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On the construction of the Feynman path integral for the Dirac equation

By

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Abstract

The Feynman path integral for the Dirac equation is determined mathematically, in the form of the sum-over-histories, satisfying the superposition principle. That is, it is given by the "sum" of the probability amplitudes with a common weight, over all possible paths that go in any direction at any speed forward and backward in time. It has been expected by Feynman himself for a long time that the Feynman path integral for the Dirac equation is represented in this form.

§ 1. Introduction

In the present paper the Feynman path integral for the Dirac equation in the general dimensional space-time is determined mathematically, in the form of the sum-over-histories, satisfying the superposition principle. That is, it is given by the "sum" of the probability amplitudes with a common weight, over all possible paths that go in any direction at any speed forward and backward in time. It has been expected by Feynman himself for a long time that the Feynman path integral for the Dirac equation is represented in this form.

Moreover, we will show other mathematical results and some remarks in the present paper We will not give a detailed proof of our results and so recommend readers interested in our results to see papers [17], [18] and [19].

We denote the electric strength and the magnetic strength tensor by $E(t,x) = (E_1,\dots,E_d) \in \mathbb{R}^d$ and $(B_{jk}(t,x))_{1\leq j<k\leq d} \in \mathbb{R}^{d(d-1)/2}$ for $(t,x) = (t,x_1,\dots,x_d) \in$

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$\mathbb{R}^{d+1}$, respectively. We introduce an electromagnetic potential $(V(t,x), A(t,x)) = (V, A_1, \dots, A_d) \in \mathbb{R}^{d+1}$, i.e.

$$E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x}, \tag{1.1}$$
$$B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad (1 \le j < k \le d),$$

where $\partial V/\partial x = (\partial V/\partial x_1, \dots, \partial V/\partial x_d)$.

Let $t_i \in \mathbb{R}$ be an initial time and $f(x) = {}^t(f_1(x), \dots, f_N(x)) \in \mathbb{C}^N$ an initial probability amplitude. We consider a more general equation than the Dirac equation

$$i\hbar \frac{\partial u}{\partial t}(t) = H(t)u(t) \tag{1.2}$$
$$:= \left[c\sum_{j=1}^{d} \widehat{\alpha}^{(j)} \left(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - eA_j(t,x) \right) + \widehat{\beta} mc^2 + eV(t,x) \right] u(t)$$

with $u(t_i) = f$ as in (11) of §67, p.257 of [4], where $u(t) = {}^t(u_1(t), \dots, u_N(t)) \in \mathbb{C}^N$, $\widehat{\alpha}^{(j)} (j = 1, 2, \dots, d)$ and $\widehat{\beta}$ are constant $N \times N$ Hermitian matrices, c is the velocity of light, $\hbar$ is the Planck constant and e is the charge of an electron. For the sake of simplicity we suppose $\hbar = 1$ and $e = 1$ hereafter. We note that through the present paper constant matrices $\widehat{\alpha}^{(j)}$ $(j = 1, 2, \dots, d)$ and $\widehat{\beta}$ are assumed to be simply Hermitian.

Let us take the Hamiltonian function

$$\mathcal{H}(t,x,p) = c\sum_{j=1}^{d} \widehat{\alpha}^{(j)}(p_j - A_j(t,x)) + \widehat{\beta} mc^2 + V(t,x) \tag{1.3}$$

as in (23) of §69, p.261 of [4], where $p \in \mathbb{R}^d$ is the canonical momentum. We write the kinetic momentum as $\xi := p - A(t,x) \in \mathbb{R}^d$. Then the Lagrangian function is given by

$$\mathcal{L}(t,x,\dot{x},\xi) = p \cdot \dot{x} - \mathcal{H}(t,x,p) \tag{1.4}$$
$$= \xi \cdot \dot{x} + \dot{x} \cdot A(t,x) - V(t,x) - (c\widehat{\alpha} \cdot \xi + \widehat{\beta} mc^2),$$

where $\dot{x} \in \mathbb{R}^d, p \cdot \dot{x} = \sum_{j=1}^d p_j \dot{x}_j, \widehat{\alpha} = (\widehat{\alpha}^{(1)}, \dots, \widehat{\alpha}^{(d)})$ and $\widehat{\alpha} \cdot \xi = \sum_{j=1}^d \widehat{\alpha}^{(j)} \xi_j$.

In the present paper we will determine the Feynman path integral in phase space mathematically in terms of the Lagrangian function (1.4). Let $\tau_j \in \mathbb{R}$ $(j = 1, 2, \dots, \nu-1)$ and define a time division $\Delta := \{\tau_j\}_{j=1}^{\nu-1}$. We don't necessarily assume $\tau_j < \tau_{j+1}$. It is possible that $\tau_j \ge \tau_{j+1}$ for some j hold. We set $\tau_0 = t_i$ and $\tau_\nu = t$. Let $x \in \mathbb{R}^d$ be fixed. We take arbitrary points $x^{(j)} \in \mathbb{R}^d$ $(j = 0, 1, \dots, \nu-1)$ and determine a piecewise linear path $(\Theta_\Delta, q_\Delta(x^{(0)}, \dots, x^{(\nu-1)}, x))$ in $\mathbb{R}^{d+1}$ joining $(\tau_j, x^{(j)})$ $(j = 0, 1, \dots, \nu, x^{(\nu)} = x)$ in order. We also take arbitrary points $\xi^{(j)} \in \mathbb{R}^d$ $(j = 0, 1, \dots, \nu-1)$ and determine a piecewise constant path $(\Theta_\Delta, \xi_\Delta(\xi^{(0)}, \dots, \xi^{(\nu-1)}))$ in $\mathbb{R}^{d+1}$ by using ξ_Δ that takes value

$\xi^{(j)}$ $(j = 0, 1, \ldots, \nu - 1)$ for $\theta \in [\tau_j, \tau_{j+1}]$ if $\tau_j \le \tau_{j+1}$ or $\theta \in [\tau_{j+1}, \tau_j]$ if $\tau_{j+1} < \tau_j$. We note that the paths $(\Theta_\Delta, q_\Delta)$ and $(\Theta_\Delta, \xi_\Delta)$ go in any direction forward and backward in time and that q_Δ has any speed, even the infinite speed.

