathsf{s}(S_r)\right]}{\left(\mathbb{E}\left[\mathsf{s}(S_r)\right]\right)^2} = O\left(r^{-\delta}\right) \quad \text{as } r \to \infty \text{ for some } \delta > 0. \tag{B.4}$$

Conditions (B.1)–(B.3) relate to the jump rate and the growth rate of the density of particles. Condition (B.4) is necessary to control the fluctuation of the number of particles in S_r. Moreover we remark that the LHS of (B.4) is represented by the 1 and 2-correlation functions of μ by the following:

$$\frac{\mathrm{Var}\left[\mathsf{s}(S_r)\right]}{\left(\mathbb{E}\left[\mathsf{s}(S_r)\right]\right)^2} = \frac{\int_{S_r} \rho^1(x)dx - \int_{S_r^2} \left(\rho^1(x_1)\rho^1(x_2) - \rho^2(x_1, x_2)\right) dx_1 dx_2}{\left(\int_{S_r} \rho^1(x)dx\right)^2}.$$

By the expression we can check that (B.4) holds if μ is the Poisson random point field with respect to Lebesgue measure or μ is a determinantal point field. Hence condition (B.4) is mild.

Now we state an our main theorem:

Theorem 1. *Suppose that (A.0)–(A.2), (B.1)–(B.4) hold. Then $(\mathscr{E}, \mathscr{D})$ is a quasi-regular Dirichlet form on $L^2(\mathsf{S}, \mu)$. Therefore there exists a special standard process $\{\mathbb{P}_\mathsf{s}\}_{\mathsf{s} \in \mathsf{S}}$ associated with $((\mathscr{E}, \mathscr{D}), L^2(\mathsf{S}, \mu))$. Moreover $\{\mathbb{P}_\mathsf{s}\}_{\mathsf{s} \in \mathsf{S}}$ is reversible with invariant measure μ.*

Remark 1. Condition (B.1) and (B.2) imply that

$$\int_S \rho^1(x) p(x, A) dx < \infty, \tag{4}$$

for all compact subset A. The property (4) is necessary to construct the infinite particle systems of independent jump type processes. Hence Condition (B.1) and (B.2) are natural.

3 Sketch of proof of Theorem 1

In this section we give the sketch of the proof of the quasi-regularity of $(\mathscr{E}, \mathscr{D})$. For the reader's convenience we give the definition of quasi-regular Dirichlet form. We refer to Ma and Röckner [1] for detail and related notions. A symmetric Dirichlet form $(\mathscr{E}, \mathscr{D})$ on $L^2(\mathsf{S}, \mu)$ is called quasi-regular if $(\mathscr{E}, \mathscr{D})$ satisfies the following:

(Q.1) There exists an $\mathscr{E}$-nest consisting of compact sets.

(Q.2) There exists an $||\cdot||_1$-dense subset of F whose elements have $\mathscr{E}$-continuous μ-versions. Here $||f||_1^2 = \mathscr{E}(f,f) + ||f||^2_{L^2(\mathsf{S},\mu)}$.

(Q.3) There exist $u_n \in \mathscr{D}$, $n \in \mathbb{N}$, having $\mathscr{E}$-continuous μ-versions $\widetilde{u}_n$, and an $\mathscr{E}$-exceptional set N such that $\{\widetilde{u}_n\}$ separates the points of $\mathsf{S} - N$.

We can check (Q.2) and (Q.3) by the similar way used in [2]. Hence it is sufficient that we check (Q.1).

Lemma 1. *Assume (B.4). Let* $\boldsymbol{a}_n = \{n2^{(d+\kappa)r}\}_{r\in\mathbb{N}}$, $n \in \mathbb{N}$. *Then we have*

$$\mu\left(\bigcup_{n=1}^{\infty} \mathsf{S}[\boldsymbol{a}_n]\right) = 1.$$

where $\mathsf{S}[\boldsymbol{a}] = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S_{2^r}) \le a_r \text{ for all } r\}$ *for* $\boldsymbol{a} = \{a_r\}_{r\in\mathbb{N}}$.

It is known that $\mathsf{S}[\boldsymbol{a}]$ is a compact set for all $\boldsymbol{a} = \{a_r\}_{r\in\mathbb{N}}$. Hence Lemma 1 says that there exists a family of compact subsets whose union has probability one.

