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Path integrals and Stochastic Analysis with Bernstein processes

By

Jean-Claude ZAMBRINI*

Abstract

After a summary of the main mathematical interpretations of Feynman's path integrals we shall describe another one ("Stochastic Deformation") founded on a probabilistic interpretation of his concept of transition amplitude. The relation between this and an old problem formulated by E. Schrödinger will be described. Its solution provides, in fact, the looked for interpretation of transition amplitude (and elements). The principal features of Feynman's approach will be revisited in this new probabilistic context. The Hamilton Least Action principle is reinterpreted using tools of stochastic optimal control, in Lagrangian and Hamiltonian forms.

Various illustrations of the method of Stochastic Deformation are mentioned: the deformation of Jacobi's integration method and the analysis of loops, in particular, as some present research in progress. The relations with some other approaches are also described.

§ 1. Path integrals: one informal idea, many interpretations

We summarize briefly the main ways to look at Feynman's idea, for a (non relativistic) Hamiltonian system of the form

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + V(q)$$

acting on $L^2(\mathbb{R})$, V being a (bounded below) scalar potential and $\hbar$ a positive constant.

The solution $\psi = \psi(q,t)$ of the associated Schrödinger equation is represented by the "Path Integral"

$$\psi(q,t) = (e^{-\frac{i}{\hbar}t\hat{H}}\psi_0)(q) = \int_{\Omega^{q,t}} \psi_0(\omega(0)) \;\; e^{\frac{i}{\hbar}S_L[\omega]} \;\; \mathcal{D}\omega$$

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This work was supported by JSPS. KAKENHI(C) 24540193
*GFM Univ. Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal.
E-mail: jczambrini@fc.ul.pt

where $\Omega^{q,t} = \{\omega \in C([0,t],\mathbb{R}) \quad \text{s.t.} \quad \omega(t) = q\}$

The function ψ solves

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi, \qquad \psi(q,0) = \psi_0(q)$$

For smooth $\omega(\cdot)$ the Action associated with $\hat{H}$ is

$$S_L[\omega] = \int_0^t \frac{1}{2}(\frac{d\omega}{d\tau})^2 d\tau - \int_0^t V(\omega(\tau))d\tau := \int_0^t L\,(\frac{d\omega}{d\tau},\omega)d\tau$$

Although $\mathcal{D}\omega$ was treated by Feynman as a countably additive measure on the path space Ω^{qt}, it was soon understood that $\mathcal{D}\omega$ does not exist as a mathematical object. Still, a number of deep mathematical results have been obtained along the years, notably by S. Albeverio, S. Mazzucchi (Oscillatory integrals), D. Fujiwara, N. Kumano-Go (Time slicing approximation), for a large class of potentials V. A simpler approach (from the viewpoint of the mathematical interpretation of $\mathcal{D}\omega$) is to do a "Wick rotation" $t \to -it$ in Feynman's representation of $\psi(q,t)$. This is the "voie royale" of Path integration starting with N. Wiener, who built the appropriated countably additive measure on the space of continuous paths $\omega(\cdot)$ of the Brownian motion. Then M. Kac used the same measure to express the effect of the potential V as a perturbation:

$$\eta_*(q,t) := (e^{-\frac{1}{\hbar}t\hat{H}}f)(q) = E[f(W^{t,q}(0)e^{-\frac{1}{\hbar}\int_0^t V(W^{t,q}(\tau))d\tau}]$$

solving

$$-\hbar\frac{\partial\eta_*}{\partial t} = \hat{H}\eta_*, \qquad \eta_*(q,0) = f(q)$$

We shall be interested here by real (and even positive) boundary condition f for the associated heat equation with potential V. The above notation, $*$, therefore, does not refer to any complex conjugacy. It is meant simply as a suggestive analogy whose justification will become clear later on.

Now we are dealing with a time expectation over Wiener (or Brownian) process $W^{t,q}(\tau)$, of variance $\hbar\tau$, conditioned to be in q at a final time t: $W^{tq}(t) = q$. Formally, Wiener measure is built from distributions over absolutely continuous paths $\omega(\cdot)$ with $\|\omega\|^2 = \int_0^t (\frac{d\omega}{d\tau})^2 d\tau < \infty$ and Feynman original Action functional is turned into

$$\int_0^t \{\frac{1}{2}(\frac{d\omega}{d\tau})^2 + V(\omega(\tau))\}d\tau := -\int_0^t L_E(\dot{\omega},\omega)d\tau$$

where E stands for "Euclidean", a traditional terminology in physics, motivated by quantum field theory.

This way to interpret Feynman's path integral method has been considerably generalized and, in particular, to large classes of Markovian processes beyond Wiener. For instance, it is known that, for magnetic relativistic Hamiltonians $\hat{H}$ the path integral is associated with some Lévy processes (c.f. T. Ichinose).