Let t and s be in $\mathbb{R}$ and $t \neq s$. For x and y in $\mathbb{R}^d$ we define

$$q^{t,s}_{x,y}(\theta) := y + \frac{\theta - s}{t - s}(x - y) \tag{1.5}$$

in $s \le \theta \le t$ or $t \le \theta \le s$. Let $\xi \in R^d$. We consider a path $(q^{t,s}_{x,y}(\theta), \xi) \in \mathbb{R}^{2d}$ in phase space. Then the classical action is given by

$$\begin{aligned}(1.6)\quad S(t, s; x, \xi, y) &:= \int_s^t \mathcal{L}(\theta, q^{t,s}_{x,y}(\theta), \dot{q}^{t,s}_{x,y}(\theta), \xi)d\theta = (x - y)\cdot\xi \\ &+ \int_s^t \left\{\dot{q}^{t,s}_{x,y}(\theta)\cdot A(\theta, q^{t,s}_{x,y}(\theta)) - V(\theta, q^{t,s}_{x,y}(\theta))\right\}d\theta - (t - s)(c\widehat{\alpha}\cdot\xi + \widehat{\beta}mc^2) \\ &= (x - y)\cdot\xi + (x - y)\cdot\int_0^1 A(t - \theta\rho, x - \theta(x - y))d\theta \\ &- \rho\int_0^1 V(t - \theta\rho, x - \theta(x - y))d\theta - \rho(c\widehat{\alpha}\cdot\xi + \widehat{\beta}mc^2), \quad \rho = t - s\end{aligned}$$

from (1.4), where $\dot{q}^{t,s}_{x,y}(\theta) = dq^{t,s}_{x,y}(\theta)/d\theta$. From (1.6) we define $S(s, s; x, \xi, y)$ by

$$S(s, s; x, \xi, y) := (x - y)\cdot\xi + (x - y)\cdot\int_0^1 A(s, x - \theta(x - y))d\theta, \tag{1.7}$$

which we write $\int_s^s \mathcal{L}(\theta, q^{s,s}_{x,y}(\theta), \dot{q}^{s,s}_{x,y}(\theta), \xi)d\theta$ formally.

We take $\chi \in C_0^\infty(\mathbb{R}^d)$, i.e. an infinitely differentiable function in $\mathbb{R}^d$ with compact support, such that $\chi(0) = 1$. The approximation $K_{D\Delta}(t, t_i)f$ of the Feynman path integral $K_D(t, t_i)f$ for the Dirac equation (1.2) is determined by

$$\begin{aligned}(1.8)\quad K_{D\Delta}(t, t_i)f &= \iint e^{*iS(t, q_\Delta, \xi_\Delta)} f(x^{(0)})\mathcal{D}q_\Delta\mathcal{D}\xi_\Delta \\ &:= \lim_{\epsilon\to+0}\int\cdots\int e^{*iS(t, q_\Delta, \xi_\Delta)} f(x^{(0)})\prod_{j=0}^{\nu-1}\left\{\chi(\epsilon x^{(j)})\chi(\epsilon\xi^{(j)})\right\}dx^{(0)}\cdots dx^{(\nu-1)} \\ &\cdot đ\xi^{(0)}\cdots đ\xi^{(\nu-1)}\end{aligned}$$

for $f = {}^t(f_1, \cdots, f_d) \in \mathcal{S}(\mathbb{R}^d)^N$, i.e. the Schwartz rapidly decreasing function, where $đ\xi^{(j)} = (2\pi)^{-d}d\xi^{(j)}$ and the probability amplitude $\exp *iS(t, q_\Delta, \xi_\Delta)$ for a path $(\Theta_\Delta, q_\Delta, \xi_\Delta)$

is defined as a product of matrices in terms of the Lagrangian function (1.4) by

(1.9)
$$\exp i\int_{\tau_{\nu-1}}^{t}\mathcal{L}(\theta, q^{t,\tau_{\nu-1}}_{x,x^{(\nu-1)}}(\theta), \dot{q}^{t,\tau_{\nu-1}}_{x,x^{(\nu-1)}}(\theta), \xi^{(\nu-1)})d\theta\cdot\exp i\int_{\tau_{\nu-2}}^{\tau_{\nu-1}}\mathcal{L}(\theta, q^{\tau_{\nu-1},\tau_{\nu-2}}_{x^{(\nu-1)},x^{(\nu-2)}}(\theta),$$
$$\dot{q}^{\tau_{\nu-1},\tau_{\nu-2}}_{x^{(\nu-1)},x^{(\nu-2)}}(\theta), \xi^{(\nu-2)})d\theta\cdots\exp i\int_{t_i}^{\tau_1}\mathcal{L}(\theta, q^{\tau_1,t_i}_{x^{(1)},x^{(0)}}(\theta), \dot{q}^{\tau_1,t_i}_{x^{(1)},x^{(0)}}(\theta), \xi^{(0)})d\theta.$$

It will be proved in Theorem 1.1 below that $K_{D\Delta}(t,t_i)f$ is determined independently of the choice of χ. The last equation in (1.8) is called the oscillatory integral and often written as

$$\mathrm{Os}-\int\cdots\int e^{*iS(t,q_\Delta,\xi_\Delta)}f(x^{(0)})dx^{(0)}\cdots dx^{(\nu-1)}đ\xi^{(0)}\cdots đ\xi^{(\nu-1)}$$

(cf. p. 45 of [21]).

Let $L^2(\mathbb{R}^d)$ denote the space of all square integrable functions in $\mathbb{R}^d$ with inner product $(f,g) := \int f(x)\overline{g(x)}dx$ and norm $\|f\|$, where $\overline{g(x)}$ denotes the complex conjugate of $g(x)$. We denote the product Hilbert space of N copies of $L^2(\mathbb{R}^d)$ by $L^2(\mathbb{R}^d)^N$ and write its norm as $\|f\| = \sqrt{\sum_{j=1}^d \|f_j\|}$ for $f = {}^t(f_1,\dots,f_d)$.

For an $x = (x_1,\dots,x_d) \in \mathbb{R}^d$ and a multi-index $\alpha = (\alpha_1,\dots,\alpha_d)$ we write $|\alpha| = \sum_{j=1}^d \alpha_j$, $x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$, $\partial_{x_j} = \partial/\partial x_j$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$. The main theorem in the present paper is the following.

Theorem 1.1 ([19]). *Let $\partial_x^\alpha E_j(t,x)$ $(j = 1,2,\dots,d)$, $\partial_x^\alpha B_{jk}(t,x)$ $(1 \le j < k \le d)$ and $\partial_t B_{jk}(t,x)$ be continuous in $\mathbb{R}^{d+1}$ for all α. We assume the adiabatic hypothesis: There exists a sufficient large $T_0 > 0$ such that*

$$E(t,x) = 0,\ B_{jk}(t,x) = 0\ (1 \le j < k \le d) \tag{1.10}$$

for $|t| \ge T_0$ (p. 93 in [23]). In addition, we assume

$$|\partial_x^\alpha E_j(t,x)| \le C_\alpha, \quad |\alpha| \ge 1, \tag{1.11}$$

$$|\partial_x^\alpha B_{jk}(t,x)| \le C_\alpha < x >^{-(1+\delta_\alpha)}, \quad |\alpha| \ge 1 \tag{1.12}$$

in $\mathbb{R}^{d+1}$ with constants $\delta_\alpha > 0$ for $j,k = 1,2,\dots,d$. Let $(V, A_1,\dots,A_d)$ be an electromagnetic potential inducing $E(t,x)$ and $(B_{jk}(t,x))_{1\le j<k\le d}$ via equation (1.1) *such that $V, \partial_{x_j}V, \partial_t A_k$ and $\partial_{x_j}A_k$ $(j,k = 1,2,\dots,d)$ are continuous in $\mathbb{R}^{d+1}$.*