Here we introduce a function $\chi[\boldsymbol{a}]$ defined by

$$\chi[\boldsymbol{a}](\mathsf{s}) = \rho \circ d_{\boldsymbol{a}}(\mathsf{s}), \quad d_{\boldsymbol{a}}(\mathsf{s}) = \sum_{r=1}^{\infty} \sum_{j\in J_{r,\mathsf{s}}} \frac{(2^r - |s_j|) \wedge 2^{r-1}}{2^{r-1}a_r},$$

where $(s_j)_{j\in\mathbb{N}}$ is a sequence such that $|s_j| \le |s_{j+1}|$ for all $j \in \mathbb{N}$, $\mathsf{s} = \sum_j \delta_{s_j}$ and

$$J_{r,\mathsf{s}} = \{j; j > a_r, s_j \in S_{2^r}\}.$$

$\rho : \mathbb{R} \to [0,1]$ is the function defined by

$$\rho(t) = \begin{cases} 1 & \text{if } t < 0, \\ 1 - t & \text{if } 0 \le t \le 1 \\ 0 & \text{if } 1 < t, \end{cases}$$

(see Figure 1). For the function $\chi[\boldsymbol{a}]$ we can see the following lemma by the straightforward calculation (see Figure 2).

Lemma 2. *For any* $\boldsymbol{a} = \{a_r\}_{r\in\mathbb{N}}$ *we have*

$$\chi[\boldsymbol{a}](\mathsf{s}) = \begin{cases} 1 & \text{if } \mathsf{s} \in \mathsf{S}[\boldsymbol{a}], \\ 0 & \text{if } \mathsf{s} \in \mathsf{S}[2\boldsymbol{a}_+]^c, \end{cases}$$

where we set $2\boldsymbol{a}_+ = \{2a_{r+1}\}_{r\in\mathbb{N}}$.

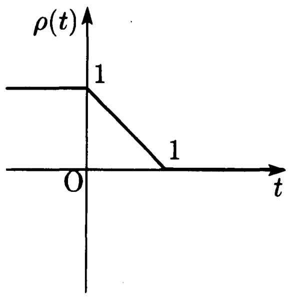

Figure 1: $\rho(t)$

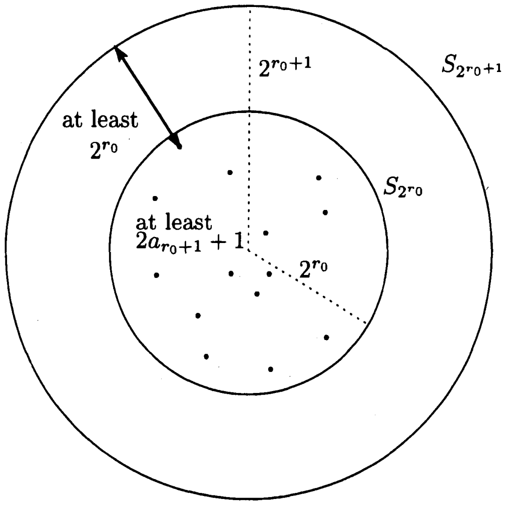

Figure 2: an example of a configuration in $\mathsf{S}[2\boldsymbol{a}_+]^c$

From lemma 2 we can call $\chi[\boldsymbol{a}]$ a cut off function on $\mathsf{S}[\boldsymbol{a}]$.

The next lemma is a key lemma of the proof of Theorem 1. This lemma is proved by the lemma 2 and some additional arguments.

Lemma 3. *Suppose* $\boldsymbol{a}_n = \{a_{n,r}\}_{r\in\mathbb{N}} = \{n2^{(d+\kappa)r}\}_{r\in\mathbb{N}}$. *Let* $\alpha > \kappa$, $0 < \beta < 2$. *Then there exists* $C = C_{d,\alpha,\beta,\kappa}$ *such that*

$$\int_{\mathsf{S}} \mathbb{D}[\chi[\boldsymbol{a}_n], \chi[\boldsymbol{a}_n]](\mathsf{s}) f^2(\mathsf{s}) d\mu \leq C \int_{\mathsf{A}(\boldsymbol{a}_n)} f^2(\mathsf{s}) d\mu \quad \textit{for all } n \in \mathbb{N} \textit{ and } f \in \mathscr{D}_\infty.$$