Let us observe that this traditional Euclidean interpretation transforms quantum models, which are time symmetric for conservative systems, into irreversible ones typical of statistical mechanics since we are using semigroup theory.

We are going to describe a distinct "Euclidean" approach called "Stochastic deformation". It preserves the time-symmetry of quantum theory and holds as well for a large class of Hamiltonians $\hat{H}$, for instance associated with Lévy processes (in momentum representation).

From now on the state (or, better, "configuration") space of the processes, denoted by $\mathcal{M}$, will be $\mathbb{R}^n$ or a n-dimensional Riemannian manifold. Our basic claim is that Path integral representations of solutions of Cauchy problems are not so essential in Feynman's approach to quantum dynamics. His key notions are the ones of "Transition amplitude" and "Transition element" on a given time interval $I = [s, u]$, namely

$$\int\int\int_{\Omega_{x,s}^{z,u}} \psi_s(x) e^{\frac{i}{\hbar} S_L[\omega(\cdot); u-s]} \mathcal{D}\omega \bar{\varphi}_u(z) dx dz := < \varphi, \mathrm{Id}\ \psi >_{S_L} \qquad \in \mathbb{C}$$

for all $\psi_s, \bar{\varphi}_u$ "states" in L^2 and where Id denotes the identity. For F "any" functional,

$$\int\int\int_{\Omega_{x,s}^{z,u}} \psi_s(x) e^{\frac{i}{\hbar} S_L[\omega(\cdot); u-s]} F[\omega(\cdot)] \mathcal{D}\omega \bar{\varphi}_u(z) dx dz := < \varphi, F\ \psi >_{S_L}$$

The two boundary states, here ψ_s and $\bar{\varphi}_u$ (where the bar denotes the complex conjugacy) can be interpreted respectively as initial and final boundary conditions of two adjoint Schrödinger equations for a given system with Hamiltonian $\hat{H}$.

§ 2. An "unrelated" old problem of Schrödinger (1931)

Let us consider, with Schrödinger, a system of 1-d (for simplicity) Brownian particles $X_t = \hbar^{\frac{1}{2}} W_t$, $\hbar > 0$, observed during a time interval $I = [s, u]$. For a given initial distribution $\mu_s(dx) = \rho_s(x)dx$, it is well known that the probability $P(X_u \in dz) = \eta_u^*(z)dz$, where $\eta_t^*(q)$ solves the free heat equation

$$-\hbar \frac{\partial \eta^*}{\partial t} = \hat{H}_0 \eta^*,$$

$t \in I$, $\hat{H}_0 = -\frac{\hbar^2}{2}\Delta$, $\eta_s^*(x) = \rho_s(x)$.

The spreading described by η_t^* is the archetype of irreversible phenomena (heat dissipation).

Now Schrödinger wondered about the qualitative effect of an additional final distribution given arbitrarily and, in particular, distinct from η_u^*:

$$\mu_u(dz) = \rho_u(z)dz$$

Since we are analyzing diffusion phenomena, such a data can make perfect sense for instance if we describe relatively rare events.

Schrödinger's problem (SP):

Find the most probable evolution of the probability distribution $\rho_t(q)dq$ for X_t, $s \leq t \leq u$, compatible with those data.[I]

S. Bernstein understood (SP) as an indication that a Markovian framework was not appropriate.

Let $\mathcal{P}_t$, $\mathcal{F}_t$ be, respectively, the increasing sigma-algebra representing the past information about a process and the decreasing one representing its future information. The usual formulation of Markov property is the familiar one in statistical mechanics:

If $B \in \mathcal{F}_t$, $P(B|\mathcal{P}_t) = P(B|\mathcal{P}_t \cap \mathcal{F}_t)$, where $\mathcal{P}_t \cap \mathcal{F}_t$ is the present σ-algebra.

For our purpose, the time symmetric version is more natural:

If, in addition, $A \in \mathcal{P}_t$, $\quad P(A\ B|\mathcal{P}_t \cap \mathcal{F}_t) = P(A|\mathcal{P}_t \cap \mathcal{F}_t).P(B|\mathcal{P}_t \cap \mathcal{F}_t)$.

Bernstein suggested, in 1932, the weaker time-symmetric version that he called "reciprocal":

For $A \in \mathcal{P}_s \cup \mathcal{F}_t$, $\quad B \in \sigma_{(s,t)}$ (the sigma-algebra on this interval),

$$P(A.B|X_s, X_t) = P(A|X_s, X_t).P(B|X_s, X_t), \qquad \forall s \leq t$$

Of course, this property reappeared in a variety of contexts after 1932. It has been called "Markov field" or "Local Markov" in the context of Quantum field theory ($\sim$ 1970), or "Two-sided" or "Quasi-Markov" (for instance by Hida's school).

