数理解析研究所講究録
第 1960 巻 2015 年 93-100

Ordering of groups as a tool to understand random 3-manifolds and knots

Tetsuya Ito
Research Institute for Mathematical Sciences, Kyoto University

1 Introduction

This is a slightly expanded version of the paper [Ito3], where we observed various properties of random open books and closed braids. In this article we add more explanations on the background materials and some new results.

The question we address in this paper is the following.

Question 1. What property does a random 3-manifolds and links have ?

Of course, to answer the question we need to clarify the meaning of the "random 3-manifolds and links". In this paper, as a model of random 3-manifolds and links, we use random open books and random closed braids.

Let G be the mapping class group or the braid group of an oriented compact surface S with connected boundary. Throughout the paper we assume that ∂S is connected, for a sake of simplicity. All results, expect the results concerning taut foliations and tight contact structures, can be generalized for the case ∂S is not connected, with appropriate modifications.

An *open book* is a pair (S,ϕ) consisting of a surface S and an element of the mapping class group $\phi \in MCG(S)$. The *open book manifold* $M_{(S,\phi)}$ is a 3-manifold defined by

$$M_{(S,\phi)} = M_\phi \cup (D^2 \times S^1)$$

where $M_\phi = M \times [0,1] \,/\, (x,1) \sim (\phi(x),0)$ is the mapping torus of ϕ and the solid torus $D^2 \times S^1$ is glued along $\partial M_\phi = S^1 \times \partial S$ so that the circle $S^1 \times \{\text{a point on } \partial S\}$ bounds the disc in $D^2 \times S^1$. An n-braid $\beta \in B_n(S)$ of the surface S is represented as strings in $S \times [0,1]$. By taking its image under the map $S \times [0,1] \to M_\phi \subset M_{(S,\phi)}$, one obtains an oriented link in the open book manifold $M_{(S,\phi)}$. We call this link the *closure of* β and denote by $\widehat{\beta}$.

Let μ be a probability measure on G with finite support. We denote the sub semi-group of G generated by the support of μ (the set of elements of G with $\mu(\{g\}) \neq 0$) by H_μ. Consider the simple random walk with respect to μ starting from the identity: The transition probability $p(x, y)$, the probability that a point $x \in G$ at the time k moves to a point $y \in G$ at the time $k+1$ is given by $p(x, y) = \mu(yx^{-1})$. We denote the random variable representing the position of a point at the time k by g_k.

Example 2. Here is the simplest, but crucial example of random walk. let $G = \mathbb{Z}$ be the infinite cyclic group, and consider the probability measure μ given by $\mu(\{\pm 1\}) = \frac{1}{2}$. In this case, the probability that at the time k the point lies on i is given by

$$P(g_k = i) = \frac{1}{2^k}\binom{k}{\frac{1}{2}(k + |i|)}$$

so in particular, for large k, the probability that $g_k = 0$ is asymptotically given by

$$P(g_{2k} = 0) = \frac{1}{2^{2k}}\binom{2k}{k} \underset{k\to\infty}{\sim} C\frac{1}{\sqrt{k}}$$

where C is a constant which is not important here. This shows that the probability that g_k lies in the bounded interval $[-M, M]$ goes to zero as $k \to \infty$. One can see that this is true for more general probability measure μ so schematically saying, a random integer is unbounded, as we naively expect.

A random walk $\{g_k\}$ can be regarded as a process of generating a random element of G, hence by taking an open book or a closed braid one obtains a random (contact) 3-manifold or a random oriented link in a 3-manifold. We will see that by using the fractional Dehn twist coefficient (FDTC), which is related to a left-ordering of G, one can easily show various non-trivial properties of random open books and closed braids.

2 Background material I: Quasi-morphism

Definition 3. A map $\phi : G \to \mathbb{R}$ is a *quasimorphism* if

$$D_\phi = \sup\{g, h \in G \mid |\phi(gh) - \phi(g) - \phi(h)|\} \leq \infty.$$

The constant D_ϕ is called the *defect* of ϕ.

