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Generalized Feynman-Kac formulae for the solution of high order heat-type equations

By

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Abstract

The construction of generalized Feynman-Kac formulae representing the solution of high order heat-type equations is presented.

§ 1. Introduction

The connection between the solution of parabolic equations associated to second-order elliptic operators and the theory of stochastic processes is a largely studied topic [9]. The main instance is the *Feynman-Kac formula*, providing a representation of the solution of the heat equation with potential $V \in C_\infty(\mathbb{R}^d)$ (the continuous functions vanishing at infinity)

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x), & t \in \mathbb{R}^+, x \in \mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

in terms of an integral with respect to the Wiener measure [26], the probability measure associated to the Wiener process $\{W(t), t \geq 0\}$, the mathematical model of the Brownian motion [21]:

$$u(t,x) = \mathbb{E}[e^{-\int_0^t V(W(s)+x)ds}u_0(W(t)+x)]. \tag{1.2}$$

On the other hand, if one considers more general PDEs, such as, for instance, the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}u(t,x) = -\frac{1}{2\hbar^2}\Delta u(t,x) + V(x)u(t,x), \qquad t \in \mathbb{R}^+, x \in \mathbb{R}^d \tag{1.3}$$

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or heat-type equations associated to higher order differential operators, as for instance:

$$\frac{\partial}{\partial t}u(t,x) = c\frac{\partial^p}{\partial x^p}u(t,x) + V(x)u(t,x), \qquad t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{1.4}$$

where $c \in \mathbb{C}$ is a complex constant and $p \in \mathbb{N}$, with $p > 2$, then the traditional theory cannot be applied. In fact it is not possible to define a stochastic Markov process $\{X(t), t \geq 0\}$, that plays for Eq. (1.3) or Eq. (1.4) the same role that the Brownian motion plays for the heat equation and construct a "generalized Feynman Kac formula" of the form:

$$u(t,x) = \mathbb{E}[e^{\int_0^t V(X(s)+x)ds}u(0, X(t)+x)], \tag{1.5}$$

representing the solution of the initial value problem in terms of a (Lebesgue integral) with respect to a probability measure P on $\mathbb{R}^{[0,t]}$ associated to the process $\{X(t), t \geq 0\}$. In fact, such a formula cannot be proved for semigroups whose generator does not satisfy the maximum principle, as in the case of ∂_x^p with $p > 2$ [27]. Further, in the case of the Schrödinger equation (1.3) the problem of the construction of an integral representation for the solution is deeply connected with the mathematical definition of Feynman path integrals [20, 24].

One can obtain a deeper understanding of this negative result by analyzing one of the proofs of the Feynman-Kac formula (1.2) (see, e.g., [26]). In order to simplify the notation, we shall restrict ourselves to the case where the space variable x is one-dimensional, but our reasonings can be generalized to the case where $x \in \mathbb{R}^d$, with $d > 1$.

Let us consider the Hilbert space $L^2(\mathbb{R})$ and the strongly continuous contraction semigroup $T(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $t \geq 0$, generated by the operator sum of the Laplacian $-\frac{\Delta}{2}$ (regarded as a positive self-adjoint operator with domain $H^2(\mathbb{R})$) and the bounded multiplication operator associated to the potential $V \in C_\infty(\mathbb{R}^d)$, i.e. defined as $Vu(x) = V(x)u(x)$, $u \in L^2(\mathbb{R})$. By the Trotter product formula the semigroup $T(t)$, formally written as $e^{t(\frac{\Delta}{2}-V)}$, can be computed in terms of the strong limit

$$T(t)u = \lim_{n\to\infty}(e^{\frac{t}{n}\frac{\Delta}{2}}e^{-\frac{t}{n}V})u.$$

By passing to a subsequence and introducing the fundamental solution G of the heat equation, namely $e^{t\frac{\Delta}{2}}u(x) = \int G(t, x-y)u(y)dy$, one obtains that for a.e. $x \in \mathbb{R}$ the action of semigroup $T(t)$ can be described in terms of the limit of a sequence of integrals of the form:

$$T(t)u(x) = \lim_{n\to\infty}\int_{\mathbb{R}^n} u(x+x_0)e^{-\sum_{j=1}^n V(x+x_j)\frac{t}{n}}\Pi_{j=1}^n G(t/n, x_j - x_{j-1})dx_j. \tag{1.6}$$

By introducing the explicit form of the Green function of the heat equation, i.e.

$$G(t, x-y) = \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}},$$

which in fact is the density of the (Gaussian) transition probability of the Wiener process $\{W(t), t \geq 0\}$, the integrals appearing on the right hand side of (1.6) assume the following form

$$\lim_{n\to\infty}\int_{\mathbb{R}^n} u(x+x_0)e^{-\sum_{j=1}^n V(x+x_j)\frac{t}{n}}\frac{e^{-\sum_{j=1}^n\frac{(x_j-x_{j-1})^2}{2(t/n)}}}{(2\pi t/n)^{n/2}}dx_j$$

and can be regarded as the cylindrical approximations of a Wiener integral. By taking the limit for $n\to\infty$ one obtains the following integral representation of the solution of (1.1), namely the Feynman-Kac formula:

$$T(t)u(x) = \mathbb{E}[u(x+W(t))e^{-V(W(s)+x)ds}].$$

Let us consider now the Schrödinger equation (1.3) or the high order heat equation (1.4) (in the case where the constant c satisfies the inequality $c(ix)^p \leq 0$ for all $x \in \mathbb{R}$), and let $A : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator defined by