Let us summarize the construction of Bernstein processes. To stay close to Markovian construction, the transition probability should become a 3 points Bernstein transition $\mathcal{B} \ni A \to Q(s, x, t, A, u, z)$, $\quad s \leq t \leq u$.

We look for properties of Q s.t. for $X_u = z$ fixed, Q reduces to a usual forward Markovian transition, for $X_s = x$ to a backward one. Clearly the data of initial Markovian transition should be substituted by a joint one $dM(x, z)$.

Theorem 2.1 (Jamison (1974)). *Given* Q, M

a) $\exists!$ *Prob. measure* P_M *s.t., under* P_M*,* X_t *is Bernstein*

b) $P_M(X_s \in A_s,\ X_u \in A_u) = M(A_s \times A_u)$

c) $P_M(X_s \in A_s,\ X_{t_1} \in A_1, ..., X_{t_n} \in A_n,\ X_u \in A_u) =$

$\int_{A_s \times A_u} dM(x, z) \int_{A_1} Q(s, x, t_1, dq_1, u, z) \int_{A_2} \cdots \int_{A_n} Q(t_{n-1}, q_{n-1}, t_n, dq_n, u, z)$,

$s < t_1 < t_2 < \ ... < t_n < u$.

Let us stress that, for most joint probabilities dM, the resulting process X_t is only Bernstein reciprocal and not Markovian. In fact, only one class of dM, denoted dM_m, provides a Markovian process:

Let $\hat{H}$ be the lower bounded Hamiltonian generator of a (strongly continuous, contraction) semigroup generalizing Schrödinger's $\hat{H}_0$ (A Pseudo-differential $\hat{H}$ is also OK, c.f. Privault-JCZ ([17]). M_m takes the form

$$M_m(A_s \times A_u) = \int_{A_s \times A_u} \eta_s^*(x) \underbrace{\left(e^{-\frac{1}{\hbar}(u-s)\hat{H}}\right)(x,z)}_{h(s,x,u,z)} \eta_u(z) dx dz$$

η_s^*, $\eta_u > 0$ (not necessarily bounded) real valued functions.

Given $\hat{H}$ (i.e. h), a Markovian Bernstein transition is

$$Q(s,x,t,dq,u,z) = h^{-1}(s,x,u,z) h(s,x,t,q) h(t,q,u,z) dq$$

Substituting into c),

$\rho_n(dx_1, t_1, dx_2, t_2, \ ..., dx_n, t_n)$
$= \int_{A_s \times A_u} \eta_s^*(x) h(s, dx, t_1, dx_1) ... h(t_n, dx_n, u, dz) \eta_u(z)$

NB: If η_u^* and η_u are complex elements of L^2 and h becomes the integral kernel for Schrödinger equation (after Wick rotation), dM_m is turned into Feynman's transition amplitude. Where $n = 1$, in particular,

$$P(X_t \in A) = \int_A \eta_t^* \eta_t(q) dq, \qquad \eta_t^*, \ \eta_t \ > 0$$

solving the two adjoint equations

$$t \in [s,u] : \quad \begin{cases} -\hbar \frac{\partial \eta^*}{\partial t} = \hat{H}^+ \eta^* & \hbar \frac{\partial \eta}{\partial t} = \hat{H} \eta \\ \eta^*(s,q) = \eta_s^*(q) & \eta(u,q) = \eta_u(q) \end{cases}$$

This means in particular that the Markovian answer to Schrödinger's problem is

$$\rho_t(q) dq = \eta_t^* \eta_t(q) dq$$

The two adjoint heat equations and this Markovian answer justify our notations for their associated solutions. Informally, for a self-adjoint Hamiltonian $\hat{H}$ and when t becomes $\sqrt{-1}t$, ρ_t is turned into Born's interpretation of the wave function ψ. The product form of ρ_t reintroduces a version of time symmetry in a problem of classical statistical physics ! We are therefore entitled to interpret the probability measures in question as the Euclidean version of Feynman's quantum mechanical "measure". [II]

From now on we shall consider a specific example, sufficient to show the main features of this non conventional Euclidean approach, namely

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + \hbar\, A\nabla + \frac{\hbar}{2}\nabla.A - \frac{1}{2}|A|^2 + V$$

for potentials $A : \mathbb{R}^n \to \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ bounded below.