Let us define $K_{D\Delta}(t,t_i)f$ for $f \in \mathcal{S}^N$ by (1.8) for a time division Δ. We define

$$\sigma(\Delta) := \sum_{j=0}^{\nu-1}{}'(\tau_{j+1}-\tau_j)^2, \tag{1.13}$$

where $\sum'$ means the sum excluding the term $(\tau_{j+1} - \tau_j)^2$ such that $\tau_j, \tau_{j+1} \geq T_0$ or $\tau_j, \tau_{j+1} \leq -T_0$. Then we have: (1) $K_{D\Delta}(t, t_i)$ on $\mathcal{S}^N$ is determined independently of the choice of χ and can be extended to a bounded operator on $(L^2)^N$. We have

$$\|K_{D\Delta}(t,t_i)f\| \leq e^{K_0\sigma(\Delta)}\|f\| \tag{1.14}$$

for all t, t_i in $\mathbb{R}$ with a constant $K_0 \geq 0$. (2) Let $f \in (L^2)^N$. Then, as $\sigma(\Delta) \to 0$, $K_{D\Delta}(t,t_i)f$ converges in $(L^2)^N$ uniformly with respect to $t \in \mathbb{R}$ and $t_i \in \mathbb{R}$. We call this limit the Feynman path integral and write it $K_D(t,t_i)f$. (3) $K_D(t,t_i)f$ for $f \in (L^2)^N$ belongs to $\mathcal{E}_t^0(\mathbb{R};(L^2)^N)$ and is the solution to the Dirac equation (1.2) *in distribution sense with $u(t_i) = f$, where $\mathcal{E}_t^j(\mathbb{R};(L^2)^N)$ $(j = 0, 1, \dots)$ denotes the space of all $(L^2)^N$-valued j-times continuously differentiable functions in $t \in \mathbb{R}$. (4) Let $t_i < t_1 < t$. Then we have the rule for two events:*

$$K_D(t,t_i)f = K_D(t,t_1)K_D(t_1,t_i)f, \ \ K_D(t,t_1)f = K_D(t,t_i)K_D(t_i,t_1)f$$

for $f \in (L^2)^N$. (5) Let $\psi(t,x)$ be a real-valued function such that $\partial_{x_j}\partial_{x_k}\psi(t,x)$ and $\partial_t\partial_{x_j}\psi(t,x)$ $(j,k = 1,2,\dots,d)$ are continuous in $\mathbb{R}^{d+1}$ and consider the gauge transformation

$$V' = V - \frac{\partial\psi}{\partial t}, \quad A'_j = A_j + \frac{\partial\psi}{\partial x_j}. \tag{1.15}$$

We write (1.8) *for this (V', A') as $K'_{D\Delta}(t,t_i)f$. Then we have the formula*

$$K'_{D\Delta}(t,t_i)f = e^{i\psi(t,\cdot)}K_{D\Delta}(t,t_i)\left(e^{-i\psi(t_i,\cdot)}f\right) \tag{1.16}$$

for all $f \in (L^2)^N$. (6) Let us define the subset Δ' of Δ with the same ordering as in Δ by the compliment of $\{\tau_j \in \Delta \ (j \geq 1); \tau_{j-1}, \tau_j, \tau_{j+1} \geq T_0$ or $\tau_{j-1}, \tau_j, \tau_{j+1} \leq -T_0\}$. Then we have

$$K_{D\Delta}(t,t_i)f = K_{D\Delta'}(t,t_i)f.$$

We could say from (1.8) that the Feynman path integral $K_D(t,t_i)f$ is written in the form of the sum-over-histories, satisfying the superposition principle. That is, it is given by the "sum" of the probability amplitudes with a common weight over all possible paths that go in any direction at any speed forward and backward in time.

This form of the Feynman path integral is the one that Feynman stated repeatedly. F. Dyson says the following on p.376 of [5]: Thirty-one years ago, Dick Feynman told me about his "sum over history" version of quantum-mechanics. "The electron does anything it likes," he said. "It just goes in any direction at any speed, forward or backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function." I said to him, "You're crazy." But he wasn't.

We recommend readers interested in this fact to see also p.752 of [6], p.772 of [7] and p.163 of [9]. We should note that at present, in the physical theory positrons are represented as electrons going back in time (cf. p.61 of [23], p.54 of [25], and pp.150 and 240 of [28]).

It is stated on p.38 of [8] that in the relativistic theory of the electron we shall not find it possible to express the amplitude for a path as e^{iS}, or in any other simple way. Moreover, it is stated by Feynman on p.169 of [9] that And, so I dreamed that if I were clever, I would find a formula for the amplitude of a path ..., which would be equivalent to the Dirac equation, ... I have never succeeded in that either.

On the other hand, we note that our way of representing the amplitude of an electron in terms of the Lagrangian function, that I stated in Theorem 1.1 in the present paper, is enough simple.

Now we go back to the past studies of the Feynman path integral for the Dirac equation. There seems to be no past studies of the Feynman path integral in phase space. All studies were made of that in configuration space. Consider the Dirac equation in two dimensional spacetime

$$i\frac{\partial u}{\partial t}(t) = \left[c\widehat{\alpha}^{(0)}\left(\frac{1}{i}\frac{\partial}{\partial x} - A(t,x)\right)u + \widehat{\beta}^{(0)}mc^2 + V(t,x)\right]u(t), \tag{1.17}$$

where $u(t) = {}^t(u_1(t), u_2(t)) \in \mathbb{C}^2$ and and

$$\widehat{\alpha}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widehat{\beta}^{(0)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Let $(V, A) = 0$ in (1.17). Suppose that the interval $[t_i, t]$ $(t_i < t)$ is divided into small equal steps of length $\epsilon_0 > 0$. We consider all zigzags in the spacetime of straight segments with velocity c that go only forward in time. The amplitude for each zigzag is given by $(i\epsilon_0)^R$, where R is the number of its reversals. It follows from the superposition principle that the Feynman path integral was determined by (2-27), p.35 of [8]. See Appendix E, p.118 of [27] in detail. In [10] and Theorem 2.1 of [11] the solution to a general (1.17) was written in terms of a measure on the space of all continuous paths in $[t_i, t]$. In Theorem, p.8 of [1] and p.221 of [3] the solution to (1.17) was written in terms of a Poisson process. All results in [1], [3], [10] and [11] does not satisfy the superposition principle and also don't consider paths that go backward in time.

Remark. We consider inhomogeneous Lorentz transformations. We set $x_0 = ct$. Take a j such that $1 \le j \le d$ and a constant $0 \le \beta < 1$. The Lorentz transformation called a boost is given by

$$x_0' = (x_0 - \beta x_j)/\sqrt{1-\beta^2}, \quad x_j' = (x_j - \beta x_0)/\sqrt{1-\beta^2}, \quad x_k' = x_k \ (k \ne j).$$

In addition, we know a rotation in the configuration space $\mathbb{R}_x^d$, the space reflection, the time reversal and the translation in the configuration space $\mathbb{R}_x^d$ as other elementary Lorentz transformations (cf. p.40 of [25]). It is also well known (e.g. Theorem 2 and a canonical coordinate system of the second kind of the Lie group in §10 of Chapter IV of [24]) that all inhomogenious Lorentz transformations can be represented as a product of a finite number of elementary Lorentz transformations.