$$A = \frac{i\hbar}{2}\Delta, \qquad D(A) = H^2(\mathbb{R}),$$

in the case of Eq. (1.3), or as

$$A = c\frac{\partial^p}{\partial x^p}, \qquad D(A) = H^p(\mathbb{R}),$$

in the case of Eq. (1.4). If $V : \mathbb{R} \to \mathbb{R}$ is a bounded function, then the contraction semigroup $T(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ generated by the operator sum $A+V : D(A) \subset L^2(\mathbb{R}) \to \mathbb{R}$ is still well defined. Further Trotter product formula still holds, giving a representation for $T(t)u(x) = e^{t(A+V)}u(x)$ of the form (1.6), but in these cases G stands for the Green function of the Schrödinger equation, i.e. $G(t,x,y) = \frac{e^{i\frac{(x-y)^2}{2t}}}{\sqrt{2\pi i t}}$, or the Green function of equation (1.4) with $V=0$, i.e. $G(t,x-y) = \frac{1}{2\pi}\int e^{i(x-y)\xi}e^{c(i\xi)^p t}d\xi$. In both cases G is not real and positive [17] (in contrast with the case of Eq (1.1)) and cannot be interpreted as the density of a transition probability measure. This fact has the troublesome consequence that the complex (resp. signed) finitely additive measure μ on $\Omega = \mathbb{R}^{[0,t]}$ defined on the algebra of "cylindrical sets" $I_k \subset \Omega = \{\omega : [0,\infty) \to \mathbb{R}\}$ of the form

$$I_k := \{\omega \in \Omega : \omega(t_j) \in [a_j, b_j], j = 1, \dots k\}, \quad 0 < t_1 < t_2 < \dots t_k,\ a_j, b_j \in \mathbb{R},$$

as

$$\mu(I_k) = \int_{a_1}^{b_1}\dots\int_{a_k}^{b_k}\prod_{j=0}^{k-1} G_{t_{j+1}-t_j}(x_{j+1}, x_j)dx_1\dots dx_k, \qquad x_0 \equiv 0,\ t_0 \equiv 0, \tag{1.7}$$

cannot be extended to a σ-additive measure on the σ-algebra generated by the cylindrical sets. Indeed, if this measure exists, it would have infinite total variation. This problem was pointed out by Cameron [7] in 1960 in the case of the Schrödinger equation and by Krylov [23] in the case of Eq. (1.4). These results can be regarded as particular cases of a general theorem proved by E. Thomas [28], generalizing Kolmogorov existence theorem [4] for the limit of a projective system of probability measures to the case of signed or complex measures. In fact these negative results forbid a functional integral representation of the solution of Eq. (1.3) or Eq. (1.4) in terms of a Lebesgue-type integral with respect to a σ-additive complex or signed measure with finite total variation. Consequently, the integral appearing in the generalized Feynman-Kac formula (1.5) has to be realized in a generalized weaker sense. One possibility is the definition of the "integral" in terms of a linear continuous functional on a suitable Banach algebra of "integrable functions", in the spirit of Riez-Markov theorem, that states a one to one correspondence between complex measures (on suitable topological spaces X) with finite total variation and linear continuous functional on $C_\infty(X)$ (the continuous functions on X vanishing at ∞). The systematic implementation of a generalized integration theory on infinite dimensional spaces based on these ideas is presented in [3]. In the case of the Schrödinger equation this program has been extensively implemented, giving rise to several different mathematical definitions of Feynman path integrals (see [24, 20] for a review of this topic). Analogous techniques have been proposed in the case of higher order heat-type equations, in particular for the parabolic equation associated to the *bilaplacian*, i.e. $\partial_t u(t,x) = -\Delta^2 u(t,x)$ [13, 17].

The present paper describes two new approaches for the construction of generalized Feynman-Kac formulae representing the solution of high order PDEs of the form (1.4). In section 2 the solution of Eq. (1.4) with $V = 0$ is constructed in terms of a particular scaling limit of a random walk on the complex plane. This approach has some relations with the mathematical definition of Feynman path integrals in terms of analytically continued Wiener integrals [7, 8]. In section 3 the mathematical theory of Fresnel integrals, developed in [2, 1] in connection with the representation of the solution of Schrödinger equation, is generalized to the case of polynomially growing phase functions and applied to the construction of Feynman-Kac formulae representing the solution of Eq (1.4) in the case where the potential $V \in C_b(\mathbb{R})$ is the Fourier transform of a complex bounded variation measure on $\mathbb{R}$.

§ 2. A random walk on the complex plane

Let us consider the Feynman-Kac formula (1.2), representing the solution of the heat equation in terms of the expectation with respect to the measure associated to the Wiener process $W(t)_{t\geq 0}$. By the central limit theorem, the Wiener process can be obtained as the weak limit of a sequence of jump processes $W_n(t)$, constructed as a suitable scaling limit of a random walk, namely:

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \xi_j,$$

where $\{\xi_j\}_{j\in\mathbb{N}}$ are independent identically distributed random variables, with $P(\xi_j = 1) = P(\xi_j = -1) = \frac{1}{2}$. By the weak convergence of W_n to the Wiener process W [5], the Feynman-Kac formula (1.2) for $u_0, V \in C_b(\mathbb{R})$ can be written as

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[u_0(x + W_n(t))e^{-\int_0^t V(x+W_n(s))ds}]. \tag{2.1}$$

In this section we are going to present a generalization of formula (2.1) to the case of the high order heat equation (1.4), with $c = \frac{\alpha}{p!}$, $\alpha \in \mathbb{C}$. We shall construct the solution of the associated initial value problem, for a suitable class of analytical initial data and in the case where $V = 0$, as:

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[u(0, x + W_{p,n}(t))],$$

where $W_{p,n}(t)$, $t \geq 0$, is a sequence of jump processes in the complex plane. For a detailed analysis of this problem and further applications see [6].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let α be a complex number and $p > 2$ a given integer. Let $R(p) = \{e^{2i\pi k/p},\ k = 0, 1, \dots, p-1\}$ be the roots of the unity and let us consider the random variable ξ uniformly distributed on the set $\alpha^{1/p}R(p)$:

$$\mathbb{E}[f(\xi)] = \frac{1}{p} \sum_{k=0}^{p-1} f(\alpha^{1/p} e^{2i\pi k/p}). \tag{2.2}$$

