数理解析研究所講究録
第 1960 巻 2015 年 46-58

The Goeritz groups of Heegaard splittings for 3-manifolds

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1 Introduction

This note is adapted from the talk at 2015 Intelligence of Low-dimensional Topology held in Research Institute for Mathematical Sciences, Kyoto University. We refer the readers to [6], [7], [8] and [9] for the details.

Every closed orientable 3-manifold can be decomposed into two handlebodies of the same genus, which is called a *Heegaard splitting* of the manifold. The genus of the handlebodies is called the *genus* of the splitting. The 3-sphere admits a Heegaard splitting of each genus $g \geq 0$ (see [27]), and lens spaces and $S^2 \times S^1$ admit Heegaard splittings of each genus $g \geq 1$ (see [3]).

Given a Heegaard splitting of a 3-manifold, the *Goeritz group* of the splitting is the group of isotopy classes of orientation preserving diffeomorphisms of the manifold that preserve the splitting. When a genus-g Heegaard splitting for a manifold is unique up to isotopy, we call the Goeritz group of the splitting the *genus-g Goeritz group* of the manifold without mentioning a specific splitting of the manifold. The Goeritz groups have been interesting objects in the study of Heegaard splittings. For example, some interesting questions on Goeritz groups were proposed by Minsky in [12]. A Goeritz group will be "small" when the gluing map of the two handlebodies that defines the Heegaard splitting is sufficiently complicated. Indeed, Namazi [21] showed that the Goeritz group is actually a finite group when the Heegaard splitting has "high" *Hempel distance.* Here, we just mention that the Hempel distance is a measure of complexity of the gluing map that defines the splitting. We refer to [13] for its precise definition. Finite generating set of Goeritz groups have been obtained for the following manifolds:

[1]The first-named author is supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF-2015R1A1A1A05001071) funded by the Ministry of Science, ICT and Future Planning.
[2]The second-named author is supported in part by Grant-in-Aid for Young Scientists (B) (No. 26800028), Japan Society for the Promotion of Science.

- The genus-3 Heegaard splitting of the 3-torus $S^1 \times S^1 \times S^1$ (see [16]).
- The genus-$(g+1)$ Heegaard splitting of the genus-g handlebody (see [25]).
- Heegaard splittings obtained by once-stabilizing Heegaard splittings of sufficiently large Hempel distance (see [17]).
- Heegaard splittings obtained by connecting the genus-1 Heegaard splitting of $S^2 \times S^1$ and Heegaard splittings of sufficiently large Hempel distance (see [8]).

On the other hand, finite presentations of Goeritz groups have been obtained for:

- The genus-2 Heegaard splitting of the 3-sphere S^3 (see [11], [24], [1] and [4]).
- The genus-2 Heegaard splittings of the lens spaces $L(p, 1)$ (see [5]).
- The genus-2 Heegaard splittings of $S^2 \times S^1$ (see [6]).
- The genus-2 Heegaard splittings of non-prime 3-manifolds (see [7]).
- The genus-2 Heegaard splittings of lens spaces $L(p, q)$, where $1 \leq q \leq p/2$ and $p \equiv \pm 1 \pmod q$ (see [9]).

In this note, we survey finite presentations of the Goeritz groups of the genus-2 Heegaard splittings of $S^2 \times S^1$, some lens spaces, and non-prime 3-manifolds. We then explain some applications to the theory of unknotting tunnels and the spaces of Heegaard splittings.

Throughout the note, $(V, W; \Sigma)$ will denote a genus-2 Heegaard splitting of a given 3-manifold M. That is, V and W are genus-2 handlebodies such that $V \cup W = M$ and $V \cap W = \partial V = \partial W = \Sigma$ is a genus-2 closed orientable surface, which is called a Heegaard surface in M. Any disks in a handlebody are always assumed to be properly embedded, and their intersection is transverse and minimal up to isotopy. For convenience, we will not distinguish disks (or union of disks) and homeomorphisms from their isotopy classes in their notation. Finally, $\mathrm{Nbd}(X)$ will denote a regular neighborhood of X, where the ambient space will always be clear from the context.