Let us take an initial point $(t_i, x^{(0)})$ and a final point (t, x) arbitrarily, and fix them. We consider the space of all paths $(\Theta_\Delta, q_\Delta(x^{(0)}, \ldots, x^{(\nu-1)}))$ taking arbitrary Δ and arbitrary $x^{(j)} \in \mathbb{R}^d$ $(j = 1, 2, \ldots, \nu - 1)$. Then, we can easily see from the definition of a piecewise linear path $(\Theta_\Delta, q_\Delta)$ that this space is invariant under the Lorentz transformation. So is the space of all zigzags in [8] with velocity c. On the other hand, the space of all continuous paths in $[t_i, t]$, studied in [1], [3], [10] and [11], is not invariant under a Lorentz boost.

We will explain an idea for proving our results. Let t and s be in $\mathbb{R}$. Let $\epsilon > 0$ be a constant and $\chi \in C_0^\infty(\mathbb{R}^d)$ the function taken before. We define an operator

$$(G_\epsilon(t,s)f)(x) = \iint e^{iS(t,s;x,\xi,y)} f(y)\chi(\epsilon\xi)dy đ\xi \tag{1.18}$$

for $f \in \mathcal{S}(\mathbb{R}^d)^N$ in terms of (1.6) and (1.7). Then we can write

$$K_{D\Delta}(t,t_i)f = \lim_{\epsilon\to 0} G_\epsilon(t,\tau_{\nu-1})\chi(\epsilon\cdot)G_\epsilon(\tau_{\nu-1},\tau_{\nu-2})\chi(\epsilon\cdot)\cdots\chi(\epsilon\cdot)G_\epsilon(\tau_1,t_i)f \tag{1.19}$$

from (1.8) and (1.9).

We will prove that $\{G_\epsilon(t,s)\}_{0<\epsilon\le 1}$ is a bounded family of operators from $\mathcal{S}(\mathbb{R}^d)^N$ into itself, that there exists an operator $G(t,s)$ on $\mathcal{S}(\mathbb{R}^d)^N$ independent of the choice of χ satisfying

$$G(t,s)f = \lim_{\epsilon\to 0} G_\epsilon(t,s)f \tag{1.20}$$

in $\mathcal{S}(\mathbb{R}^d)^N$ for $f \in \mathcal{S}(\mathbb{R}^d)^N$, that the stability

$$\|G(t,s)f\| \le e^{K_0(t-s)^2}\|f\| \tag{1.21}$$

holds with a constant $K_0 \ge 0$ and that the consistency

$$\lim_{\epsilon\to 0}\left\|\left(i\frac{\partial}{\partial t} - H(t)\right)G_\epsilon(t,s)f\right\| \le C|t-s|\| < \cdot >^M f\| \tag{1.22}$$

holds with a constant $C \ge 0$ and a positive integer M, where $< x >= \sqrt{1+|x|^2}$ and $f \in \mathcal{S}(\mathbb{R}^d)^N$.

From (1.19) and (1.20) we can easily see

$$K_{D\Delta}(t,t_i)f = G(t,\tau_{\nu-1})G(\tau_{\nu-1},\tau_{\nu-2})\cdots G(\tau_1,t_i)f \tag{1.23}$$

for $f \in \mathcal{S}(\mathbb{R}^d)^N$, which is determined independently of the choice of χ. Let $U(t,t_i)f$ for $f \in \mathcal{S}(\mathbb{R}^d)^N$ be the solution to (1.2) with $u(t_i) = f$. From (1.21) and (1.22) we will prove that

$$\|K_{D\Delta}(t,t_i)f - U(t,t_i)f\| = \|G(t,\tau_{\nu-1})\cdots G(\tau_1,t_i)f - U(t,\tau_{\nu-1})\cdots U(\tau_1,t_i)f\|$$

for $f \in L^2(\mathbb{R}^d)^N$ converges to 0 uniformly in $t \in \mathbb{R}$ and $t_i \in \mathbb{R}$ as $\sigma(\Delta) \to 0$. It should be emphasized that the theory of pseudo-differential operators plays an important role to prove (1.21) and (1.22).

The plan of the present paper is as follows. In §2 some other theorems in addition to Theorem 1.1 and some remarks will be stated. In §§3 and 4, a roughly sketched proof of (1.21) and (1.22) will be given, respectively. In §5 a roughly sketched proof of theorems in the present paper will be given.

§ 2. Other theorems and some remarks

Let M and a be positive integers. We introduce the weighted Sobolev spaces $B^a_M(\mathbb{R}^d)^N := \{f \in L^2(\mathbb{R}^d)^N;\ \|f\|_{B^a_M} := \|f\| + \sum_{|\alpha|=aM}\|x^\alpha f\| + \sum_{|\alpha|=a}\|\partial^\alpha_x f\| < \infty\}$. Let $B^{-a}_M(\mathbb{R}^d)^N$ denote their dual spaces. We set $B^0_M(\mathbb{R}^d)^N := L^2(\mathbb{R}^d)^N$. We can easily prove

$$\mathcal{S}(\mathbb{R}^d) = \cap_{a=0}^\infty B^a_M(\mathbb{R}^d), \quad \mathcal{S}'(\mathbb{R}^d) = \cup_{a=0}^\infty B^{-a}_M(\mathbb{R}^d). \tag{2.1}$$

In the present paper we often use symbols $C, C_\alpha, C_{\alpha,\beta}$ and C_a to write down constants, though these values are different in general. We note again that throughout the present paper constant matrices $\widehat{\alpha}^{(j)}$ $(j = 1,2,\ldots,d)$ and $\widehat{\beta}$ in (1.2) are assumed to be simply Hermitian.

We can prove the following on the Feynman path integral in the weighted Sobolev spaces.

Theorem 2.1. *We assume* (1.10) - (1.12) *in Theorem 1.1. Let* (V,A) *be an electromagnetic potential inducing* $E(t,x)$ *and* $(B_{jk}(t,x))_{1\le j<k\le d}$ *via equation* (1.1). *We also assume*

$$|\partial^\alpha_x A_j(t,x)| \le C_\alpha, \quad |\alpha| \ge 1 \tag{2.2}$$

in $[-2T_0, 2T_0]\times\mathbb{R}^d$ *for* $j = 1,2,\ldots,d$ *and that there exists an integer* $M \ge 1$ *satisfying*

$$|\partial^\alpha_x V(t,x)| \le C_\alpha < x >^M, \quad |\alpha| \ge 1 \tag{2.3}$$

and

$$|\partial_x^\alpha \partial_t A_j(t,x)| \le C_\alpha < x >^M \tag{2.4}$$

for all α in $[-2T_0, 2T_0] \times \mathbb{R}^d$. Let $L_0 \ge 0$ an arbitrary constant. We consider only a family of time divisions $\Delta = \{\tau_j\}_{j=0}^{\nu-1}$ such that

$$\sum_{j=0}^{\nu-1}{}' |\tau_{j+1} - \tau_j| \le L_0. \tag{2.5}$$

We set $|\Delta| := \max_{0 \le j \le \nu-1} |\tau_{j+1} - \tau_j|$.