The associated process X_t, $s \leq t \leq u$, is a Markovian diffusion, with initial probability $\rho_s(x)dx$, forward drift:

$$D_t X = \lim_{\Delta t \downarrow 0} E_t\Big[\frac{X_{t+\Delta t} - X_t}{\Delta t}\Big] = \hbar \nabla \log \eta_t(X) - A(X)$$

and final probability $\rho_u(z)dz$ for the backward drift,

$$D_t^* X = \lim_{\Delta t \downarrow 0} E_t\Big[\frac{X_t - X_{t-\Delta t}}{\Delta t}\Big] = -\hbar \nabla \log \eta_t^*(X) - A(X)$$

The coexistence of two drifts for the same process should not be a surprise. They provide the mathematical justification of Feynman's uncertainty principle [c.f. Feynman-Hibbs (7-45)] in the form

$$E[X_j(t) \lim_{\Delta t \downarrow 0} E_t[\frac{X(t) - X(t-\Delta t)}{\Delta t}]_k - \lim_{\Delta t \downarrow 0} E_t[\frac{X(t+\Delta t) - X(t)}{\Delta t}]_k X_j(t)] = \hbar \delta_{jk}$$

where E_t denotes the conditional expectation given X_t and E the absolute one.

It will be crucial that D_t, extended by Itô calculus to an operator acting on any $f(X_t, t)$, $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$, the infinitesimal generator of X_t, kills $\mathcal{P}_t$- martingales and D_t^* (c.f. backward Itô calculus) kills $\mathcal{F}_t$- martingales. And, as well, that $D_t = D_t^* = \frac{d}{dt}$ on smooth trajectories $t \to X_t$, i.e., at the "classical limit" $\hbar = 0$ where the uncertainty principle disappears.

In the above specific example, the decomposition of the drifts into a curl free part and a divergence free one (a choice of gauge for A) is geometrically meaningful ("Helmholtz decomposition"). The complete solution of Schrödinger's (Markovian) problem requires to specify now two positive functions η_s^* and η_u in the joint probability M_m. Since the data of Schrödinger problem are $\{\rho_s, \rho_u\}$, the marginals of M_m provide a nonlinear and integral system of equations for η_s^* and η_u:

$$\begin{cases} \eta_s^*(x) \int h(s,x,u,z)\eta_u(z)dz = \rho_s(x) \\ \eta_u(z) \int \eta_s^*(x) h(s,x,u,z)dx = \rho_u(z) \end{cases}$$

Beurling has proved in 1960 that for a given integral kernel $h > 0$ positive and continuous on any locally compact configuration space $\mathcal{M}$ there is a unique pair of positive (η_s^*, η_u), not necessarily integrable, solutions of this system, for any given strictly positive $\{\rho_s,\ \rho_u\}$.

Special cases

Let us consider the one dimensional case, with $A = V = 0$ in $\hat{H}$. Then the integral kernel $h = h_0$ is the Gaussian one.

1) The following example was already considered by Schrödinger. It corresponds to $\{\eta_s^*(x) = \rho_s(x), \eta_0(z) = 1\}$ on $\mathcal{M} = \mathbb{R}$.

$$\eta_t^*(q) = \int \rho_s(x) h_0(s,x,t,q)dx \ , \qquad \eta_t(q) = 1 \ , \quad \forall t \in [s,u]$$

$$D_t X = 0 \ , \qquad D_t^* X = -\hbar \nabla \log \eta_t^*(X)$$

i.e. $X_t =$ usual Wiener W_t with $\rho_s(x)dx$.

This original example of Schrödinger is interesting. It is clear that, in spite of preserving some time symmetry via the use of two adjoint heat equations, this framework will supply much more Bernstein measures than those relevant for quantum dynamics : the relation between $\eta_t^*(q)$ and $\eta_t(q) = 1, \quad \forall t \in [s,u]$ is manifestly quite distinct from any analogue of complex conjugate in $L^2(\mathbb{R})$.

Also notice that when $\rho_s = \delta_x$ the backward drift reduces to $D_t^* X = \frac{X_t - x}{t-s}$.

2) Consider, in the same case 1), a permutation $\{\eta_s^*(x) = 1, \quad \eta_u(z) = \rho_u(z)\}$. Then, clearly, $\forall t \in [s,u]$,

$$\eta_t^*(q) = 1 \ , \qquad \eta_t(q) = \int h_0(t,q,u,z)\rho_u(z)dz$$

$$D_t X = \hbar \nabla \log \eta_t(X) \ , \qquad D_t^* X = 0$$

$X_t =$ is a "backward" Wiener W_t^* , with final probability $\rho_u(z)dz$.

3) Some call "reversible" Brownian (a bad probabilistic terminology in our context) in the same case 1), $\{\eta_s^*(x) = 1, \eta_u(z) = 1\}$ corresponding to the trivial drifts $D_t X = D_t^* X = 0$.

