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The Diamond Lemma and prime decompositions

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In 1942 M. H. A. Newman proved a simple lemma (see [3]) which turned to be useful for different branches of mathematics such as algebra and mathematical analyses. In this paper I describe a new version of this lemma suitable for topological applications.

Let Γ be an oriented graph. We say that a vertex B of Γ is a *root* of a vertex A of Γ if there is an oriented path in Γ from A to B, and B has no outgoing edges. A vertex of Γ may have one root, several roots, or no roots at all. The following question seems to be reasonable.

Question: *under what conditions each vertex of* Γ *has exactly one root?*

In order to answer this question let us formulate two properties of Γ.

1. Finiteness property (FP): *Any oriented path in* Γ *has finite length.* In other words, Γ does not contain oriented cycles and infinite oriented paths. This property implies that any vertex of Γ has at least one root. By an *angle* we mean a pair of edges of Γ emanating from the same vertex. The set of all angles of Γ we denote by $A(\Gamma)$. Let N be the set of all natural numbers.

Definition. *A map* $\mu: A(\Gamma) \to N \cup \{0\}$ *is called an angle measure if it possesses the following property* **(AM):**

1. If $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) = 0$ then there exists a vertex C of Γ such that Γ contains edges $(\overrightarrow{B_1C})$ and $(\overrightarrow{B_2C})$.
2. If $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) > 0$ then there is an edge $\overrightarrow{AC}$ such that for $i = 1, 2$ we have $\mu(\overrightarrow{AB_i}, \overrightarrow{AC}) < \mu(\overrightarrow{AB_1}, \overrightarrow{AB_2})$. We will call $\overrightarrow{AC}$ *a mediator edge.*

The Diamond Lemma. *Suppose* Γ *has properties* (FP) *and* (AM). *Then any vertex of* Γ *has a unique root.*

Outlines of the proof.

Let us call a vertex X of Γ *regular* if it has only one root. Arguing by contradiction assume that Lemma 1 is false. Then there is a singular vertex X such that the endpoints of all outgoing edges are regular. Since X is singular, there is a pair of outgoing edges such that their endpoints have different roots. Among all such pairs we take a pair having minimal angle measure μ. There are two cases: $\mu = 0$ and $\mu > 0$. In the first case we get a contradiction with item (1) of property (AM), in the second one we get a contradiction with item (2) of (AM). □

This lemma turned to be useful for proving or disproving prime decomposition theorems of topological objects. To give an example we describe a new simple proof of the famous Kneser-Milnor prime decomposition theorem for orientable 3-manifold.

Theorem. *Any connected orientable 3-manifold can be decomposed into connected sum of prime factors, and these factors are unique up to permutation.*

Proof. Let us construct a graph Γ whose vertices are finite sets of 3-manifolds. Two vertices A, B are joined by an oriented edge $\overrightarrow{AB}$ if B is obtained from A by replacing a manifold $X \in A$ by manifolds $X_1, X_2 \in B$ such that X is a connected sum of X_1 and X_2. Note that X_1, X_2 are obtained from X by so-called *spherical reduction*, i.e. cutting X along a nontrivial sphere S and filling by balls two spheres in the boundary of the resulting manifolds. It is convenient to think that $\overrightarrow{AB}$ is determined by S.

We claim that Γ possesses properties (FP) and (AM). Property (FP) follows from the famous Kneser Lemma [1]. Let us define a map $\mu: A(\Gamma) \to N \cup \{0\}$ by setting $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2})$ to be the minimal number of curves in the general position intersection of spheres S_1 and S_2 which determine $\overrightarrow{AB_1}$ and $\overrightarrow{AB_2}$. The proof that this map is an angle measure, i.e. it has property (AM) is based on the standard technics of removing intersection of surfaces.

Case 1. Suppose that $\mu(\overrightarrow{AB_1}, \overrightarrow{AB_2}) = 0$. Then the corresponding reducing spheres S_1 and S_2 are disjoint and thus each of them survives the reduction along the other. It follows that $S_1 \subset B_2$ and $S_2 \subset B_1$. By reducing B_1 along S_2 and B_2 along B_1 we obtain the same vertex C. This proves item (1) of property (AM).

Case 2. Suppose that $\mu_0 = \mu(\overrightarrow{AB_1}, \overrightarrow{AB_2})$ is positive. Then those spheres lie in the same connected manifold $Q \in A$. Among the circles in $S_1 \cap S_2$ we choose one, denoted by c, which is innermost with respect to S_1. This means that c bounds a disk D in S_1 such that $D \cap S_2 = c$. We cut S_2 along c and glue up the boundaries of the cut by two parallel copies of D. Applying a small perturbation we obtain two new spheres S' and S'' whose intersection with S_2 is empty and whose intersection with S_1 consists of a smaller number of circles (since c disappeared), see Fig. 1. At least one of these two spheres (say S') must be non-trivial in Q, since otherwise S_2 would be trivial. Therefore, reduction along S' determines a mediator edge. It follows that Γ possess property (AM). Applying the Diamond Lemma we conclude that any vertex of Γ has a unique root, which means that any oriented 3-manifold has a unique decomposition into prime factors. □

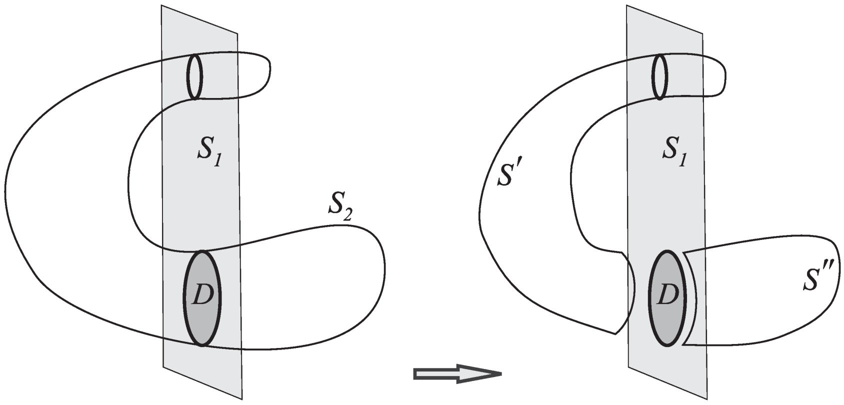

図 1: Fig.1: Surgery along an innermost circle.

The same strategy works for proving many other prime decomposition problems as well as for finding counterexamples. First we construct a graph whose vertices are collections of topological objects we are interested in, and whose edges are reductions along appropriate surfaces. As a rule, property (FP) is easy while property (AM) requires developing a new or adjusting a known technics for removing intersections of surfaces used for reductions. Let me list several results obtained by this strategy (see [2] for details).

1. The Kneser-Milnor prime decomposition theorem (new proof).
2. The Swarup theorem for boundary connected sums (new proof).
3. A spherical splitting theorem for knotted graphs in 3-manifolds (Joint work with C. Hog-Angeloni);
4. Counterexamples to prime decomposition theorems for knots in 3-manifolds and for 3-orbifolds.
5. A new theorem on annular splittings of 3-manifolds, which is independent of the JSJ-decomposition theorem.
6. An existence and uniqueness theorem for prime decompositions of knots in products of surfaces and intervals.
7. A theorem on the exact structure of the semigroup of theta-curves in 3-manifolds (joint work with V. Turaev).
8. Prime decompositions theorem for virtual knots.

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References

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