We say that a probability distribution μ is *unbounded* with respect to a quasi-morphism ϕ if $\phi(H_\mu)$ is unbounded.

Note that if ϕ is a homomoprhism, then the asymptotic behavior of $\phi(g_k)$ can be described by a random walk on $\mathbb{Z}$ (or, $\mathbb{R}$) hence as Example 2 shows, in such case

$P(|\phi(g_k)| \leq M) \underset{k\to\infty}{\sim} C\frac{1}{\sqrt{k}}$ for some constant C. Since a quasi-morphism can be seen as a homomorphism with bounded error, one may expect that this "bounded error" does not affect the asymptotic behavior. This is true as the next theorem shows.

Theorem 4 (Malyutin [Mal]). *For a non-trivial quasi-morphism $\phi : G \to \mathbb{R}$ and constant $M > 0$,*

$$P(|\phi(g_k)| \leq M) \underset{k\to\infty}{\sim} C\frac{1}{\sqrt{k}}$$

for some constant C. In particular, $P(|\phi(g_k)| \leq M)) \to 0$ $(k \to \infty)$.

3 Background material II: Nielsen-Thurston orderings and the Fractional Dehn twist coefficient

For the mapping class group or the braid group of a surface S, there is a particularly important quasi-morphism, called the *Fractional Dehn twist coefficient* (*FDTC*, in short). Here we briefly review the definition of FDTC following the formulation in [IK].

Let $\pi : \widetilde{S} \to S$ be the universal covering. Take a basepoint $* \in \partial S$, and take one of its lift $\widetilde{*} \in \pi^{-1}(*) \subset \pi^{-1}(\partial S)$. We denote by $\widetilde{C}$ the connected component of $\pi^{-1}(\partial S)$ that contains $\widetilde{*}$. By equipping an hyperbolic metric on S, $\widetilde{S}$ can be isometrically embedded into the hyperbolic plane $\mathbb{H}^2$. We compactify $\widetilde{S}$ as a topological disk $\overline{S}$ by attaching the points at infinity.

For a homeomorphism of $\phi : S \to S$ which fixes ∂S pointwise, Take a lift $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ so that $\widetilde{\phi}(\widetilde{*}) = \widetilde{*}$. Then $\widetilde{\phi}$ extends to the homeomorphism of $\overline{\phi} : \overline{S} \to \overline{S}$. A crucial point is that two homeomorphisms ϕ and ψ are isotopic if and only if the action of their lifts on the boundary $\partial\overline{S}$ are the same. Thus, by identifying $\partial\overline{S} - \widetilde{C}$ with the real line $\mathbb{R}$ we get an injective homeomorphism

$$\Theta : MCG(S) \to \mathrm{Homeo}^+(\mathbb{R})$$

which we call the *Nielsen-Thurston map*.

The Nielsen-Thurston map introduces a left-ordering on $MCG(S)$.

Definition 5 (Nielsen-Thurston ordering [SW]). Take a point $x \in \partial\overline{S} - \widetilde{C} \cong \mathbb{R}$. For $g, h \in MCG(S)$, we define the ordering relation $<_x$ by

$$g <_x h \iff [\Theta(g)](x) <_{\mathbb{R}} [\Theta(h)](x).$$

Here $<_{\mathbb{R}}$ denotes the standard ordering of $\mathbb{R}$. It is known that for generic x, $<_x$ is a left ordering, namely, a total ordering invariant under the left action of $MCG(S)$ itself.

Nielsen-Thurston orderings are quite interesting objects. For example, the Dehornoy ordering, the standard left-ordering of the braid group having rich combinatorial structure [DDRW], is a special one of the Nielsen-Thurston ordering [SW].