If R denotes the reference measure of this "flat" Brownian motion on $\Omega = C([s,u];\mathbb{R})$ a modern version of Schrödinger is formulated in terms of relative entropy of any measure P on Ω with respect to R, i.e.,

$$H(P|R) = \int_\Omega \log\left(\frac{dP}{dR}\right) dP$$

and Schrödinger's problem can be expressed as the minimization problem:

$$H(P|R) \to \min$$

among all the measures $P(\Omega)$ such that $P_s = \mu_s, \quad P_u = \mu_u$ are given.

Ref: H. Föllmer [8], L. Wu, A.B. Cruzeiro, J.C.Z. [18], C. Léonard, S. Roelly, J.C.Z. [21].[III]

§ 3. Key point of Feynman's Path Integral strategy : infinite dimensional version of stationary phase for $\hbar \to 0$. Hamilton Least Action principle

A fundamental aspect of Feynman's Path integral approach is his use of an infinite dimensional version of the stationary phase principle when $\hbar \to 0$, interpreted as Hamilton least Action principle for the trajectories of the underlying classical system. This aspect has been particularly investigated by S. Albeverio, R. Hoegh-Krohn and S. Mazzucchi.

In our context, the topic is now to describe the dynamics of $X_t, t \in [s,u]$, solving Schrödinger's problem. In other terms, for the Hamiltonian $\hat{H}$ of the specific example, what is the probabilistic counterpart of Feynman's equation of motion in $\mathcal{M} = \mathbb{R}^3$:

$$< \ddot{\omega}(\tau) >_{S_L} = < -\nabla V(\omega(\tau)) + \dot{\omega} \wedge \mathrm{rot} A >_{S_L}$$

where the left hand side denotes some time discretization of the acceleration, $< \cdot >_{S_L}$ its transition element and the right hand side is the classical Lorentz force ?

For us, the starting point is the following Action functional (with final boundary condition S_u)

$$\begin{aligned} J[X(\cdot)] &= E_{xt}\{\int_t^u \mathcal{L}(X_\tau, DX_\tau)d\tau + S_u(X_u)\} \\ &= E_{xt}\{\int_s^u \frac{1}{2}|D_\tau X|^2 + V(X_\tau)d\tau + \int_s^u A \circ dX_\tau + S_u(X_u)\} \end{aligned}$$

where E_{xt} is the conditional expectation $E[...|X_t = x]$. Feynman had already suggested the need of Stratonovich integral for the vector potential A.[IV]

Let $\mathcal{D}_J$ denote the domain of J:

$\mathcal{D}_J = \{$diffusions $X_. <<$ Wiener $P_W^\hbar$, with fixed diffusion matrix, and unspecified drift$\}$

$X \in \mathcal{D}_J$ is called an extremal of J iff

$$0 = E_{xt}\big[\lim_{\epsilon \to 0} \frac{J[X + \epsilon\delta X] - J[X]}{\epsilon}\big]$$

$\forall \delta X$ in the Cameron-Martin Hilbert space. Then

$$0 = E_{xt}\int_t^u (\frac{\partial \mathcal{L}}{\partial X} - D_\tau(\frac{\partial \mathcal{L}}{\partial D_\tau X}))\delta X_\tau d\tau + E_{xt}\{(\frac{\partial \mathcal{L}}{\partial D_\tau X} + \nabla S_u)\delta X_u\}$$

So X_τ extremal implies the (a.s.) Stochastic Euler-Lagrange equations.

$$(SEL)\begin{cases} D_\tau\left(\frac{\partial \mathcal{L}}{\partial D_\tau X}\right) - \frac{\partial \mathcal{L}}{\partial X} = 0 \qquad t < \tau < u \\ \\ \frac{\partial \mathcal{L}}{\partial D_\tau X}(X_u, DX_u) = -\nabla S_u(X_u) \quad , \qquad X(t) = x \end{cases}$$

In our example,

$$D_\tau D_\tau X = \nabla V(X) + D_\tau X \wedge \mathrm{rot} A + \frac{\hbar}{2} \mathrm{rot} \;\; \mathrm{rot} \; A$$

The left-hand side is a well defined version of Feynman's $< \ddot{\omega}(\tau) >_{S_L}$. The second term of the r.h.s. is the (Euclidean) Lorentz force and the last one a stochastic ($\hbar$-dependent) deformation term.

Let us define, as classically, the momentum P as

$$P = \frac{\partial \mathcal{L}}{\partial D_\tau X}(X_\tau, D_\tau X)$$

under the hypothesis det $\left(\frac{\partial^2 \mathcal{L}}{\partial DX^i \partial DX^j}\right) \neq 0$ so that this relation is solvable in $D_\tau X = \Phi(P, X)$.