Then we have: (1) $K_{D\Delta}(t,t_i)$ on $\mathcal{S}^N$ can be extended to a bounded operator on $(B_M^a)^N$ ($a = 0, \pm 1, \pm 2 \ldots$). (2) Let $f \in (B_{M+1}^a)^N$. Then, as $|\Delta| \to 0$ under the assumption (2.5), *$K_{D\Delta}(t,t_i)f$ converges to $K_D(t,t_i)f$ in $(B_{M+1}^a)^N$ uniformly in $t \in \mathbb{R}$ and $t_i \in \mathbb{R}$.*

Remark. Under the assumptions of Theorem 1.1 on $E(t,x)$ and $B_{jk}(t,x)$ we can find an electromagnetic potential (V, A) satisfying (2.2) - (2.4) with $M = 1$. See Lemma 6.1 in [14].

Remark. Let us compare the result of Theorem 2.1 to Theorem 1.1 in the case of $L^2(\mathbb{R}^d)^N$. We can easily see that Theorem 1.1 gives a generalization of Theorem 2.1 because of

$$\sigma(\Delta) = \sum_{j=0}^{\nu-1}{}' (\tau_{j+1} - \tau_j)^2 \le |\Delta| L_0$$

from (2.5).

Let $\lambda_j(\xi)$ ($j = 1, 2, \ldots, N$) be the eigenvalue of $\widehat{\alpha} \cdot \xi$ and set

$$\lambda_{\max} = \max_{j=1,2,\ldots,N} \sup_{|\xi|=1} \lambda_j(\xi). \tag{2.6}$$

We can easily see $\lambda_{\max} \ge 0$ because of $\lambda_j(s\xi) = s\lambda_j(\xi)$ ($s \in \mathbb{R}$).

We can prove the following two theorems on causality of the Feynman path integral $K_D(t,t_i)f$. That is, $K_D(t,t_i)f$ has the propagation speed not exceeding the velocity $c\lambda_{\max}$.

Theorem 2.2. *Let $f \in (L^2)^N$ and $K_D(t,t_i)f$ the Feynman path integral determined in Theorem 1.1. Then, $K_D(t,t_i)f$ has the propagation speed not exceeding the velocity $c\lambda_{\max}$. That is, if supp $f(\cdot) \subset \{x \in \mathbb{R}^d; |x - b| \le R\}$ for $b \in \mathbb{R}^d$, then we have supp $K_D(t,t_i)f(\cdot) \subset \{x \in \mathbb{R}^d; |x - b| \le c\lambda_{\max}|t - t_i| + R\}$.*

Theorem 2.3. *Let $f \in (B^a_{M+1})^N$ $(a = 0, \pm 1, \pm 2, \dots)$ and $K_D(t, t_i)f$ the Feynman path integral determined in Theorem 2.1. Then, $K_D(t, t_i)f$ has the propagation speed not exceeding the velocity $c\lambda_{\max}$.*

Example 2.4. Let $T_0 > 0$ be the constant in Theorem 1.1. Let $n \geq 1$ be an arbitrary integer and take an arbitrarily large t_k $(k = 1, 2, \dots, n)$ such that $t_k > T_0$ and an arbitrarily small t'_k $(k = 1, 2, \dots, n)$ such that $t'_k < -T_0$. Then we can easily determine a time division $\Delta_n = \{\tau_j\}_{j=1}^{\nu-1}$ $(\nu = n^3)$ such that $t_k, t'_k \in \Delta_n$ $(k = 1, 2, \dots, n)$ by taking

$$|\tau_1 - \tau_0| \leq 4T_0 n^{-2}, \quad |\tau_{j+1} - \tau_j| = 4T_0 n^{-2}$$

if $\tau_j \in [-T_0, T_0]$ or $\tau_{j+1} \in [-T_0, T_0]$. We can easily see

$$\sigma(\Delta_n) = \sum_{j=0}^{\nu-1}{}' (\tau_{j+1} - \tau_j)^2 \leq n^3 (4T_0)^2 n^{-4} = (4T_0)^2 \frac{1}{n},$$

which shows $\sigma(\Delta_n) \to 0$ as $n \to \infty$. Hence it follows from Theorem 1.1 that $K_{D\Delta_n}(t, t_i)f \to K_D(t, t_i)f$ in $(L^2)^N$ as $n \to \infty$ for $f \in (L^2)^N$.

Remark. It is stated by Feynman on p.163 of [9]: Professor Wheeler telephoned to Feynman that "suppose that the world lines ... were a tremendous knot, and then, when we cut through the knot, by the plane corresponding to a fixed time, we would see many, many world lines and that would represent many electrons, except for one thing. ...."

Now, let us consider the time divisions Δ_n determined in Example 2.4 and cut thorough the knot by the plane corresponding to a time t such that $|t| \leq T_0$. Then we see more than n electrons and more than n positrons. Letting $n \to \infty$, we can see countably infinite electrons and positrons.

Remark. Let us consider the time divisions Δ_n determined in Example 2.4 again. Let $f \in (L^2)^N$. We consider the limit of $K_{D\Delta_n}(t, t_i)f$ as $t_k \to \infty, t'_l \to -\infty$ $(j, l = 1, 2, \dots, n)$, which we write $K_{D\widehat{\Delta}_n}(t, t_i)f$. It follows from (6) in Theorem 1.1 that $K_{D\Delta_n}(t, t_i)f$ is equal to $K_{D\Delta'_n}(t, t_i)f$ and so, $K_{D\widehat{\Delta}_n}(t, t_i)f = K_{D\Delta'_n}(t, t_i)f$. Hence $K_{D\widehat{\Delta}_n}(t, t_i)f$ converges to $K_D(t, t_i)f$ in $(L^2)^N$ as $n \to \infty$. We note that the path integral $K_{D\widehat{\Delta}_n}(t, t_i)f$ is defined by the paths going across the infinite past and future n times.