The random variable ξ has some interesting properties, indeed its analytic moments have the following form:

$$\mathbb{E}[\xi^m] = \begin{cases} \alpha^{m/p}, & m = np,\ n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

Let us construct a random walk on the complex plane associated to the complex random variable ξ. More precisely, let $\{\xi_j,\ j \in \mathbb{N}\}$ be a sequence of i.i.d. random variables having uniform distribution on the set $\alpha^{1/p}R(p)$ as in (2.2). Let S_n be the random walk defined by the $\{\xi_j\}$, i.e.,

$$S_n = \sum_{j=1}^{n} \xi_j,$$

and let $\tilde{S}_n$ be the the normalized random walk

$$\tilde{S}_n = \frac{1}{n^{1/p}} S_n. \tag{2.4}$$

The following result allows to interpret formally the distribution of $\tilde{S}_n$ as an approximation of a stable distribution of order p.

Theorem 2.1.

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda\tilde{S}_n)] = \exp\left(\frac{i^p\alpha}{p!}\lambda^p\right). \tag{2.5}$$

Proof.

$$\mathbb{E}[e^{i\lambda\tilde{S}_n}] = \mathbb{E}[e^{i\lambda\frac{1}{n^{1/p}}\sum_{j=1}^n \xi_j}] = \Pi_{j=1}^n \mathbb{E}[e^{\frac{i\lambda\xi_j}{n^{1/p}}}] = (\mathbb{E}[e^{\frac{i\lambda\xi}{n^{1/p}}}])^n.$$

Now, one has that

$$\mathbb{E}[e^{\frac{i\lambda\xi}{n^{1/p}}}] = \frac{1}{p}\sum_{k=0}^{p-1} e^{i\frac{\lambda}{n^{1/p}}\alpha^{1/p}e^{ik\frac{2\pi}{p}}}$$

and

$$(\mathbb{E}[e^{\frac{i\lambda\xi}{n^{1/p}}}])^n = e^{n\log\left(\mathbb{E}[e^{\frac{i\lambda\xi}{n^{1/p}}}]\right)}.$$

For $n \to \infty$,

$$\begin{aligned}
\log\left(\mathbb{E}[e^{\frac{i\lambda\xi}{n^{1/p}}}]\right) &= \log\left(1 + \frac{1}{p}\sum_{k=0}^{p-1} e^{i\frac{\lambda}{n^{1/p}}\alpha^{1/p}e^{ik\frac{2\pi}{p}}} - 1\right) \\
&\sim \frac{1}{p}\sum_{k=0}^{p-1}\left(e^{i\frac{\lambda}{n^{1/p}}\alpha^{1/p}e^{ik\frac{2\pi}{p}}} - 1\right) \\
&\sim \frac{1}{p}\sum_{k=0}^{p-1}\frac{1}{M!}\left(i\frac{\lambda}{n^{1/p}}\alpha^{1/p}e^{ik\frac{2\pi}{p}}\right)^p \\
&= \frac{1}{p}\sum_{k=0}^{p-1}\frac{1}{p!}\frac{(i)^p\lambda^p\alpha}{n} = \frac{1}{p!}\frac{(i)^p\lambda^p\alpha}{n}.
\end{aligned}$$

In particular one has

$$\lim_{n\to\infty}\mathbb{E}[e^{i\lambda\tilde{S}_n}] = \lim_{n\to\infty} e^{n\log\left(\mathbb{E}[e^{\frac{i\lambda\xi}{n^{1/p}}}]\right)} = \exp\left(\frac{(i)^p\lambda^p\alpha}{p!}\right).$$

□

Further, let us fix $m \in \mathbb{N}$ and assume that n is large (i.e. $n > m$). Then the m-moment of $\tilde{S}_n$ satisfies:

$$\mathbb{E}[(\tilde{S}_n)^m] = \begin{cases} \left(\frac{\alpha}{p!}\right)^{m/p} \frac{m!}{(m/p)!} + R_n, & m = Mp,\ M \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

where $\lim_{n\to\infty} R_n = 0$. For a detailed proof see [6].

Remark. In the case where $p > 2$, because of the particular scaling exponent in the denominator $n^{1/p}$ appearing in (2.4), the sequence of random variables $\tilde{S}_n$ does not converge weakly to a well defined random variable $\tilde{S}$. Indeed, by the central limit theorem, one has that $\frac{1}{n^{1/2}} S_n = \frac{n^{1/p}}{n^{1/2}} \tilde{S}_n$ has a Gaussian limit, hence $\tilde{S}_n$ cannot converge. On the other hand a stable distribution of order p with $p > 2$ cannot exists.

In case $p = 2$, the limit of the random walk $\tilde{S}_n$ is a Wiener process; to be precise, since in the definition of $\tilde{S}_n$ no time is involved, it converges to the Wiener process at time $t = 1$. It is possible to extend this result to general times; however, it is not possible to talk about the limit process in case $N > 2$. Nevertheless, in the following we are going to construct a family of random walks $W_{p,n}(t)$ that generalizes, in a suitable sense, $\tilde{S}_n$ to a continuous time process.