2 Primitive disk complexes

Since our main target in this note is a finite presentation of each Goeritz group, we begin with recalling a specialized version of Bass-Serre Structure Theorem, which is actually the key to obtain a presentation of each Goeritz group.

Theorem 2.1 (Serre [26]). *Suppose that a group G acts on a tree $\mathcal{T}$ without inversion on the edges. If there exists a subtree $\mathcal{L}$ of $\mathcal{T}$ such that every vertex (every edge, respectively) of $\mathcal{T}$ is equivalent modulo G to a unique vertex (a unique edge, respectively) of $\mathcal{L}$. Then G is the free product of the isotropy groups G_v of the vertices v of $\mathcal{L}$, amalgamated along the isotropy groups G_e of the edges e of $\mathcal{L}$.*

Due to this theorem, our plan is to construct a simplicial complex on which the Goeritz group acts simplicially and co-compactly, without edge inversions.

Let V be a handlebody of genus $g \geq 2$. The *disk complex* $\mathcal{K}(V)$ of V is defined to be the simplicial complex whose vertices are the isotopy classes of essential disks in V such that the collection of distinct $k+1$ vertices spans a k-simplex if and only if they admit a set of pairwise disjoint representatives. The disk complex is $(3g-4)$-dimensional and is not locally finite.

The following is a key property of a disk complex.

Theorem 2.2 ([20], [4]). *If $\mathcal{L}$ is a full subcomplex of the disk complex $\mathcal{K}(V)$ satisfying the following condition, then $\mathcal{L}$ is contractible.*

- *Let E and D be disks in V representing vertices of $\mathcal{L}$. If they intersect each other transversely and minimally, then at least one of the disks from surgery on E along an outermost subdisk of D cut off by $D \cap E$ represents a vertex of $\mathcal{L}$.*

From the theorem, we see that the disk complex itself is contractible, and its full subcomplex spanned by the vertices of non-separating disks, whcih we call the *non-separating disk complex*, is also contractible. We denote by $\mathcal{D}(V)$ the non-separating disk complex of V.

Consider the case that M is a genus-2 handlebody V. Then the complex $\mathcal{D}(V)$ is 2-dimensional, and every edge of $\mathcal{D}(V)$ is contained in infinitely but countably many 2-simplices. For any two non-separating disks in V which intersect each other transversely and minimally, it is easy to see that "both" of the two disks obtained from surgery on one along an outermost subdisk of another cut off by their intersection are non-separating. This implies, from Theorem 2.2, that $\mathcal{D}(V)$ and the link of any vertex of $\mathcal{D}(V)$ are all contractible. Thus the complex $\mathcal{D}(V)$ deformation retracts to a tree in the barycentric subdivision of it. Actually, this tree is a dual complex of $\mathcal{D}(V)$. A portion of the non-separating disk complex of V together with its dual tree is described in Figure 1.

Now we return to the genus-2 Heegaard splitting $(V, W; \Sigma)$ of M, where M is S^3, $S^2 \times S^1$ or a lens space. An essential disk E in V is called *primitive* if there exists an essential disk E' in W such that ∂E intersects $\partial E'$ transversely in a single point. Such a disk E' is called a *dual disk* of E, which is also primitive in W having a dual disk E. Note

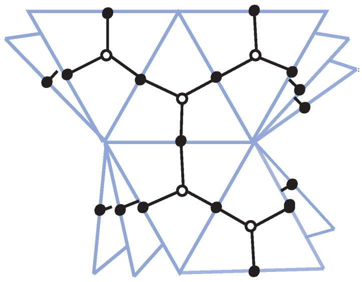


図 1: A portion of the non-separating disk complex $\mathcal{D}(V)$ of a genus-2 handlebody V with its dual tree.

that both $W \cup \mathrm{Nbd}(E)$ and $V \cup \mathrm{Nbd}(E')$ are solid tori. Primitive disks are necessarily non-separating.