Define the Hamiltonian by

$$\mathcal{H}(X, P) = P\Phi(P, X) - \mathcal{L}(X, \Phi(P, X))$$

For instance, in our example,

$$\mathcal{H} = \frac{1}{2}|P - A(X)|^2 - V(X) - \frac{\hbar}{2}\nabla.A$$

Then the stochastic Hamiltonian (a.s.) equations hold:

$$(\mathrm{SH}) \qquad \begin{cases} D_\tau X = \frac{\partial \mathcal{H}}{\partial P} \\ \\ D_\tau P = -\frac{\partial \mathcal{H}}{\partial X} \end{cases}$$

In our example,

$$\begin{cases} D_\tau X = P - A \\ \\ D_\tau P = D_\tau X \wedge \mathrm{rot} A + D_\tau X.\nabla A + \frac{\hbar}{2} \; \mathrm{rot} \; \mathrm{rot} \; A + \frac{\hbar}{2}\Delta A \; + \nabla V \end{cases}$$

Those (stronger) counterparts of Feynman's equations of motion hold true in the method of Stochastic deformation. The definitions are such that, at the classical limit $\hbar = 0$, we recover classical Lagrangian and Hamiltonian mechanics.

Although (SEL) and (SH) involve only the increasing filtration and, therefore, break manifestly the invariance under time reversal, this one can be re-established using the information contained in the other filtration. The same processes are described, in this sense, from different perspectives (and boundary conditions).[V]

We shall consider the stochastic deformation of classical system on $\mathcal{M} = \mathbb{R}^n$, with Hamiltonian $H(p,q) = \frac{1}{2}|p|^2 + V(q)$ for V smooth and bounded below. In this case, the stochastic deformation of the associated Hamilton-Jacobi equation is known as Hamilton-Jacobi-Bellman equation for a scalar field $S = S(q,t)$:

$$-\frac{\partial S}{\partial t} + \frac{1}{2}|\nabla S|^2 - V + \frac{\hbar}{2}\Delta S = 0 \qquad (HJB)$$

whose only stochastic deformation appears, in fact, in the last term. The relation of this equation with the above Action functional $J[X(\cdot)]$ (for $A = 0$) is, of course, well known in Stochastic Control theory (C.f. H. Fleming, H.M. Soner) but our approach will be quite different, founded on a dynamical interpretation.

It follows from those classical results of stochastic control that the relation between S and critical points X_t of J is very simple.

The extremal (in fact minimizer) of J has the drift

$$D_t X = -\nabla S(X,t)$$

for S a smooth classical solution of (HJB). Of course such strong regularity conditions on S are not necessary, but they will be sufficient for our present purpose, more geometric in nature.

It follows indeed that the gradient of HJB equation coincides with SEL equation for the Lagrangian of our example when $A = 0$ This is the stochastic deformation of a classical integrability condition for smooth trajectories (C.f. J.C.Z., Journal of Geometric Mechanics, 2009).

A complete solution of HJB equation is defined as $S(q,t,\alpha) := S_\alpha(q,t)$, $\alpha = (\alpha^1, \dots, \alpha^n)$, n real parameters such that det $\left(\frac{\partial^2 S}{\partial q^i \partial \alpha^k}\right) \neq 0$ and $S(q,t,\alpha)$ solves (HJB) for all α.

The following stochastic deformation of Jacobi's integration Theorem can be proved:

To find a solution $X_t^\alpha(M_t)$, $P_t^\alpha(M_t)$ of (SH):

$$\begin{cases} D_t X = \frac{\partial \mathcal{H}}{\partial P} \\ D_t P = -\frac{\partial \mathcal{H}}{\partial X} \end{cases}$$

where $M_t = (M_t^1, \dots, M_t^n) =$ are martingales of $X_t^\alpha(M_t)$ we have to:

a) Solve n implicit equations $\frac{\partial S_\alpha}{\partial \alpha^i}(q,t) = -M_t^i$ as $q = X_t^\alpha(M_t)$;

b) Supplement $X_t^\alpha(M_t)$ by $P_t^\alpha = P_t^\alpha(M_t) = -\nabla_q S_\alpha(X_t^\alpha(M_t), t)$.

Then $(X_t^\alpha \ , \ P_t^\alpha)$ solve (SH) equations.

The proof will be given somewhere else.

Notice that, at the classical limit $\hbar = 0$ of smooth trajectories, the martingales M_t reduce to a collection of numbers ("first integrals" of the system) and the whole statement to the original Theorem of Jacobi (C.f. Giaquinta, Hildebrandt).

It follows, in particular, that for a large class of starting Hamiltonians, (SH) is integrable in the sense that we have enough martingales. But, in general, this is not the case.