Let us start by considering the time interval $[0, 1]$. Let $\{\xi_j\}$ be a sequence of independent copies of the random variable ξ defined in (2.2). Then for any $n \in \mathbb{N}$ we set

$$\begin{aligned} &X_n(0) = 0; \\ &X_n(\tfrac{k}{n}) = \frac{1}{n^{1/p}} \sum_{j=1}^{k} \xi_j, \qquad k = 1, \dots, n, \\ &X_n(t) = X_n(\tfrac{k}{n}), \qquad t \in [\tfrac{k}{n}, \tfrac{k+1}{n}). \end{aligned} \tag{2.6}$$

Let us extend now $X_n(t)$ to a process $W_{p,n}(t)$ for values of $t \in (-\infty, +\infty)$. For $t > 0$ we set $\lfloor t \rfloor$ the integer part of t and

$$\begin{aligned} &W_{p,n}(t) = X_n^{(1)}(t), && t \in [0, 1], \\ &W_{p,n}(t) = W_{p,n}(1) + X_n^{(2)}(t - 1), && t \in [1, 2], \end{aligned} \tag{2.7}$$

and, in general,

$$W_{p,n}(t) = W_{p,n}(\lfloor t \rfloor) + X_n^{(\lfloor t \rfloor + 1)}(t - \lfloor t \rfloor), \qquad t \geq 0.$$

while for negative times we set (with the convention that $\lfloor -t \rfloor = -\lfloor t \rfloor$)

$$\begin{aligned} &W_{p,n}(-t) = e^{i\pi/p} X_n^{(-1)}(t), && t \in [0, 1], \\ &W_{p,n}(-t) = W_n(-1) + e^{i\pi/p} X_n^{(-2)}(t - 1), && t \in [1, 2], \end{aligned} \tag{2.8}$$

and, in general,

$$W_{p,n}(-t) = W_{p,n}(\lfloor -t \rfloor) + e^{i\pi/p} X^{(\lfloor -t \rfloor -1)}(\lfloor -t \rfloor - (-t)), \qquad t \geq 0,$$

where $X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(-1)}, X_n^{(-2)}, \dots$ are i.i.d. copies of X_n in (2.6).

We remark that $W_{p,n}$ can be seen as the extension to continuous time of the random walk $\{\tilde{S}_n\}$, since we have the following identity for the laws

$$W_{p,n}(t) \overset{\mathcal{L}}{=} \left(\frac{\lfloor nt \rfloor}{n}\right)^{1/p} \tilde{S}_{\lfloor nt \rfloor}, \qquad \text{for } t > 0. \tag{2.9}$$

In the following, in order to simplify the notation, we shall set $W_{p,n} \equiv W_n$, by skipping the explicit dependence on $p \in \mathbb{N}$.
The sequence of processes $\{W_n\}$ should converge in a very weak sense to a p-stable process (which, we note again, does not exist for $p > 2$). The result is analog to what is proved for the normalized random walk $\tilde{S}_n$.

Theorem 2.2. *For any $t \in (-\infty, +\infty)$ and $\lambda \in \mathbb{C}$,*

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda W_n(t))] = \exp\left(i^p \frac{\lambda^p}{p!}\alpha t\right). \tag{2.10}$$

The previous result allows to construct a probabilistic representation for the solution of the initial value problem associated to the following complex valued parabolic PDE

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\alpha}{p!}\frac{\partial^p}{\partial x^p}u(t,x), \\ u(t_0,x) &= f(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{2.11}$$

showing that for a suitable class of initial data f, the limit

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[f(x + W_n(t-t_0))] \tag{2.12}$$

is well defined for any $x \in \mathbb{R}$ and $t \in \mathbb{R}$ and it provides a representation for the solution of (2.11).

Let us consider the set D of functions $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = \int_{\mathbb{R}} e^{ixy} d\mu(y)$, where μ is a complex Borel measure on $\mathbb{R}$ with finite total variation. We shall also assume that the measure μ has compact support.

Under this assumptions, any function $f \in D$ can be extended to an analytic function, denoted again by f, which is defined as

$$f(z) := \int_{\mathbb{R}} e^{izy} d\mu(y), \qquad z \in \mathbb{C},$$

the integral being absolutely convergent.

Theorem 2.3. *Let $f \in D$. Then the classical solution of the Cauchy problem* (2.11) *is given by* (2.12)

Proof. For $f \in D$ the integral $\mathbb{E}[f(x + W_n(t - t_0))$ is well defined and given by

$$\mathbb{E}[f(x + W_n(t - t_0)) = \mathbb{E}[\int_{\mathbb{R}} e^{i(x+W_n(t-t_0))y} d\mu(y)] = \int_{\mathbb{R}} e^{ixy}\mathbb{E}[e^{iW_n(t-t_0)y}]d\mu(y)]$$

By the dominated convergence theorem

$$\lim_{n\to\infty} \mathbb{E}[f(x + W_n(t - t_0)) = \int_{\mathbb{R}} e^{ixy} \lim_{n\to\infty} \mathbb{E}[e^{iW_n(t-t_0)y}]d\mu(y)] = \int_{\mathbb{R}} e^{ixy} e^{i^p \frac{y^p}{p!}\alpha(t-t_0)} d\mu(y)$$

In fact, by the assumptions on μ, one can directly verify that the function

$$u(t, x) = \int_{\mathbb{R}} e^{ixy} e^{i^p \frac{y^p}{p!}\alpha(t-t_0)} d\mu(y)$$

is a classical solution of the Cauchy problem (2.11). □

This kind of result can be extended to a more general class of initial data. Let $D(T_1, T_2)$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \int_{\mathbb{R}} e^{ixy}\, \mathrm{d}\mu(y),$$

where μ is a measure of bounded variation on $\mathbb{R}$ such that there exists a time interval (T_1, T_2), with $T_1 < t_0 < T_2 \in \mathbb{R}$, such that

$$\int_{\mathbb{R}} \left| \exp\left(i^p \alpha \frac{x^p}{p!}(t - t_0)\right)\right| \mathrm{d}|\mu|(x) < \infty$$

for all $t \in (T_1, T_2)$, where $|\mu|$ stands for the total variation of μ. In this case for any $R \in \mathbb{R}^+$ the compactly supported measure $\mu_R := \chi_{[-R,R]}\mu$ belongs to the set D and the solution of (2.11) with initial datum f_R, with

$$f_R(x) = \int_{-\mathbb{R}} e^{ixy}\, \mathrm{d}\mu_R(y) = \int_{-R}^{R} e^{ixy}\, \mathrm{d}\mu(y),$$

is given by the probabilistic representation (2.12). By the assumptions on $f \in D(T_1, T_2)$ and the dominated convergence theorem, the solution of (2.11) with initial datum f is given for any $t \in (T_1, T_2)$ by the pointwise limit

$$u(t, x) = \lim_{R\to\infty} \lim_{n\to\infty} \mathbb{E}[f_R(x + W_n(t - t_0))].$$