The *primitive disk complex* $\mathcal{P}(V)$ for the splitting $(V, W; \Sigma)$ is defined to be the full subcomplex of $\mathcal{D}(V)$ spanned by the vertices of primitive disks in V. From the structure of $\mathcal{D}(V)$, we observe that every connected component of any full subcomplex of $\mathcal{D}(V)$ is contractible. In the following, we will see that the primitive disk complex is 1-dimensional or 2-dimensional, depending on the manifold M, and it is actually suitable for finding a finite presentation of the Goeritz group.

3 The Goeritz groups

3.1 The 3-sphere

Let $(V, W; \Sigma)$ be the genus-2 Heegaard splitting $(V, W; \Sigma)$ of S^3. In this case, the following holds:

Lemma 3.1 ([4]). *The primitive disk complex $\mathcal{P}(V)$ is 2-dimensional and contractible. The complex $\mathcal{P}(V)$ is actually isomorphic to $\mathcal{D}(V)$.*

Using this complex (more precisely, the barycentric subdivision of the dual complex of $\mathcal{P}(V)$, which is a tree), one has the following presentation of the Goeritz group by Theorem 2.1:

Theorem 3.2 ([11], [24], [1] and [4]). *The Goeritz group of the genus-2 Heegaard splitting $(V, W; \Sigma)$ of S^3 has the following presentation:*

$$\langle\ \alpha \mid \alpha^2\ \rangle \oplus \langle\ \beta, \gamma, \delta \mid \gamma^2,\ \delta^3,\ \gamma\beta\gamma\beta^{-1}\alpha,\ \gamma\delta\gamma\delta^{-1}\ \rangle.$$

Figure 2 illustrates the generators α, β, γ and δ in the above presentation of the Goeritz group.

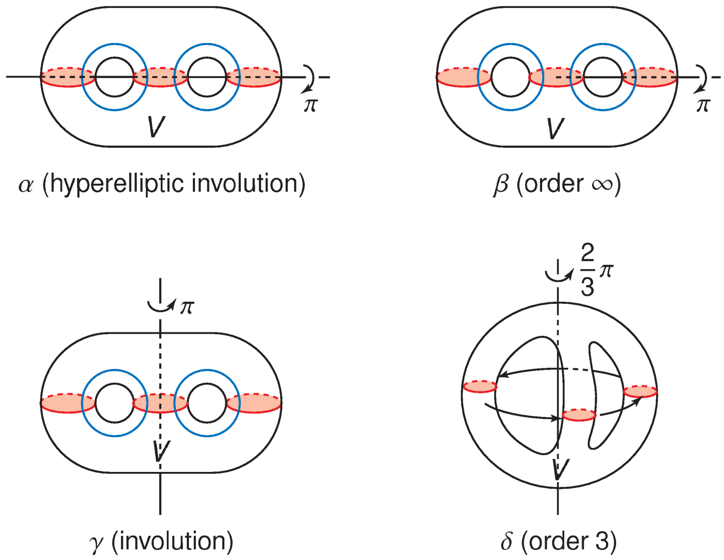


図 2: The four generators α, β, γ and δ of the Goeritz group.

3.2 $S^2 \times S^1$

Let $(V, W; \Sigma)$ be the genus-2 Heegaard splitting of $S^2 \times S^1$. In this case, we can show that there exists a unique non-separating disk E_0 in V such that ∂E_0 also bounds a disk in W. Then we can show the following:

Lemma 3.3 ([6]). *The primitive disk complex $\mathcal{P}(V)$ is exactly the link of E_0 in $\mathcal{D}(V)$. In particular, the complex $\mathcal{P}(V)$ is a tree.*

Using the barycentric subdivision of $\mathcal{P}(V)$, we can obtain the following presentation of the Goeritz group by Theorem 2.1:

Theorem 3.4 ([6]). *The Goeritz group of the genus-2 Heegaard splitting of $S^2 \times S^1$ has the following presentation:*

$$\langle\ \epsilon\ \rangle \oplus \langle\ \alpha \mid \alpha^2\ \rangle \oplus \langle\ \beta, \gamma, \sigma \mid \gamma^2,\ \sigma^2,\ (\gamma\beta\sigma)^2\ \rangle.$$

The element ϵ is the Dehn twist about the disk E_0, which extends to a diffeomorphism of the whole of $S^2 \times S^1$ since ∂E_0 bounds a disk also in W. The other generators are almost the same as in the case of S^3.

3.3 Lens spaces

The structure of primitive disk complexes for lens spaces is much more complicated than the previous cases.

Recall that the fundamental group of the genus-2 handlebody is the free group $\mathbb{Z} * \mathbb{Z}$ of rank two. An element of $\mathbb{Z} * \mathbb{Z}$ *primitive* if it is a member of a generating pair of $\mathbb{Z} * \mathbb{Z}$. Primitive elements of $\mathbb{Z} * \mathbb{Z}$ have been well understood by [22].

A simple closed curve in the boundary of a genus-2 handlebody W represents elements of $\pi_1(W) = \mathbb{Z} * \mathbb{Z}$. We call a pair of essential disks in W a *complete meridian system* for W if the union of the two disks cuts off W into a 3-ball. Given a complete meridian system $\{D, E\}$, assign symbols x and y to the circles ∂D and ∂E respectively. Suppose that an oriented simple closed curve l on ∂W that meets $\partial D \cup \partial E$ transversely. Then l determines a word in terms of x and y which can be read off from the the intersections of l with ∂D and ∂E (after a choice of orientations of ∂D and ∂E), and hence l represents an element of the free group $\pi_1(W) = \langle x, y \rangle$.

Let $(V, W; \Sigma)$ be the genus-2 Heegaard splitting of a lens space $L = L(p, q)$. Any simple closed curve on the boundary of the solid torus W represents an element of $\pi_1(W)$ which is the free group of rank two. We interpret primitive disks algebraically as follows, which is a direct consequence of [11].

Lemma 3.5. *Let D be a non-separating disk in V. Then D is primitive if and only if ∂D represents a primitive element of $\pi_1(W)$.*

Due to Lemma 3.5, we can use the classical combinatorial group thoery (the Ozborn-Zieschang's criterion [22]) to study the structure of the primitive disk complex. In particular, we obtain the following:

Lemma 3.6. *Given a lens space $L(p, q)$, $1 \leq q \leq p/2$, with a genus-2 Heegaard splitting $(V, W; \Sigma)$, suppose that $p \equiv \pm 1 \pmod q$. Let D and E be primitive disks in V which intersect each other transversely and minimally. Then at least one of the two disks from surgery on E along an outermost subdisk of D cut off by $D \cap E$ is primitive.*

Remark that Lemma 3.6 and Theorem 2.2 imply that primitive disk complex $\mathcal{P}(V)$ for $L(p, q)$, $1 \leq q \leq p/2$ is contractible provided $p \equiv \pm 1 \pmod q$. Actually, we can show that this is the only case for $\mathcal{P}(V)$ to be connected:

Lemma 3.7. *For a lens space $L(p, q)$ with $1 \leq q \leq p/2$, the primitive disk complex $\mathcal{P}(V)$ is contractible if and only if $p \equiv \pm 1 \pmod q$. If $p \not\equiv \pm 1 \pmod q$, $\mathcal{P}(V)$ consists of infinitely many trees.*

Figure 3 shows the shape of primitive disk complexes $\mathcal{P}(V)$ for $L(p, q)$, $1 \leq q \leq p/2$. As we can see in the figure, the primitive disk complex is 1-dimensional or 2-dimensional, depending on the parameter (p, q) of a lens space. The number on each edge shows the number of "common" dual disks of the two end points, which are primitive disks.