After suitable normalization, the Nielsen-Thurston map produces a quasi-morphism which is extremely useful and plays a crucial role in 3-dimensional contact geometry. We normalize the identification $\partial\overline{S} - \widetilde{C} \cong \mathbb{R}$ so that $\Theta(T_{\partial S})$, the action of the Dehn twist along the boundary is the translation map $x \mapsto x+1$. Since $T_{\partial S}$ is a central element of $MCG(S)$, under this normalization, the Nielsen-Thurston action is an injection to smaller subgroup of $\mathrm{Homeo}^+(\mathbb{R})$,

$$\Theta : MCG(S) \to \widetilde{\mathrm{Homeo}}^+(S^1).$$

Here $\widetilde{\mathrm{Homeo}}^+(S^1)$ is a subgroup of $\mathrm{Homeo}^+(\mathbb{R})$ consisting of a lift of an orientation preserving homeomorphism of S^1.

Definition 6 (Fractional Dehn twist coefficient). The Fractional Dehn twist coefficient (FDTC) is the map

$$\mathrm{FDTC} = \tau \circ \Theta : MCG(S) \to \mathbb{R}$$

where $\tau : \widetilde{\mathrm{Homeo}}^+(S^1) \to \mathbb{R}$ is the translation number $\tau(f) = \lim_{n\to\infty} \frac{f^n(0)}{n} \in \mathbb{R}$.

Since the translation map is a quasi-morphism, so is the FDTC map. As the definition shows, the FDTC can be regarded as a numerical approximation of Nielsen-Thurston orderings. In fact, by using Nielsen-Thruston orderings one can compute the value of FDTC.

Remark 7.

1. The first definition of the FDTC in [HKM1] is based on the Nielsen-Thurston classification, the dynamics of surface automorphisms.
2. Although translation number can be irrational in general, the image of FDTC map is always rational.
3. The FDTC plays a fundamental role in contact geometry. For example, the open book (S, ϕ) supports an overtwisted contact structure if $\mathrm{FDTC}(\phi) < 0$.
4. The (normalized) Nielsen-Thurston map Θ is far from unique: in the construction we have various choices, like a hyperbolic metric or an identification $\partial\overline{S} - \widetilde{C} \cong \mathbb{R}$ that affects the resulting Nielsen-Thurston map. On the other hand, the FDTC map is uniquely determined and indepedent of the various choices involved in the construction of Θ.

Why we considser FDTC ? The answer is simple:

Key principle. For $\phi \in MCG(S)$, if its (absolute value of) FDTC is sufficiently large, then the corresponding (contact) 3-manifold $M_{(S,\phi)}$ has various nice properties.

This Key principle, combining with Theorem 4 says that:

Consequence. A random open book (S, ϕ) has large $|\,\mathrm{FDTC}\,|$ (large with respect to Nielsen-Thruston ordering) so a random 3-manifold $M_{(S,\phi)}$ has various nice properties.

4 Conclusions: Properties of random open books and closed braids

Now we are ready to present various properties of random open books and closed braids. First of all, we recall that a random element of the mapping class group is pseudo-Anosov.

Theorem 8. *[Mah],[Mal, Corollary 0.6]. Let us fix an element $\phi \in G$. If the probability measure μ is non-elementary, that is, H_μ contains pseudo-Anosov elements with distinct fixed points on the Thurston boundary of the Teichmüller space, then the probability that $g_k\phi$ is pseudo-Anosov goes to one as $k \to \infty$.*

From now on, we will always assume that the probability measure μ is chosen so that it is non-elementary and unbounded (with respect to FDTC).

The first result justifies our naive expectation for "generic" 3-manifolds – one can expect a random 3-manifold admits various nice structures.

Theorem 9. *Let us fix $\phi \in G$. As $k \to \infty$, the probability that an open book $(S, g_k\phi)$ has the following properties goes to one.*

(a) $M_{(S,g_k\phi)}$ is hyperbolic. (In particular, $M_{(S,g_k\phi)}$ is irreducible and atoroidal.)

(b) For a fixed $C > 0$, $M_{(S,g_k\phi)}$ contains no incompressible surface of genus less than C.

(c) Either $(S, g_k\phi)$ or $(S, (g_k\phi)^{-1})$ supports a weakly symplectically fillable and universally tight contact structure, which is a perturbation of a co-oriented taut foliation. (In particular, $M_{(S,g_k\phi)}$ admits a co-oriented taut foliation).