We have, up to now, considered only Feynman's transition amplitude, whose Euclidean version is the Markovian joint probability M_m of Bernstein processes. But, of course, general Bernstein processes are not Markovian. And many of those are interesting in physics. For instance, loops are relevant for periodic phenomena. Start from the one dimensional Markovian bridge, $X_t = X_{x,0}^{z,1}(t)$ for $V = A = 0$. As a Bernstein it solves two stochastic differential equations (SDE), for $t \in [0,1]$, $\hbar = 1$,

$$dX_t = \frac{z - X_t}{1-t}dt + dW_t \quad , \quad d_*X_t = \frac{X_t - x}{t}dt + d_*W_t^*$$

where d, d_* denote the differentials under the expectations of D_t, D_t^*. The law P^{xz} of this Markovian bridge is built from the Gaussian kernel h_0. The process is Gaussian with expectation and covariance given by

$$X_t \sim \mathcal{N}((1-t)x + tz, t(1-t)), \quad 0 \leq t \leq 1$$

Observe, in particular, that it is invariant under the time reversal $t \to 1-t$.

The loop is defined as the special case $x = z$, i.e. $\mathcal{N}(x, t(1-t))$. It is still Markovian. But if $X_0 = X_1$ become random, with $\mathcal{N}(0,1)$ for instance, we have a degenerate joint probability M and X_t can be called a periodic loop, whose law is of the form

$$P^{\text{ per}} = \int P^{xx}\mathcal{N}(0,1)dx$$

The associated SDE are the randomized versions of the Markovian bridge ones:

$$dX_t = \frac{X_0 - X_t}{1-t}dt + dW_t \quad , \quad dX_t = \frac{X_t - X_0}{t} + d_*W_t^*$$

X_t cannot be Markovian since the drifts depend on X_t and X_0, but the process is still Bernstein. In fact, although the form of its covariance, $C(X_s, X_t) = s(1-t) + 1$ ruins the Markov property, it preserves the symmetry under time reversal.

Notes.

[I] Schrödinger's idea has been often quoted, since its publication. Kolmogorov, for instance, mentioned it (1937) in relation with reversibility of the statistical laws of nature (but considered only stationary situations, insufficient for the solution of Schrödinger problem).

Later on N. Nagasawa (1964) developed this notion of reversibility. On the physical side, references to Schrödinger's idea can be found, generally in the context of stochastic analogies with quantum physics (Max Jammer's "The philosophy of Quantum Mechanics: The interpretations of quantum mechanics in historical perspective" (1974) provides a good summary in Chap. 9). But, most of the time, Schrödinger's original observation was mentioned only as a physically superficial analogy between heat equation and Schrödinger equation (C.f., for instance, N. Saitô, M. Namiki, "On the quantum-mechanics-like description of the theory of the Brownian motion and quantum statistical mechanics" (1956)). Feynman's Path integrals theory was also often referred to allusively along the way. The few mathematicians who really tackled the mathematical content of Schrödinger's idea before 85-86 are S. Bernstein, R. Fortet, A. Beurling and B. Jamison.

Let us stress that Schrödinger's problem is clearly an (unconventional) Euclidean problem, treated as such in our approach of stochastic deformation.

[II] In the traditional Euclidean approach, the analytical counterpart of our Markovian joint probability M_m, for η_s^* and η_u real valued and bounded is the scalar product in the L^2 quantum Hilbert space, $< \eta_s^* \mid e^{-\frac{1}{\hbar}(u-s)\hat{H}}\eta_u >_2$ or, expressed in terms of Wiener measure μ_W,

$$\eta_s^*(\omega(s))\big(\exp -\frac{1}{\hbar}\int_s^u V(\omega(\tau))d\tau\big)\eta_u(\omega(u))d\mu_W(\omega).$$

It has been used for versions of Feynman-Kac formula at least from E. Nelson (1964). But although the two boundary terms η_s^*, η_u are arbitrary, in those versions, our appropriate pair $(\eta_s^*,\ \eta_u)$ must solve Schrödinger's nonlinear integral system of equations whose solution was shown by Beurling in 1960. This will have deep consequences in the dynamical structure of our stochastic deformation.

[III] In his St. Flour entropic reinterpretation of the diffusions constructed by us along Schrödinger's strategy, H. Föllmer (1988) called them "Schrödinger's bridge". We prefer to reserve this name for those singular cases corresponding to two Dirac boundary probability densities.