Example 2.5. Let $T_0 > 0$ be the constant in Theorem 1.1. We take an L_0 such that $L_0 \geq 4T_0$. Let $l_0 \geq 1$ be the greatest integer less than or equal to $L_0/(4T_0)$ and take an arbitrarily large t_k $(k = 1, 2, \dots, l_0)$ such that $t_k > T_0$ and an arbitrarily small t'_k $(k = 1, 2, \dots, l_0)$ such that $t'_k < -T_0$. Let $n \geq 1$ be an arbitrary integer. Then we can easily see determine a time division $\Delta_n = \{\tau_j\}_{j=1}^{\nu-1}$ $(\nu = nl_0)$ such that

$t_k, t'_k \in \Delta_n$ $(k = 1, 2, \ldots, l_0)$ by taking

$$|\tau_1 - \tau_0| \le 4T_0 n^{-1}, \quad |\tau_{j+1} - \tau_j| = 4T_0 n^{-1} \quad (j = 1, 2, \ldots, \nu - 1)$$

if $\tau_j \in [-T_0, T_0]$ or $\tau_{j+1} \in [-T_0, T_0]$. We can easily see

$$\sum_{j=0}^{\nu-1}{}' |\tau_{j+1} - \tau_j| \le n l_0 \cdot \frac{4T_0}{n} \le L_0,$$

which satisfies (2.5) and $|\Delta_n| \to 0$ as $n \to \infty$. Hence it follows from Theorem 2.1 that $K_{D\Delta_n}(t, t_i)f \to K_D(t, t_i)f$ in $(B^a_M)^N$ as $n \to \infty$ for $f \in (B^a_M)^N$.

Let us consider the scattering problem as in §6-4 of [8]. Let $U_0(t, t_i)f$ be the solution with $u(t_i) = f$ to the free Dirac equation (1.2), i.e. with $(V, A) = 0$. Let $T_0 > 0$ be the constant in Theorem 1.1. We consider the scattering operator

$$Sf = (W_+)^* W_- f := \lim_{t\to\infty} U_0(t, 0)^{-1} U(t, 0) \lim_{t_i \to -\infty} U(t_i, 0)^{-1} U_0(t_i, 0) f \tag{2.7}$$

as in p.527 of [20]. Then we have

Theorem 2.6. *Let Δ be time divisions such that $T_0 \in \Delta$ and $-T_0 \in \Delta$. Then under the assumptions of Theorem 1.1 we have*

$$Sf = U_0(T_0, 0)^* \lim_{\sigma(\Delta)\to 0} K_{D\Delta}(T_0, -T_0) U_0(0, -T_0)^* f \tag{2.8}$$

for $f \in (L^2)^N$.

§ 3. A roughly sketched proof of (1.21)

Hereafter, where no confusion can arise, we write $\mathcal{S}(\mathbb{R}^d)^N, L^2(\mathbb{R}^d)^N$ and $B^a_M(\mathbb{R}^d)^N$ as $\mathcal{S}(\mathbb{R}^d), L^2(\mathbb{R}^d)$ and $B^a_M(\mathbb{R}^d)$, respectively for the sake of simplicity, omitting the superscript N.

The following gives a formula of derivatives of a matrix-valued function. We will use this formula repeatedly.

Lemma 3.1. *Let $A(w)$ $(w \in \mathbb{R}^d)$ be an $N \times N$ matrix whose all components are continuously differentiable with respect to w. Then we have*

$$\frac{\partial}{\partial w_j} e^{A(w)} = \int_0^1 e^{(1-\tau)A(w)} \frac{\partial A}{\partial w_j}(w) e^{\tau A(w)} d\tau. \tag{3.1}$$

Proof. We set $u(t;w) = e^{tA(w)}$. Then

$$\frac{\partial u}{\partial t}(t;w) = A(w)u(t;w).$$

So

$$\frac{d}{dt}\frac{\partial u}{\partial w_j}(t;w) = A(w)\frac{\partial u}{\partial w_j}(t;w) + \frac{\partial A}{\partial w_j}(w)u(t;w)$$

with $\partial u(0;w)/\partial w_j = 0$. Consequently we have

$$\frac{\partial u}{\partial w_j}(t;w) = \int_0^t e^{(t-\tau)A(w)}\frac{\partial A}{\partial w_j}(w)e^{\tau A(w)}d\tau,$$

which shows (3.1). □

Let us write

$$\mathbf{q}^{t,s}_{x,y} : \mathbf{q}^{t,s}_{x,y}(\theta) = (\theta, q^{t,s}_{x,y}(\theta)) \in \mathbb{R}^{d+1} \ (s \le \theta \le t \text{ or } t \le \theta \le s). \tag{3.2}$$

Lemma 3.2. *Let t and s be in $\mathbb{R}$ such that $t \ne s$. Then we have*

$$\left(\int_{\mathbf{q}^{t,s}_{y,x}} - \int_{\mathbf{q}^{t,s}_{y,z}}\right)(A \cdot dx - Vdt) = (x - z) \cdot \Psi(t,s;x,y,z), \tag{3.3}$$

where $\Psi = (\Psi_1, \dots, \Psi_d) \in \mathbb{R}^d$ and

$$\begin{aligned}\Psi_j(t,s;x,y,z) = &-\int_0^1 A_j(s, z + \theta(x-z))d\theta \\ &+ (t-s)\int_0^1\int_0^1 \sigma_1 E_j(t - \sigma_1(t-s), y + \sigma_1(z-y) + \sigma_1\sigma_2(x-z))d\sigma_1 d\sigma_2 \\ &+ \sum_{k=1}^d (y_k - z_k)\int_0^1\int_0^1 B_{jk}(t - \sigma_1(t-s), y + \sigma_1(z-y) + \sigma_1\sigma_2(x-z))d\sigma_1 d\sigma_2.\end{aligned} \tag{3.4}$$

Proof. We can prove Lemma 3.2 from the Stokes theorem

$$\left(\int_{\mathbf{q}^{t,s}_{y,x}} - \int_{\mathbf{q}^{t,s}_{y,z}} + \int_{\mathbf{q}^{s,s}_{x,z}}\right)(A \cdot dx - Vdt) = \iint_\Lambda d(A \cdot dx - Vdt) \tag{3.5}$$

and

$$d(A \cdot dx - Vdt) = -\sum_{j=1}^d E_j(t,x)dt \wedge dx_j + \sum_{1 \le j < k \le d} B_{jk}dx_j \wedge dx_k, \tag{3.6}$$

where Λ is the 2-dimensional plane in $\mathbb{R}^{d+1}$ with oriented boundary consisting of $\mathbf{q}^{t,s}_{y,x}$, $-\mathbf{q}^{t,s}_{y,z}$ and $\mathbf{q}^{s,s}_{x,z}$. □

To avoid the complexity we suppose hereafter that χ in (1.8) and (1.18) is real-valued.

Proposition 3.3. *Let t and s be in $\mathbb{R}$ such that $t \neq s$. Let $\Psi = \Psi(t, s; x, y, z)$ be the function defined by* (3.3). *Then for $f \in \mathcal{S}$ we have*

$$(3.7) \quad (G_\epsilon(t,s)^* G_\epsilon(t,s)f)(x) = \iint e^{i(x-z)\cdot\xi} dz đ\xi \iint e^{-i\eta\cdot w} e^{i(t-s)(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2)} \times e^{-i(t-s)(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2 - c\widehat{\alpha}\cdot\eta)} \chi(\epsilon(\xi+\Psi))\chi(\epsilon(\xi+\Psi-\eta)) f(z) dw đ\eta$$

with $\Psi = \Psi(t, s; x, w+z, z)$, where $\eta \in \mathbb{R}^d$, $w \in \mathbb{R}^d$ and $G_\epsilon(t,s)^*$ denotes the formally adjoint operator of $G_\epsilon(t,s)$.