(d) $M_{(S,g_k\phi)}$ is not a Heegaard-Floer L-space.

Proof. (a) follows from [IK, Theorem 8.3]: $M_{(S,g\phi)}$ is hyperbolic if $g\phi$ is pseudo-Anosov with $|\mathrm{FDTC}(g\phi)| > 1$. (b) follows from [IK, Theorem 7.2]: an existence of incompressible surface of genus $C > 0$ implies $|\mathrm{FDTC}(g\phi)| \leq C$. (c) follows from [HKM2, Theorem 1.2]: $(S, g\phi)$ supports a desired contact structure if $g\phi$ is pseudo-Anosov with $\mathrm{FDTC}(g\phi) > 1$.

(c) and [OS, Theorem 1.4], that asserts that an L-space does not admit a co-oriented taut foliation, prove (d). □

Note that when we take ϕ and μ so that $M_{(S,\phi)}$ is an integral homology sphere and that $\mathrm{supp}(\mu)$ is contained in the Torelli group, then $M_{(S,\phi g_k)}$ is always an integral homology sphere so we get a notion of random integral homology sphere. The fundamental group of a atroidal integral homology sphere M is left-orderable if M admits a co-oriented taut foliation [CD], hence we get the following.

Corollary 1. *A random integral homology sphere M has the following properties*

1. *M is not a Heegaard Floer L-space.*
2. *M admits a co-oriented taut foliation.*
3. *$\pi_1(M)$ is left-orderable.*

This gives a supporting evidence for L-space conjecture [BGW], that asserts the three properties in the corollary are equivalent for all rational homology 3-sphere.

Next we study a random link in a fixed 3-manifold. Fix a 3-manifold M and its open book decomposition (S,ϕ). We regard an n-braid $\beta \in B_n(S)$ and the monodromy ϕ as an element of $MCG(S - \{n \text{ points}\})$ and consider their product $\beta\phi$. We define the FDTC of a (closed) braid $\widehat{\beta}$ as the FDTC of $\beta\phi$, viewed as an element of $MCG(S - \{n \text{ points}\})$ (See [IK, Section 4] for details).

The first part of the next result generalizes [Ma].

Theorem 10. *As $k \to \infty$, the probability that $\widehat{\beta}_k$, the closure of a random braid β_k, is a hyperbolic link in $M_{(S,\phi)}$ goes to one as $k \to \infty$. Moreover, if $\widehat{\beta}_k$ is null-homologous (for example, when $M_{(S,\phi)}$ is an integral homology sphere), then for any fixed constant $C > 0$, the probability that $g(\widehat{\beta}_k) \leq C$ goes to zero as $k \to \infty$. Here $g(\widehat{\beta}_k)$ denotes the genus of $\widehat{\beta}_k$.*

Proof. This follows from [IK, Theorem 8.4, Corollary 7.13]: $\widehat{\beta}_k$ is hyperbolic if $\beta_k\phi$ is pseudo-Anosov with $|\mathrm{FDTC}(\beta_k\phi)| > 1$, and that $|\mathrm{FDTC}(\beta_k\phi)|$ gives an lower bound of $g(\widehat{\beta}_k)$. □

We analyse more precise structures of a random classical closed braid in S^3.

Theorem 11. *The probability that two random braids $\alpha_k, \beta_l \in B_n$ are non-conjugate but represent the same link goes to zero as $k, l \to \infty$.*

Proof. This follows from [Ito, Theorem 2.8], based on a deep result of Birman-Menasco [BM]: There is a constant $r(n)$ such that for n-braids α, β with $|\mathrm{FDTC}| > r(n)$ the closures of α and β are the same if and only if they are conjugate. □

Note that this also says that the closures of two random braids are *transverse* isotopic if they are *topologically* isotopic. Thus, a random closed braid model of random transverse links are the same as a random closed braid model of random topological links.

We also address a question concerning the transient properties. For $g \in \mathbb{Z}_{\geq 0}$, let $S(n,g)$ be the subset of the braid group B_n consisting of a braid whose closure represents a link of genus $\leq g$. The following result was conjectured in [Mal].