Remark. In the case where $p = 2$ and $\alpha = i\hbar$, equation (2.11) is the Schrödinger equation $i\hbar\frac{\partial}{\partial t}\psi(t,x) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\psi(t,x)$. The complex variable ξ, defined in (2.2), is uniformly distributed on the set $\sqrt{i\hbar}R(2) = \{\pm\sqrt{\hbar}e^{i\pi/4}\}$. In this case, for an initial datum $\psi(0,x) = f(x)$, $x \in \mathbb{R}$, with $f \in D$, formula (2.12) gives

$$\psi(t,x) = \lim_{n\to\infty} \mathbb{E}[\psi(0, x + W_n(t))] = \mathbb{E}[\psi(0, x + \sqrt{i\hbar}B(t))]$$

and one obtains the Feynman path integral representation in terms of analytically continued Wiener integrals, as developed e.g. in [7, 8].

§ 3. Infinite dimensional Fresnel integrals

In this section we propose an alternative construction of the solution of a higher order parabolic equation of the form:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = (-i)^p\alpha\frac{\partial^p}{\partial x^p}u(t,x) + V(x)u(t,x) \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, t \in [0,+\infty) \end{cases} \tag{3.1}$$

where $p \in \mathbb{N}$, $p \geq 2$, and $\alpha \in \mathbb{C}$ is a complex constant such that $|e^{\alpha t x^p}| \leq 1$ for all $x \in \mathbb{R}, t \in [0,+\infty)$, while $V : \mathbb{R} \to \mathbb{C}$ is a bounded continuous function.

We shall extend the theory of infinite dimensional Fresnel integrals developed in [2], where the application to the mathematical theory of Feynman path integrals and the functional integral representation of the solution of the Schrödinger equation are extensively studied.

Fresnel integrals on finite dimensional vector spaces, i.e. integrals of the form

$$\int_{\mathbb{R}^n} e^{\frac{i}{\epsilon}\|x\|^2} f(x)dx, \tag{3.2}$$

where $\epsilon \in \mathbb{R}$ is a real parameter and $f : \mathbb{R}^n \to \mathbb{C}$ a Borel bounded function, are extensively studied, in particular in connections with the theory of wave diffraction. The mathematical theory of more general oscillatory integrals, their asymptotic behavior when $\epsilon \to 0$ and the relations with the theory of Fourier integral operators has been developed, e.g., in [18, 19]. In the following we are going to present an extension of integrals (3.2) to the case where $\mathbb{R}^n$ is replaced by a real separable infinite dimensional Hilbert space.

Given a Schwartz test function $f \in S(\mathbb{R}^n)$, the Fresnel integral $\int_{\mathbb{R}^n} \frac{e^{\frac{i}{2}\|x\|^2}}{(2\pi i)^{n/2}} f(x)dx$ can be computed in terms of the following Parseval's identity:

$$\int_{\mathbb{R}^n} \frac{e^{\frac{i}{2}\|x\|^2}}{(2\pi i)^{n/2}} f(x)dx = \int_{\mathbb{R}^n} e^{-\frac{i}{2}\|x\|^2}\hat{f}(x)dx. \tag{3.3}$$

Let $(\mathcal{H}, \langle\, ,\, \rangle)$ be a real separable Hilbert space and let $\mathcal{M}(\mathcal{H})$ be the Banach space of complex Borel measures on $\mathcal{H}$ with finite total variation, endowed with the total variation norm, denoted by $\|\mu\|_{\mathcal{M}(\mathcal{H})}$. $\mathcal{M}(\mathcal{H})$ is a commutative Banach algebra under convolution, where the unit is the δ point measure. Let us consider the space $\mathcal{F}(\mathcal{H})$ of complex functions on $\mathcal{H}$ of the form:

$$f(x) = \hat{\mu}(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu(y), \qquad x \in \mathcal{H} \tag{3.4}$$

for some $\mu \in \mathcal{M}(\mathcal{H})$. By introducing on $\mathcal{F}(\mathcal{H})$ the norm $\|f\|_{\mathcal{F}} = \|\mu\|_{\mathcal{M}(\mathcal{H})}$, where $f \in \mathcal{F}(\mathcal{H})$ is Fourier transform of $\mu \in \mathcal{M}(\mathcal{H})$, the map (3.4) becomes an isometry and $\mathcal{F}(\mathcal{H})$ endowed with the norm $\|\ \|_{\mathcal{F}}$ a commutative Banach algebra of continuous functions.

Definition 3.1. Let $f \in \mathcal{F}(\mathcal{H})$. The infinite dimensional Fresnel integral of f, denoted by $\widetilde{\int} e^{\frac{i}{2}\|x\|^2} f(x)dx$, is defined as:

$$\widetilde{\int} e^{\frac{i}{2}\|x\|^2} f(x)dx := \int_{\mathcal{H}} e^{-\frac{i}{2}\|x\|^2} d\mu(x), \tag{3.5}$$

where $f(x) = \int_{\mathcal{H}} e^{i\langle x,y\rangle} d\mu(y)$.