where we set $\mathsf{A}(\boldsymbol{a}) = \mathsf{S}[2\boldsymbol{a}_+ + \mathbf{1}] \setminus \mathsf{S}[\boldsymbol{a} - \mathbf{1}]$ *for* $\boldsymbol{a} = \{a_r\}_{r\in\mathbb{N}}$, $2\boldsymbol{a}_+ + \mathbf{1} = \{2a_{r+1} + 1\}_{r\in\mathbb{N}}$ *and* $\boldsymbol{a} - \mathbf{1} = \{a_r - 1\}_{r\in\mathbb{N}}$.

From Lemma 1 and Lemma 3 and some additional arguments, we can prove the following lemma.

Lemma 4. *For all* $f \in \mathscr{D}_\infty$, *we have*

$$\chi[\boldsymbol{a}_n] f \to f \quad \textit{in } \|\cdot\|_1 \quad \textit{as } n \to \infty.$$

From Lemma 4 and some additional arguments, we can check the condition (Q.1).

4 Examples

We set that μ is the Dyson random point field or the Ginibre random point field. It is known that these random point fields are quasi-Gibbs measures, i.e. (A.0)–(A.2) hold. For these random point fields we can see ρ^1 is a constant function. Then the assumption (B.1) is satisfied for $\kappa = 0$. Hence we can take $0 < \alpha, \gamma < 2$. Therefore we can construct interacting symmetric α-stable processes for any $0 < \alpha < 2$.

On the other hand we set that μ is the Airy random point field. It is known that the Airy random point is a quasi-Gibbs measure, i.e. (A.0)–(A.2) hold. For the Airy random point field we can see $\rho^1(x) = O(|x|^{1/2})$. as $x \to -\infty$. Then the assumption (B.1) is satisfied for $\kappa = \frac{1}{2}$. Hence we can take $\frac{1}{2} < \alpha < 2$, $0 < \gamma < 2$. Therefore we can construct interacting symmetric α-stable processes for any $\frac{1}{2} < \alpha < 2$.

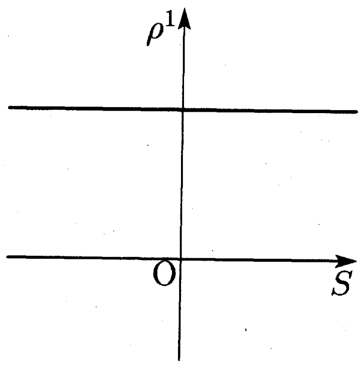

Figure 3: ρ^1 of Dyson or Ginibre

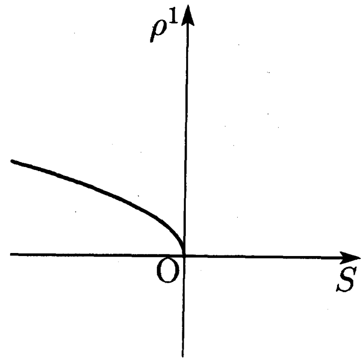

Figure 4: ρ^1 of Airy

References

[1] Ma, Z.-M., Röckner, M.: Introduction to the theory of (non-symmetric) Dirichlet forms. Springer-Verlag, Berlin, 1992.

[2] Osada, H.: Dirichlet form approach to infinitely dimensional Wiener processes with singular interactions. Comm. Math. Phys., **176** (1996), 117-131.

[3] Osada, H.: Tagged particle processes and their non-explosion criteria. J. Math. Soc. Japan, **62**, No. **3**, 867-894 (2010).

[4] Osada, H.: Infinite-dimensional stochastic differential equations related to random matrices. Probab. Theory Related Fields **153**, 471-509 (2012)

[5] Osada, H.: Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. Ann. Probab. **41**, 1-49 (2013)

[6] Osada, H.: Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II: Airy random point field. Stochastic Process. Appl. **123**, 813-838 (2013)

[7] Osada, H., Shirai, T.: Absolute continuity and singularity of Palm measures of the Ginibre point process. arXiv:math.PR/1406.3913.

Department of Mathematics and Informatics, Faculty of Science
Chiba University
Chiba 263-8522
JAPAN
E-mail address: sesaki@graduate.chiba-u.jp