Theorem 12. *$S(n,g)$ is transient for the random walk $\{g_k\}$ on B_n.*

Proof. In the proof of [Ito2, Theorem 1.2], it is shown that for $\beta \in B_n$, if $g(\widehat{\beta}) \leq g$ then β is conjugate to a braid represented by a word W over the standard generator $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$ such that the number of $\sigma_1^{\pm 1}$ in W is at most $2g$. This shows that such a braid β is written as a product of at most $4g$ reducible braids. Let $T_n \subset B_n$ be the set of all non pseudo-Anosov n-braids. Then $S(n,g) \subset T_n^{4g}$. By [Mal, Corollary 0.7], T_n^{4g} is transient for the random walk $\{g_k\}$ hence so is $S(n,g)$. □

Finally, we give another application of quasi-morphism technique.

Theorem 13. *As $k \to \infty$, the probability that $\widehat{\alpha_k}$ is an alternating link goes to zero. Similarly, the probability that $\widehat{\alpha_k}$ is slice goes to zero.*

Proof. It is known that the signature σ, the Rasmussen s-invariant and their difference $[\sigma - s]$ yield a non-trivial quasi-morphism of the braid group [Bra]. Since for an alternating knot the signature and the Rasmussen s-invariant is equal. Hence by Theorem 4,

$$P(\widehat{\alpha_k} \text{ is alternating}) \leq P([\sigma - s](\widehat{\alpha_k}) = 0) \to 0 \quad (k \to \infty).$$

The latter assertion follows from the fact that signature is zero if $\widehat{\alpha_k}$ is slice. □

References

[BM] J. Birman and W. Menasco, *Stabilization in the braid groups. I. MTWS.* Geom. Topol. **10** (2006), 413–540.

[BGW] S. Boyer, C. Gordon and L. Watson, *On L-spaces and left-orderable fundamental groups,* Math. Ann. (2013) **356** 1213–1245.

[CD] D. Calegari and N. Dunfield, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. **152** (2003), 149–204.

[Bra] M. Brandenbursky, *On quasi-morphisms from knot and braid invariants,* J. Knot Theory Ramifications **20** (2011), 1397–1417.

[DDRW] P. Dehornoy, I.Dynnikov, D.Rolfsen and B.Wiest, *Ordering Braids,* Mathematical Surveys and Monographs **148**, Amer. Math. Soc. 2008.

[HKM1] K. Honda, W. Kazez, and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary,* Invent. math. **169**, (2007), 427-449.

[HKM2] K. Honda, W. Kazez, and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary II,* Geom. Topol. 12 (2008), no. 4, 2057-2094.

[Ito] T. Ito, *Braid ordering and the geometry of closed braid,* Geom. Topol. **15** (2011), 473–498.

[Ito2] T. Ito, *Braid ordering and knot genus,* J. Knot Theory Ramifications, **20** (2011), 1311–1323.

[Ito3] T. Ito, *On a structure of random open books and closed braids,* arXiv:1504.04446.

[IK] T. Ito, and K. Kawamuro, *Essential open book foliation and fractional Dehn twist coefficient*, arXiv:1208.1559.

[Ma] J. Ma, *The closure of a random braid is a hyperbolic link*, Proc. Amer. Math. Soc **142** (2014), 695–701.

[Mah] J. Maher, *Random walks on the mapping class group*, Duke Math. J. **156** (2011), 429–468.

[Mal] A. Malyutin, *Quasimorphisms, random walks, and transient subsets in countable groups*, J. Math. Sci. (N.Y). **181** (2012), 871–885.

[OS] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds.* Geom. Topol. 8 (2004), 311–334.

[SW] H. Short and B. Wiest, *Ordering of mapping class groups after Thurston,* Enseign. Math. **46**, (2000), 279-312.

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
JAPAN
E-mail address: `tetitoh@kurims.kyoto-u.ac.jp`

京都大学・数理解析研究所　伊藤哲也