The right hand side of (3.5) is a well defined (absolutely convergent) Lebesgue integral. Moreover the application $f \mapsto \widetilde{\int} e^{\frac{i}{2}\|x\|^2} f(x)dx$ is a linear continuous functional on $\mathcal{F}(\mathcal{H})$. In [2] it has been applied to the construction of a representation for the solution of the Schroedinger equation (1.3) in the cases where the potential V belongs to $\mathcal{F}(\mathbb{R}^d)$. In order to generalize these techniques to the realization of a Feynman-Kac type formula for the representation of the solution of higher order heat type equations (3.1), we present here a generalization of definition 3.1 in the case where the quadratic phase function $\Phi(x) = \frac{\|x\|^2}{2}$, $x \in \mathcal{H}$, is replaced by an higher order polynomial function (see [25] for further details).

Let us consider a real separable Banach space $(\mathcal{B}, \|\ \|)$ and let $\mathcal{M}(\mathcal{B})$ be the Banach algebra (under convolution) of complex bounded variation measures on $\mathcal{B}$, endowed with the total variation norm. Let $\mathcal{B}^*$ be the topological dual of $\mathcal{B}$ and let $\mathcal{F}(\mathcal{B})$ be the set of complex-valued functions $f : \mathcal{B}^* \to \mathbb{C}$ of the form

$$f(x) = \int_{\mathcal{B}} e^{i\langle x,y\rangle} d\mu(y), \qquad x \in \mathcal{B}^*,$$

for some $\mu \in \mathcal{M}(\mathcal{B})$, where $\langle\, ,\, \rangle$ denotes the dual pairing between $\mathcal{B}$ and $\mathcal{B}^*$. The set $\mathcal{F}(\mathcal{B})$, endowed with the total variation norm $\|f\|_{\mathcal{F}} := \|\mu\|_{\mathcal{M}(\mathcal{B})}$ and the pointwise multiplication, is a Banach algebra of functions.

Definition 3.2. Let $p \in \mathbb{N}$ and let $\Phi_p : \mathcal{B} \to \mathbb{C}$ be a continuous homogeneous map of order p, i.e. such that:

1. $\Phi_p(\lambda x) = \lambda^p \Phi_p(x)$, forall $\lambda \in \mathbb{R}$, $x \in \mathcal{B}$,
2. $\mathrm{Re}(\Phi_p(x)) \leq 0$ for all $x \in \mathcal{B}$.

The infinite dimensional Fresnel integral on $\mathcal{B}^*$ with phase function Φ_p is the functional $I_{\Phi_p} : \mathcal{F}(\mathcal{B}) \to \mathbb{C}$, given by

$$I_{\Phi_p}(f) := \int_{\mathcal{B}} e^{\Phi_p(x)} d\mu(x), \qquad f \in \mathcal{F}(\mathcal{B}),\ f(x) = \int_{\mathcal{B}} e^{i\langle x,y\rangle} d\mu(y). \tag{3.6}$$

By construction the functional $I_{\Phi_p} : \mathcal{F}(\mathcal{B}) \to \mathbb{C}$ is linear and continuous in the $\mathcal{F}(\mathcal{B})$-norm, indeed:

$$|I_{\Phi_p}(f)| \leq \int_{\mathcal{B}} |e^{\Phi_p}| d|\mu|(x) \leq \|\mu\| = \|f\|_{\mathcal{F}}.$$

These results can be summarized in the following proposition.

Proposition 3.3. *The space $\mathcal{F}(\mathcal{B})$ is a Banach function algebra in the norm $\| \, \|_{\mathcal{F}}$. The infinite dimensional Fresnel integral with phase function Φ_p is a continuous bounded linear functional I_{Φ_p} on $\mathcal{F}(\mathcal{B})$ such that $|I_{\Phi_p}(f)| \leq \|f\|_{\mathcal{F}}$ and normalized, i.e. $I_{\Phi_p}(1) = 1$.*

The following example shows the possible applications of the functional I_{Φ_p} and its connections with the solution of higher order PDEs.
Let $p \in \mathbb{N}$, with $p \geq 2$, and let $\mathcal{B}_p$ be the Banach space of absolutely continuous maps $\gamma : [0,t] \to \mathbb{R}$, with $\gamma(t) = 0$ and a weak derivative $\dot\gamma$ belonging to $L^p([0,t])$, endowed with the norm:

$$\|\gamma\|_{\mathcal{B}_p} = \left(\int_0^t |\dot\gamma(s)|^p ds \right)^{1/p}.$$

The Banach space $\mathcal{B}_p$ is naturally isomorphic to $L^p([0,t])$, indeed the application $T : \mathcal{B}_p \to L^p([0,t])$ mapping an element $\gamma \in \mathcal{B}_p$ to its weak derivative $\dot\gamma$ is an isomorphism with inverse $T^{-1} : L^p([0,t]) \to \mathcal{B}_p$ given by:

$$T^{-1}(v)(s) = -\int_s^t v(u) du \qquad v \in L^p([0,t]). \tag{3.7}$$

Similarly the dual space $\mathcal{B}_p^*$ is isomorphic to $L_q([0,t]) = (L_p([0,t]))^*$, where $\frac{1}{p} + \frac{1}{q} = 1$, and the pairing between an element $\eta \in \mathcal{B}_p^*$ and $\gamma \in \mathcal{B}_p$ is given by $\langle \eta, \gamma \rangle = \int_0^t \dot\eta(s)\dot\gamma(s) ds$, where $\dot\eta \in L_q([0,t])$ and $\gamma \in \mathcal{B}_p$. Further, by means of the map (3.7), it is simple to verify that $\mathcal{B}_p^*$ is isomorphic to $\mathcal{B}_q$.