[IV] The idea of such a stochastic version of calculus of variations is due to the japanese theoretical physicist Kunio Yasue, in the context of Nelson "stochastic mechanics". Nelson approach was a radical (real time) attempt to interpret quantum theory in terms of classical stochastic processes ("Derivation of the Schrödinger equation from Newtonian Mechanics", 1966). What Yasue did was to introduce the associated

Lagrangian mechanics and variational principles. The mathematical problems of his approach were, in fact, mainly due to the dynamical problems of Nelson theory.

Until 1985-6 Nelson's "real time" framework was regarded as independent from Schrödinger's Euclidean one.

For his variational principle Yasue needed to assume that the two boundary random variables X_s and X_u are fixed during the variation, an hypothesis seemingly involving the data of their joint probability $dM(x, z)$. This is the origin of my own interest in Schrödinger's idea and its mathematical interpretation.

It had been suspected by many that Nelson's mechanics and Schrödinger framework were related. For instance by B. Jamison himself (private communication), whose NSF project in this direction was rejected, or by L. de la Penã-Auerbach (1967) quoted in Jammer's book. All such attempts were pointing toward the influence on Nelson's processes of a nonlocal potential in addition to the physical ones (A and V in our example). And indeed, in 1985-6 (JMP) it was shown that, at the expense of the introduction of such a "modified potential", Schrödinger's method allowed a completely different (and more explicit) reconstruction of Nelson's diffusions. This was verified many times afterwards (R. Carmona (1985), M. Nagasawa (1989), P. Cattiaux, C. Léonard (1995)...).

The hidden role, in Nelson mechanics, of this nonlocal potential (well known as the "Bohm potential", c.f. [10]) has dramatic dynamic consequences. Consider Feynman one dimensional computation of the transition element $< \omega(s)\omega(t) >_{S_L}$ for a free particle on the time interval $[0, T]$ in terms of the underlying classical (free) trajectories $t \to q(t)$. He found (Feynman, Hibbs p. 179-80):

$$< \omega(s)\omega(t) >_{S_L} = \begin{cases} q(s)q(t) \ + \ i\hbar \frac{s(T-t)}{T} & \text{if} \quad 0 \leq s \leq t \leq T \\ q(s)q(t) \ + \ i\hbar \frac{t(T-s)}{T} & \text{if} \quad 0 \leq t \leq s \leq T \end{cases}$$

where he interpreted the last terms as a quantum deformation. In our stochastic deformation inspired by Schrödinger, we are, of course, dealing with the Brownian bridge, the simplest of all Bernstein diffusions associated with $V = 0$. The result is

$$E[X_s X_t] = \begin{cases} q(s)q(t) \ + \ \hbar \frac{s(T-t)}{T} & \text{if} \quad 0 \leq s \leq t \leq T \\ q(s)q(t) \ + \ \hbar \frac{t(T-s)}{T} & \text{if} \quad 0 \leq t \leq s \leq T \end{cases}$$

i.e. the expected Euclidean counterpart of Feynman's computation.

Now consider the same Brownian bridge in Nelson's stochastic mechanics. It is not associated with $V = 0$ but with a complicated quadratic and time dependent potential, depending also on the starting and ending points x and z. In fact, the physical potential is still $V = 0$ but Bohm nonlocal one is not. The predictions of stochastic mechanics have little to do with those of quantum mechanics as soon as more than one time is involved. On this basis Nelson renounced to his original theory in 1985 and never came back to it except to stress the "mystery of stochastic mechanics" ([11]). To paraphrase a famous comment of Wolfgang Pauli (C.f. Wikipedia "Not even wrong") Nelson's stochastic mechanics was mathematically consistent and physically wrong , but on the interesting side. Nevertheless it contained technical tools (for instance the need of two filtrations) without which I would not have been able to understand and develop what is known today as the Schrödinger problem and its consequences. It would be ridiculous, nevertheless, to attribute the paternity of Nelson's stochastic mechanics to Schrödinger.

[V] The first almost sure equations of this kind for diffusion processes were due to E. Nelson (1967). They were different from ours, mixing up the informations of the two filtrations. They have never been used constructively in stochastic mechanics. In particular, no stochastic counterpart of the classical notion of first integrals has ever been found in this context.

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Prospects.

- Recently, ideas on Bernstein processes were used in the context of Optimal transport theory :

C. Léonard , "A survey of Schrödinger problem and some of its connections with optimal transport", Discrete Cont. Dynam. Syst. A , 34, 4, 1533 (2014) (and references therein)

- The community of Geometric Mechanics (field inspired by V. Arnold, S. Smale, J.M. Souriau, Abraham and Marsden) started recently to "randomize" classical mechanics. Our Stochastic deformation can be regarded in this way.

In particular (with applications to Navier-Stokes) M. Arnaudon, X. Chen, A. B. Cruzeiro, "Stochastic Euler-Poincaré reduction", J. Math. Physics , 55, 081507 (2014).

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