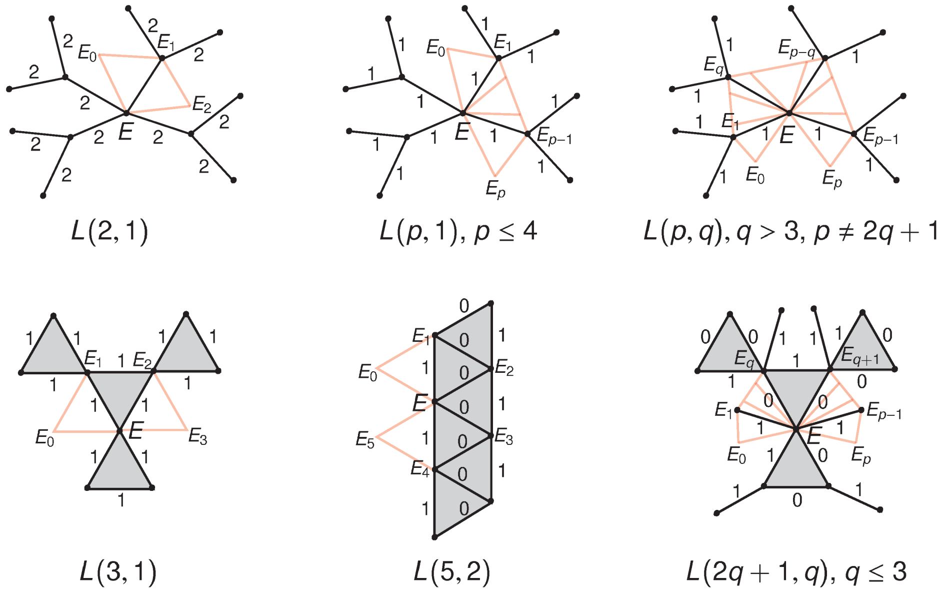


図 3: A portion of each primitive disk complex $\mathcal{P}(V)$.

Considering the action of the Goeritz groups on the primitive disk complexes in detail, we finally get the following:

Theorem 3.8. *The genus-2 Goeritz group of a lens space $L(p,q)$, $1 \leq q \leq p/2$, with $p \equiv \pm 1 \pmod q$ has the following presentations:*

1. *If $q = 1$, then we have:*
   - (a) $\langle \beta, \rho, \gamma \mid \rho^4, \gamma^2, (\gamma\rho)^2, \rho^2\beta\rho^2\beta^{-1} \rangle$ *if $p = 2$;*
   - (b) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \delta, \gamma \mid \delta^3, \gamma^2, (\gamma\delta)^2 \rangle$ *if $p = 3$;*
   - (c) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma^2, \sigma^2 \rangle$ *if $p \geq 4$;*
2. *If $q > 1$, then we have:*
   - (a) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta_1, \beta_2, \gamma_1, \gamma_2 \mid {\gamma_1}^2, {\gamma_2}^2 \rangle$ *if $p = 5$;*
   - (b) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma \mid {\gamma_1}^2, {\gamma_2}^2, \sigma^2 \rangle$ *if $p = 2q + 1$ and $q \geqslant 3$, or $p > 5$ and $q = 2$;*
   - (c) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \sigma_1, \sigma_2 \mid \gamma^2, {\sigma_1}^2, {\sigma_2}^2 \rangle$ *if $q^2 \equiv 1 \pmod p$;*
   - (d) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma_1, \sigma_2 \mid {\gamma_1}^2, {\gamma_2}^2, {\sigma_1}^2, {\sigma_2}^2 \rangle$ *otherwise.*