Let us consider now the space $\mathcal{F}(\mathcal{B}_p)$ of functions $f : \mathcal{B}_q \to \mathbb{C}$ of the form

$$f(\eta) = \int_{\mathcal{B}_p} e^{i\int_0^t \dot{\eta}(s)\dot{\gamma}(s)ds} d\mu_f(\gamma), \qquad \eta \in \mathcal{B}_q,$$

for some $\mu_f \in \mathcal{M}(\mathcal{B}_p)$.

Let $\Phi_p : \mathcal{B}_p \to \mathbb{C}$ be the polynomial phase function defined as $\Phi_p(\gamma) := (-1)^p\alpha \int_0^t \dot{\gamma}(s)^p ds$, where $\alpha \in \mathbb{C}$ is a complex constant such that

$$\text{Re}(\alpha) \leq 0 \qquad \text{if } p \text{ is even,} \tag{3.8}$$

$$\text{Re}(\alpha) = 0 \qquad \text{if } p \text{ is odd.} \tag{3.9}$$

Let us consider the infinite dimensional Fresnel integral $I_{\Phi_p} : \mathcal{F}(\mathcal{B}_p) \to \mathbb{C}$ with phase Φ_p, i.e.:

$$I_{\Phi_p}(f) = \int_{\mathcal{B}_p} e^{(-1)^p\alpha \int_0^t \dot{\gamma}(s)^p ds} d\mu_f(\gamma), \qquad f \in \mathcal{F}(\mathcal{B}_p), \quad f = \hat{\mu}_f. \tag{3.10}$$

The following lemma shows an interesting connection between the functional I_{Φ_p} and the PDE

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = (-i)^p\alpha\frac{\partial^p}{\partial x^p}u(t,x) \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, t \in [0,+\infty) \end{cases} \tag{3.11}$$

For a detailed proof see [25].

Lemma 3.4. *Let $f : \mathcal{B}_q \to \mathbb{C}$ be a cylindrical function of the following form:*

$$f(\eta) = F(\eta(t_1), \eta(t_2), ..., \eta(t_n)), \qquad \eta \in \mathcal{B}_q,$$

with $0 \leq t_1 < t_2 < ... < t_n < t$ and $F : \mathbb{R}^n \to \mathbb{C}$, $F \in \mathcal{F}(\mathbb{R}^n)$:

$$F(x_1, x_2, ..., x_n) = \int_{\mathbb{R}^n} e^{i\sum_{k=1}^n y_k x_k} d\nu_F(y_1, ..., y_n), \qquad \nu_F \in \mathcal{M}(\mathbb{R}^n).$$

Then $f \in \mathcal{F}(\mathcal{B}_p)$ and its infinite dimensional Fresnel integral with phase function Φ_p is given by

$$I_{\Phi_p}(f) = \int_{\mathbb{R}^n} F(x_1, x_2, ..., x_n)\Pi_{k=1}^n G^p_{t_{k+1}-t_k}(x_{k+1}, x_k)dx_k, \tag{3.12}$$

where $x_{n+1} \equiv 0$, $t_{n+1} \equiv t$ and G^p_s is the fundamental solution of the high order heat-type equation (3.11), *i.e.*

$$G^p_s(t,x) = \frac{1}{2\pi}\int e^{ikx}e^{\alpha t k^p} dk. \tag{3.13}$$

Further the integral (3.12) *is absolutely convergent.*

Remark. A detailed asymptotic analysis of the behavior of the fundamental solution (3.13) of Eq (3.11) in the case where p is an even integer and $\alpha \in \mathbb{R}$, with $\alpha < 0$, can be found in [17], while the general case with $p \in \mathbb{N}$, $p > 2$ and $\alpha \in \mathbb{C}$ satisfying assumptions (3.8) or (3.9) has been studied in [25].

A direct consequence of lemma 3.4 is the following representation for the solution of the initial value problem associated to equation (3.11).

Theorem 3.5. *Let $u_0 \in \mathcal{F}(\mathbb{R})$. Then the cylindrical function $f_0 : \mathcal{B}_q \to \mathbb{C}$ defined by*

$$f_0(\eta) := u_0(x + \eta(0)), \qquad x \in \mathbb{R}, \eta \in \mathcal{B}_q,$$

belongs to $\mathcal{F}(\mathcal{B}_p)$ and its infinite dimensional Fresnel integral with phase function Φ_p provides a representation for the solution of the Cauchy problem (3.11), *in the sense that for all $t \geq 0$ and $x \in \mathbb{R}$ the function defined as:*

$$u(t, x) := I_{\Phi_p}(f_0)$$

has the form

$$u(t, x) = \int_{\mathbb{R}} G_t^p(x, y) u_0(y) dy, \tag{3.14}$$

where G^p is given by (3.13).

The integral on the right hand side of (3.14) is absolutely convergent, since $u_0 \in \mathcal{F}(\mathbb{R})$ is bounded and for all $t > 0$ and $p \geq 3$ the distribution G_t^p belongs to $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ (see [25]). In the case where p is an odd integer and α satisfies assumption (3.9), the representation (3.14) for the solution of (3.11) is valid for all values of the time variable $t \in \mathbb{R}$.

The following proposition provides a generalized Feynman-Kac formula for the representation of the solution of the Cauchy problem (3.1). Let us consider indeed the Hilbert space $L^2(\mathbb{R})$ and the self-adjoint operator $\mathcal{D}_p : D(\mathcal{D}_p) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$D(\mathcal{D}_p) := H^p,$$
$$\widehat{\mathcal{D}_p u}(k) := k^p \hat{u}(k),\ u \in D(\mathcal{D}_p),$$

($\hat{u}$ denoting the Fourier transform of u). Let $B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the bounded multiplication operator defined by

$$Bu(x) = V(x)u(x), \qquad u \in L^2(\mathbb{R}).$$

For $\alpha \in \mathbb{C}$ satisfying assumption (3.8) or (3.9), one has that the operator $A := \alpha \mathcal{D}_p$ generates a strongly continuous semigroup $(e^{tA})_{t \geq 0}$ on $L^2(R)$. Analogously the operator sum $A + B : D(A) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $L^2(\mathbb{R})$.