Proof. Since $S(t, s; x, \xi, y)$ is a Hermitian matrix from (1.6), $G_\epsilon(t,s)^*$ is written as

$$(G_\epsilon(t,s)^* f)(x) = \iint e^{-iS(t,s;y,\xi,x)} f(y)\chi(\epsilon\xi) dy đ\xi$$

from (1.18). From this formula we can prove Proposition 3.3 directly. □

Proposition 3.4. *Under the assumptions* (1.10) - (1.12) *and* (2.2) *we have*

$$(3.8) \quad |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \Psi_j(t, s; x, y, z)| \le C_{\alpha,\beta,\gamma}, \ |\alpha+\beta+\gamma| \ge 1$$

in $s, t \in [-2T_0, 2T_0]$ and $x, y, z \in \mathbb{R}^d$ for $j = 1, 2, \ldots, d$.

Proof. Proposition 3.4 follows from (3.4) and the assumptions. □

Lemma 3.5. *Let A and B be $N \times N$ matrices. Then we have*

$$e^{A+B} = e^A + \int_0^1 d\theta \int_0^1 e^{(1-\tau)(A+\theta B)} B e^{\tau(A+\theta B)} d\tau.$$

Proof. We can prove Lemma 3.5 from Lemma 3.1. □

From Proposition 3.3 and Lemma 3.5 we can prove

$$(3.9) \quad (G_\epsilon(t,s)^* G_\epsilon(t,s)f)(x) = \iint e^{i(x-z)\cdot\xi} f(z) dz đ\xi \iint e^{-i\eta\cdot w} \chi(\epsilon(\xi+\Psi)) \times \chi(\epsilon(\xi+\Psi-\eta)) dw đ\eta + c(t-s) \iint e^{i(x-z)\cdot\xi} dz đ\xi \int_0^1 d\theta \int_0^1 d\tau \times \iint e^{i(t-s)(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2)} e^{-i(t-s)(1-\tau)(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2 - \theta c\widehat{\alpha}\cdot\eta)} e^{-i\eta\cdot w} i\widehat{\alpha}\cdot\eta \times e^{-i(t-s)\tau(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2 - \theta c\widehat{\alpha}\cdot\eta)} \chi(\epsilon(\xi+\Psi))\chi(\epsilon(\xi+\Psi-\eta)) f(z) dw đ\eta,$$

where $\Psi = \Psi(t, s; x, w+z, z)$. Then we can complete the proof of (1.21), applying the Calderón-Vaillancourt theorem to (3.9).

§ 4. A roughly sketched proof of (1.22)

Let $G_\epsilon(t,s)$ be the operator defined by (1.18).

Proposition 4.1. *Let $H(t)$ be the Dirac operator defined by (1.2). Then for $f \in \mathcal{S}$ we have*

$$\left[i\frac{\partial}{\partial t} - H(t)\right] G_\epsilon(t,s)f = R_\epsilon(t,s)f \tag{4.1}$$
$$:= \iint r(t,s;x,y)e^{iS(t,s;x,\xi,y)}f(y)\chi(\epsilon\xi)dy\bar{d}\xi,$$

where

$$r(t,s;x,y) = (x-y)\cdot\int_0^1 (1-\theta)E(t-\theta(t-s), x-\theta(x-y))d\theta \tag{4.2}$$
$$- c\sum_{j=1}^{d}\widehat{\alpha}^{(j)}\left[\int_0^1 \{A_j(t-\theta(t-s), x-\theta(x-y)) - A_j(t,x)\}d\theta\right.$$
$$+ (x-y)\cdot\int_0^1 (1-\theta)\frac{\partial A}{\partial x_j}(t-\theta(t-s), x-\theta(x-y))d\theta$$
$$\left. - (t-s)\int_0^1 (1-\theta)\frac{\partial V}{\partial x_j}(t-\theta(t-s), x-\theta(x-y))d\theta\right].$$

Proof. This proposition follows from the direct calculations. □

We note that $r(t,s;x,y)$ defined by (4.2) is a Hermitian matrix. So as in the proof of Proposition 3.3 we have

$$(R_\epsilon(t,s)^* f)(x) = \iint e^{-iS(t,s;y,\xi,x)} r(t,s;y,x)f(y)\chi(\epsilon\xi)dy\bar{d}\xi. \tag{4.3}$$

Consequently we can prove

$$\left(R_\epsilon(t,s)^* R_\epsilon(t,s)f\right)(x) = \iint e^{-iS(t,s;y,\xi,x)} r(t,s;y,x)\chi(\epsilon\xi)dy\bar{d}\xi \tag{4.4}$$
$$\times \iint r(t,s;y,z)e^{iS(t,s;y,\eta,z)}f(z)\chi(\epsilon\eta)dz\bar{d}\eta = \iint e^{i(x-z)\cdot\xi}dz\bar{d}\xi$$
$$\times \iint e^{-i\eta\cdot w}e^{i(t-s)(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2)} r(t,s;w+z,x)r(t,s;w+z,z)$$
$$\times e^{-i(t-s)(c\widehat{\alpha}\cdot\xi + c\widehat{\alpha}\cdot\Psi + \widehat{\beta}mc^2 - c\widehat{\alpha}\cdot\eta)}\chi(\epsilon(\xi+\Psi))\chi(\epsilon(\xi+\Psi-\eta))f(z)dw\bar{d}\eta$$

with $\Psi = \Psi(t,s;x,w+z,z)$ as in the proof of (3.7). Hence we can complete the proof of (1.22) as in the proof of (1.21), applying the Calderón-Vaillancourt theorem to (4.4).

§ 5. A roughly sketched proof of theorems

We can easily prove the following.

Lemma 5.1. *Let $E(t,x) = 0$ and $B_{jk}(t,x) = 0$ for all $j < k$ in $\mathbb{R}^{d+1}$. Define $G(t,s)f$ by (1.20) for $f \in \mathcal{S}$. Then we have*

$$G(t,s)f = U(t,s)f. \tag{5.1}$$

Proof. Lemma 5.1 follows from (3.6). □

The theorem below has been proved in Example 1.1, p.329 of [12].

Theorem 5.2. *Let $T > 0$ be an arbitrary constant. We assume*

$$|\partial_x^\alpha A_j(t,x)| \le C_\alpha < x >^M, \ |\alpha| \ge 1$$

in $[-T,T] \times \mathbb{R}^d$ for $j = 1,2,\dots,d$ and (2.3) with an integer $M \ge 1$. Let t and t_i be in $[-T,T]$ and consider the Dirac equation (1.2) with $u(t_i) = f \in B^a_{M+1}$ ($a = 0,\pm1,\pm2,\dots$). Then there exists a unique solution $U(t,t_i)f \in \mathcal{E}^0_t([-T,T]; B^a_{M+1}) \cap \mathcal{E}^1_t([-T,T]; B^{a-1}_{M+1})$, which satisfies

$$\|U(t,t_i)f\| = \|f\|, \quad \|U(t,t_i)f\|_{B^a_{M+1}} \le C_a(T)\|f\|_{B^a_{M+1}} \tag{5.2}$$

in $t, t_i \in [-T,T]$.