3.4 Non-prime 3-manifolds

Let $(V, W; \Sigma)$ be a genus-2 Heegaard splitting $(V, W; \Sigma)$ of a non-prime 3-manifold M. Remark that in this case M might admit several non-isotopic genus 2 Heegaard splittings. When $M = (S^2 \times S^1)\#(S^2 \times S^1)$, M is the double of the genus-2 handlebody V. This implies that the Goeritz group of $(V, W; \Sigma)$ is isomorphic to the genus-2 handlebody group, whose presentation is well-understood. Thus in the following we assume that at least one summand of the connected sum is a lens space. There is no primitive disks in V in this case, but we can use the semi-primitive disks. An essential disk $E \subset V$ is *semi-primitive* if there exists a Haken sphere P for the splitting $V \cup_\Sigma W$ disjoint from ∂E. Then the *semi-primitive disk complex* $\mathcal{SP}(V)$ is defined to be the full subcomplex of $\mathcal{D}(V)$ spanned by semi-primitive disks in V.

Lemma 3.9 ([7]). *1. If M is the connected sum of two lens spaces, then the semi-primitive disk complex $\mathcal{SP}(V)$ is a tree.*

2. If M is the connected sum of $S^2 \times S^1$ and a lens space, then the semi-primitive disk complex $\mathcal{SP}(V)$ is a cone on a tree.

A Haken sphere P of $(V, W; \Sigma)$ is said to be *reversible* if there exists an element g of $\mathcal{G}$ fixing P setwise such that g restricted to P is an orientation-reversing homeomorphism on P. We say that the splitting $(V, W; \Sigma)$ is *symmetric* if it admits a reversible Haken sphere.

Theorem 3.10. *Let M_1 be a lens space or $S^2 \times S^1$, and let M_2 be a lens space. Let $(V, W; \Sigma)$ be a genus two Heegaard splitting for $M_1 \# M_2$. Then the Goeritz group of $(V, W; \Sigma)$ has the following presentation:*

1. If M_1 is a lens space,

(a) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_1, \gamma_2 \mid {\gamma_1}^2, {\gamma_2}^2 \rangle$ if $(V, W; \Sigma)$ is not symmetric;

(b) $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma_1, \delta \mid {\gamma_1}^2, \delta^2, \delta\beta\delta = \alpha\beta \rangle$ if $(V, W; \Sigma)$ is symmetric;

2. If $M_1 = S^2 \times S^1$, $\langle \alpha \mid \alpha^2 \rangle \oplus \langle \beta, \gamma, \sigma \mid \gamma^2, \sigma^2 \rangle \oplus \langle \tau \rangle$.

4 Tree of knot tunnels

Let $(V, W; \Sigma)$ be a genus-2 Heegaard splitting $(V, W; \Sigma)$ of M, where M is S^3, $S^2 \times S^1$ or a lens space.

A knot K in M is said to be of *tunnel number*-1 if there is an arc τ meeting K only in its endpoints so that $\mathrm{Nbd}(K \cup \tau)$ is isotopic to V in M. The arc τ is called a *tunnel* for K.

Let τ be a tunnel of a tunnel number-1 knot K in M. Up to isotopy, the co-core of a thickening of τ can be regarded as a non-separating disk in V as illustrated in Figure 4.

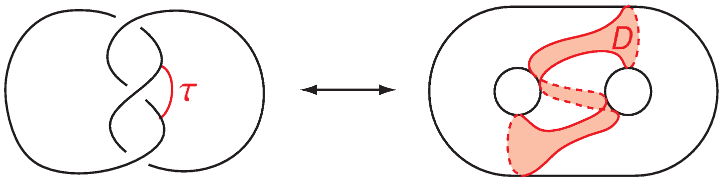


図 4: Correspondence between a tunnel and a non-separating disk in V.

Conversely, each non-separating disk D in V can be considered as a tunnel of the core loop of the solid torus cut off from V along D. For instance when $M = S^3$, a primitive disk corresponds to the trivial tunnel of the trivial knot, see Figure 5.