Theorem 3.6. *Let $u_0 \in \mathcal{F}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $V \in \mathcal{F}(\mathbb{R})$, with $u_0(x) = \int_{\mathbb{R}} e^{ixy} d\mu_0(y)$ and $V(x) = \int_{\mathbb{R}} e^{ixy} d\nu(y)$, $\mu_0, \nu \in \mathcal{M}(\mathbb{R})$. Then the functional $f_{t,x} : \mathcal{B}_q \to \mathbb{C}$ defined by*

$$f_{t,x}(\eta) := u_0(x + \eta(0)) e^{\int_0^t V(x+\eta(s))ds}, \qquad x \in \mathbb{R}, \eta \in \mathcal{B}_q, \tag{3.15}$$

belongs to $\mathcal{F}(\mathcal{B}_p)$ and its infinite dimensional Fresnel integral with phase function Φ_p provides a representation for the solution of the Cauchy problem (3.1).

Proof. Given a $u \in L^2(\mathbb{R})$, the vector $T(t)u$ can be computed by means of the convergent (in the $L^2(\mathbb{R})$-norm) Dyson series (see [16], Th. 13.4.1):

$$T(t)u = \sum_{n=0}^{\infty} S_n(t)u, \tag{3.16}$$

where $S_0(t)u = e^{tA}u$ and $S_n(t)u = \int_0^t e^{(t-s)A} V S_{n-1}(s)u ds$. By passing to a subsequence, the series above converges also a.e. in $x \in \mathbb{R}$ giving

$$T(t)u(x) = \sum_{n=0}^{\infty} \int \cdots \int_{0 \le s_1 \le \cdots \le s_n \le t} \int_{\mathbb{R}^{n+1}} V(x_1) \dots V(x_n) G_{t-s_n}(x, x_n) \tag{3.17}$$

$$\times G_{s_n - s_{n-1}}(x_n, x_{n-1}) \dots G_{s_1}(x_1, x_0) u_0(x_0) dx_0 \dots dx_n \, ds_1 \dots ds_n, \qquad \text{a.e. } x \in \mathbb{R}.$$

Under the assumption that $u_0 \in \mathcal{F}(\mathbb{R})$, one has that the cylindric function $\eta \in \mathcal{B}_q \mapsto u_0(x + \eta(0))$ is an element of $\mathcal{F}(\mathcal{B}_p)$, namely the Fourier transform of the measure $\mu_{u_0} \in \mathcal{M}(\mathcal{B}_p)$ defined by

$$\int_{\mathcal{B}_p} f(\gamma) d\mu_{u_0}(\gamma) = \int_{\mathbb{R}} e^{ixy} f(y v_0) d\mu_0(y), \qquad f \in C_b(\mathcal{B}_p).$$

Further, under the assumption that $V \in \mathcal{F}(\mathbb{R})$, let μ_V be the measure on $\mathcal{B}_p$ defined by

$$\int_{\mathcal{B}_p} f(\gamma) d\mu_V(\gamma) = \int_0^t \int_{\mathbb{R}} e^{ixy} f(y v_s) d\nu(y) ds, \qquad f \in C_b(\mathcal{B}_p),$$

where $v_s \in \mathcal{B}_p$ is the function $v_s(\tau) = \chi_{[0,s]}(\tau)(t-s) + \chi_{(s,t]}(t-\tau)s$. One has that the map $\eta \in \mathcal{B}_q \mapsto \exp(\int_0^t V(x + \eta(s))ds)$ is the Fourier transform of the measure $\nu_V \in \mathcal{M}(\mathcal{B}_p)$ given by $\nu_V = \sum_{n=0}^{\infty} \frac{1}{n!} \mu_V^{*n}$, where $*$ stands for convolution and μ_V^{*n} denotes the n-fold convolution of μ_V with itself. One can then conclude that the map $f_{t,x} : \mathcal{B}_q \to \mathbb{C}$ defined by (3.15) belongs to $\mathcal{F}(\mathcal{B}_q)$ and its infinite dimensional Fresnel integral $I_{\Phi_p}(f_{t,x})$ with phase function Φ_p is given by

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{B}_p} e^{(-1)^p \alpha \int_0^y \dot{\gamma}(s)^p ds} d\mu_{u_0} * \mu_V * \cdots * \mu_V$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t \dots \int_0^t I_{\Phi_p}(u_0(x + \eta(0)) V(x + \eta(s_1)) \dots V(x + \eta(s_n))) ds_1 \cdots ds_n.$$

By the symmetry of the integrand the latter is equal to

$$\sum_{n=0}^{\infty} \int_{0 \le s_1 \le \cdots \le s_n \le t} \cdots \int I_{\Phi_p}\big(u_0(x+\eta(0))V(x+\eta(s_1)) \dots V(x+\eta(s_n))\big) ds_1 \cdots ds_n.$$

By lemma 3.4 we finally obtain

$$\sum_{n=0}^{\infty} \int_{0 \le s_1 \le \cdots \le s_n \le t} \cdots \int \int_{\mathbb{R}^{n+1}} u_0(x+x_0)V(x+x_1)\dots V(x+x_n)G_{s_1}(x_1,x_0)$$
$$\times G_{s_2-s_1}(x_2,x_1)\dots G_{t-s_n}(0,x_n)dx_0dx_1\cdots dx_nds_1\cdots ds_n,$$

that, as one can easily verify by means of a change of variables argument, coincides with the Dyson series (3.17) for the solution of the high-order PDE (3.1). □

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