Proposition 5.3. *Under the assumptions of Theorem 2.1 we have*

$$\|G(t,s)f - U(t,s)f\|_{B^a_{M+1}} \le C_a(t-s)^2\|f\|_{B^{a+2}_{M+1}}, \ -2T_0 \le t,s \le 2T_0 \tag{5.3}$$

for $a = 0,1,2,\dots$ and $f \in \mathcal{S}$.

Proof. Let us write $\rho = t - s$ for a while. From (4.1) we can easily see

$$i\frac{G_\epsilon(s+\rho,s)f - U(s+\rho,s)f}{\rho} = \int_0^1 R_\epsilon(s+\theta\rho,s)fd\theta + \int_0^1 H(s+\theta\rho)$$
$$\cdot \{G_\epsilon(s+\theta\rho,s)f - f\}d\theta - \int_0^1 H(s+\theta\rho)\{U(s+\theta\rho,s)f - f\}d\theta.$$

Then we can complete the proof of Proposition 5.3 from (5.2) and a generalization of (1.21), which will be stated later as (5.7). □

From Lemma 5.1 we can easily see

$$\begin{aligned} K_{D\Delta}(t,t_i)f - U(t,t_i)f &= G(t,\tau_{\nu-1})\cdots G(\tau_1,t_i)f - U(t,\tau_{\nu-1})\cdots U(\tau_1,t_i)f \\ &= \sum_{j=1}^{\nu} G(t,\tau_{\nu-1})\cdots G(\tau_{j+1},\tau_j)\{G(\tau_j,\tau_{j-1}) - U(\tau_j,\tau_{j-1})\}U(\tau_{j-1},t_i)f \\ &= \sum_{j=1}^{\nu}{}' G(t,\tau_{\nu-1})\cdots G(\tau_{j+1},\tau_j)\{G(\tau_j,\tau_{j-1}) - U(\tau_j,\tau_{j-1})\}U(\tau_{j-1},t_i)f. \end{aligned} \tag{5.4}$$

Applying (1.13), (1.21), (5.2) and (5.3) with $M = 1$ to (5.4), we get

$$\begin{aligned}(5.5)\qquad \|K_{D\Delta}(t,t_i)f - U(t,t_i)f\| &\le C_0 \sum_{j=1}^{\nu}{}' e^{K_0\sigma(\Delta)}(\tau_j - \tau_{j-1})^2 \|U(\tau_{j-1},t_i)f\|_{B_2^2} \\ &\le C_0' e^{K_0\sigma(\Delta)}\sigma(\Delta)\|f\|_{B_2^2}\end{aligned}$$

with constants C_0 and C_0'.

Let $f \in L^2$ and $g \in B_2^2$. Then we have

$$\begin{aligned}(5.6)\qquad \|K_{D\Delta}(t,t_i)f - U(t,t_i)f\| &\le \|K_{D\Delta}(t,t_i)g - U(t,t_i)g\| + \|K_{D\Delta}(t,t_i)(f-g)\| \\ &+ \|U(t,t_i)(f-g)\| \\ &\le C_0' e^{K_0\sigma(\Delta)}\sigma(\Delta)\|g\|_{B_2^2} + (1 + e^{K_0\sigma(\Delta)})\|f-g\|\end{aligned}$$

from (1.21), (1.23), (5.2) and (5.5). Hence we can complete the proof of Theorem 1.1.

Let us give a proof of Theorem 2.1. We can prove

$$(5.7)\qquad \|G(t,s)f\|_{B_M^a} \le e^{K_a|t-s|}\|f\|_{B_M^a}$$

with a constant $K_a \ge 0$ for $a = 1, 2, \ldots$, which corresponds to (1.21). Using (5.7), we can prove Theorem 2.1 as in the proof of Theorem 1.1.

Let us give a proof of Theorems 2.2 and 2.3. We know

Theorem (Paley-Wiener, Theorem IX.11 in [26]). *An entire analytic function of n complex variables $g(\zeta)$ is the Fourier transform of a $C_0^\infty(\mathbb{R}^d)$ function with support in the ball $\{x \in \mathbb{R}^d; |x| \le R\}$ if and only if for each N there is a C_N so that*

$$|g(\zeta)| \le \frac{C_N e^{R|Im\ \zeta|}}{(1+|\zeta|)^N}$$

for all $\zeta \in \mathbb{C}^d$, where $Im\ \zeta$ denotes the imaginary part of ζ.

Consider the operator $G(t,s)$ defined by (1.20) with $(V, A) = 0$. We can easily see from the Paley-Wiener theorem that $G(t,s)f$ with $(V, A) = 0$ has the finite propagation speed not exceeding the velocity $c\lambda_{\max}$. Consequently, we can prove in terms of the Fourier expansion that general operators $G(t,s)f$ also have the finite propagation speed not exceeding the velocity $c\lambda_{\max}$. It follows from Theorems 1.1 and 2.1 that we have only to prove Theorems 2.2 and 2.3 for $K_{D\Delta}(t,t_i)f$ with Δ such that $t_i < \tau_1 < \ldots < \tau_{\nu-1} < \tau_\nu = t$ or $t_i > \tau_1 < \ldots > \tau_{\nu-1} > \tau_\nu = t$. Applying the result above for $G(t,s)f$ to (1.23), we can see that $K_{D\Delta}(t,t_i)f$ also has the finite propagation speed not exceeding the velocity $c\lambda_{\max}$. Thus we can prove Theorems 2.2 and 2.3 from Theorems 1.1 and 1.2, respectively.

Let us give a proof of Theorem 2.6. Let $t > T_0$ and $t_i < -T_0$. Then we can see

$$
\begin{aligned}
(5.8)\qquad & U_0(t,0)^{-1}U(t,0)U(t_i,0)^{-1}U_0(t_i,0) = U_0(0,t)U(t,0)U(0,t_i)U_0(t_i,0) \\
&= U_0(0,t)\{U(t,T_0)U(T_0,0)\}\{U(0,-T_0)U(-T_0,t_i)\}U_0(t_i,0) \\
&= U_0(0,t)\{U_0(t,T_0)U(T_0,0)\}\{U(0,-T_0)U_0(-T_0,t_i)\}U_0(t_i,0) \\
&= U_0(0,T_0)U(T_0,-T_0)U_0(-T_0,0) = U_0(T_0,0)^{-1}U(T_0,-T_0)U(0,-T_0)^{-1}.
\end{aligned}
$$

Hence we can complete the proof of Theorem 2.6 from Theorem 1.1.

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