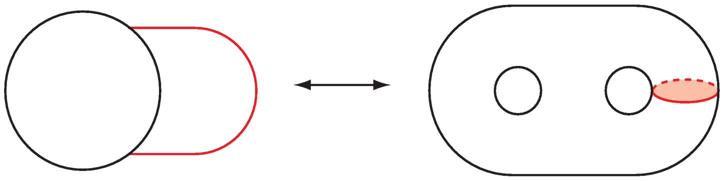


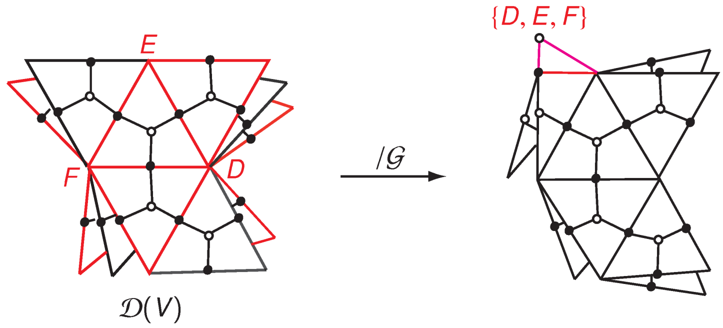
図 5: Correspondence between the trivial tunnel and a primitive disk in V.

In this way, each tunnel corresponds to a vertex of $\mathcal{D}(V)$. However, this correspondence has an indeterminacy because there are many isotopies that move the union $\mathrm{Nbd}(K \cup \tau)$ a knot K and its tunnel τ to V. In fact, each tunnel corresponds to infinitely many vertices of $\mathcal{D}(V)$. However, this indeterminancy is exactly up to the Goeritz group $\mathcal{G} = \mathcal{MCG}_+(M, V)$. Thus, there is a one-to-one correspondence between the collenction of (equivalent classes of) tunnels and the set of vertices of the quotient $\mathcal{D}(V)/\mathcal{G}$ that comes from $\mathcal{D}(V)$. This quotient complex $\mathcal{D}(V)/\mathcal{G}$ provide us a bird's- eye view of the set of tunnels of tunnel number-1 knots in M. If the Goeritz group $\mathcal{G}$ and its action on $\mathcal{D}(V)$ are well-understood, we have a precise description of the quotient $\mathcal{D}(V)/\mathcal{G}$.

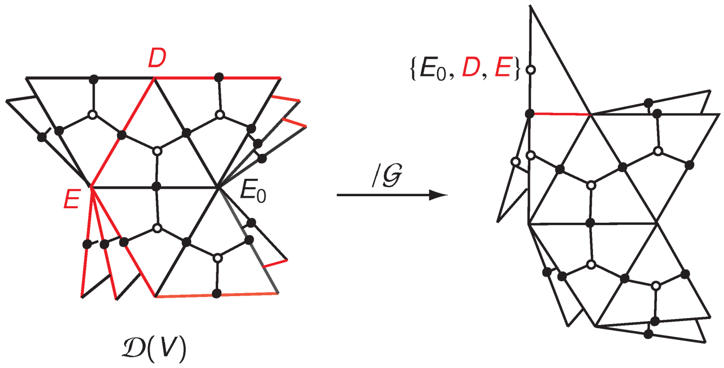
For S^3, Cho-McCullough [10] showed the following:

Theorem 4.1 ([10]). *Let $\mathcal{T}$ be the dual complex of $\mathcal{D}(V)$, which is a tree. Every tunnel for a tunnel number-1 knot in S^3 is determined uniquely (up to equivalence) by a finite sequence of consecutive vertices of the tree $\mathcal{T}/\mathcal{G}$ starting at the unique vertex coming from a triple $\{D, E, F\}$ of pairwise disjoint primitive disks in V.*

Since now we know the Goeritz group and its action on $\mathcal{D}(V)$ very well, we are ready to describe the quotient complex $\mathcal{D}(V)/\mathcal{G}$ also for $S^2 \times S^1$ and lens spaces. For example, when $M = S^2 \times S^1$, we have the follwing:



Corollary 4.2. *Let $\mathcal{T}$ be the dual complex of $\mathcal{D}(V)$, which is a tree. Every tunnel for a tunnel number-1 knot in $S^2 \times S^1$ is determined uniquely (up to equivalence) by a finite sequence of consecutive vertices of the tree $\mathcal{T}/\mathcal{G}$ starting at the unique vertex coming from a triple $\{E_0, D, E\}$ of pairwise disjoint disks in V, where E_0 is the unique disk defined in Section 3.2, and D and E are primitive disks.*



5 Space of Heegaard splittings

Let M be a closed, orientable 3-manifold, and suppose that Σ is a Heegaard surface of M. Due to [18], the space of left cosets $\mathcal{H}(M, \Sigma) := \mathrm{Diff}(M)/\mathrm{Diff}(M, \Sigma)$ is called the *space of Heegaard splittings* equivalent to (M, Σ). We note that this is a huge space and our main interest is its homotopy type. Remark that $\pi_0(\mathcal{H}(M, \Sigma))$ is exactly the set of isotopy classes of Heegaard splittings equivalent to (M, Σ).

Let $(V, W; \Sigma)$ be the genus-2 Heegaard splitting of a lens space $L = L(p, q)$ with $1 \leq q \leq p/2$. By [2] and [3], $\pi_0(\mathcal{H}(L(p, q), \Sigma))$ consists of one or two points depending on whether or not $L(p, q)$ admits an orientation-reversing diffeomorphism onto itself. For $\pi_i(\mathcal{H}(L(p, q), \Sigma))$ $(i \geq 2)$, the following holds.

Theorem 5.1 ([18]). *1. Up to the Smale Conjecture,* $\pi_i(\mathcal{H}(L(2,1),\Sigma)) \cong \pi_1(S^3 \times S^3)$ *for* $i \geq 2$.

2. If $p \geq 3$, $\pi_i(\mathcal{H}(L(p,1),\Sigma)) \cong \pi_1(S^3)$ *for* $i \geq 2$.

3. If $q \geq 2$, $\pi_i(\mathcal{H}(L(p,q),\Sigma)) \cong 0$ *for* $i \geq 2$.

On the other hand, $\pi_1(\mathcal{H}(L(p,q),\Sigma))$ remains unknown. However, by [18] we have a short exact sequence

$$1 \to \pi_1(\mathrm{Diff}(M)) \to \pi_1(\mathcal{H}(M,\Sigma)) \to G(M,\Sigma) \to 1,$$

where $G(M,\Sigma)$ is the kernel of the natural homomorphism $\mathcal{MCG}(M,\Sigma) \to \mathcal{MCG}(M)$. We remark that, in general, the group $\pi_1(\mathrm{Diff}(M))$ is not finitely-generated for a reducible 3-manifold M. In our case, due to the Smale the Smale Conjecture for the elliptic 3-manifolds by [14], $\mathrm{Diff}(L(p,q))$ is homotopy equivalent to the isometry groups of $L(p,q)$, which implies that $\pi_1(\mathrm{Diff}(L(p,q)))$ is finitely presented. Recalling the mapping class groups of lens spaces are finite by [2], we see that the Goeritz group $\mathcal{MCG}_+(L(p,q))$ is virtually isomorphic to the group $G(M,\Sigma)$. In particular, $\mathcal{MCG}_+(L(p,q))$ is finitely presented if and only if so is $G(M,\Sigma)$. Hence by Theorem 3.10, the following holds:

Corollary 5.2. *For the genus-2 Heegaard splitting* $L(p,q) = V \cup_\Sigma W$ *of a lens space* $L(p,q)$*, where* $p \equiv \pm 1 \pmod q$ *and* $1 \leq q \leq p/2$, $\pi_1(\mathcal{H}(L(p,q),\Sigma))$ *is finitely presented.*

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