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Rigorous construction of time-ordered exponential operators and its applications to quantum field theory

Kouta Usui
Department of Mathematics, Hokkaido University

Abstract

The time-ordered exponential representation of a complex time evolution operator in the interaction picture is studied. Using the complex time evolution, we prove the Gell-Mann – Low formula under certain abstract conditions, in mathematically rigorous manner.

1 Introduction

In this article, we consider a formula in quantum field theories of the type

$$\begin{aligned}&\langle \Omega, T\{\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n)\}\Omega\rangle \\ &\quad = \lim_{t\to\infty} \frac{\left\langle \Omega_0, T\{\phi_{\mathrm{I}}^{(1)}(x_1)\cdots\phi_{\mathrm{I}}^{(n)}(x_n)\exp\left(-i\int_{-t}^{t} d\tau H_1(\tau)\right)\}\Omega_0\right\rangle}{\left\langle \Omega_0, T\exp\left(-i\int_{-t}^{t} d\tau H_1(\tau)\right)\Omega_0\right\rangle}, \qquad (1.1)\end{aligned}$$

called the *Gell-Mann – Low formula* [1]. The meaning of each symbol in the formula (1.1) is as follows: the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product of a Hilbert space of quantum state vectors, $\phi^{(k)}(x_k)$ and $\phi_{\mathrm{I}}^{(k)}(x_k)$ ($k=1,...,n,\ x_k\in\mathbb{R}^4$) denote field operators in the Heisenberg and the interaction picture, respectively. For instance, in quantum electrodynamics (QED), each $\phi^{(k)}$ denotes the Dirac field ψ_l, its conjugate $\psi_l^\dagger$, or the gauge field A_μ. The symbol T denotes the time-ordering and Ω and Ω_0 the vacuum states of the interacting and the free theory, respectively. The operator

$$T\exp\left(-i\int_{-t}^{t} d\tau H_1(\tau)\right)$$

is the time evolution operator in the interaction picture, having the following series expansion:

$$\begin{aligned}&T\exp\left(-i\int_{-t}^{t} d\tau H_1(\tau)\right) \\ &\quad = 1 + (-i)\int_{-t}^{t} d\tau_1\, H_1(\tau_1) + (-i)^2\int_{-t}^{t} d\tau_1\int_{-t}^{\tau_1} d\tau_2\, H_1(\tau_1)H_1(\tau_2) + \cdots, \qquad (1.2)\end{aligned}$$

which is often called the *time-ordered exponential* or the *Dyson series* for

$$H_1(\tau) := e^{i\tau H_0} H_1 e^{-i\tau H_0} \ (\tau \in \mathbb{R}),$$

where H_0 and H_1 are the free and the interaction Hamiltonians.

This formula is a fundamental tool to generate a perturbative expansion of the *n-point correlation function*

$$\langle \Omega, T\{\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n)\}\Omega\rangle$$

with respect to the coupling constant. When the coupling is small enough (for QED, this seems valid), the first few terms of the perturbation series is expected to be a good approximation of the correlation function which gives quantitative predictions for observable variables such as scattering cross section. In QED, these predictions agree with experimental results to eight significant figures, the most accurate predictions in all of natural science. However, the mathematical proof of (1.1) is far from trivial and the derivations given in physics literatures are very heuristic and informal. The purpose of the present article is to construct a mathematically rigorous setup in which the Gell-Mann – Low formula (1.1) is adequately formulated and proved. We remark that the abstract results obtained here can be applied to the mathematical model of QED with cut-offs, which has been discussed in Ref [2].

In the original heuristic derivation of (1.1), Murray Gell-Mann and Francis Low [1] introduced *adiabatic switching* of the interaction through the time-dependent Hamiltonian of the form $H_0 + e^{-\varepsilon|t|}H_1$, where $\varepsilon > 0$ is the small parameter which eventually vanishes. We take an alternative way by sending the time t to ∞ in the imaginary direction: $t \to \infty(1 - i\varepsilon)$. The same method can be found in physics literatures (see, for example, [3, 4]). In this case, one difficulty with the mathematical proof of (1.1) is to construct the complex time evolution which possesses the following series expansion:

$$\begin{aligned} &T\exp\left(-i\int_{z'}^{z} d\zeta H_1(\zeta)\right) \\ &= 1 + (-i)\int_{z'}^{z} d\zeta_1\, H_1(\zeta_1) + (-i)^2\int_{z'}^{z} d\zeta_1 \int_{z'}^{\zeta_1} d\zeta_2\, H_1(\zeta_1)H_1(\zeta_2) + \cdots, \end{aligned} \tag{1.3}$$

($z, z' \in \mathbb{C}$) for *unbounded* H_1. We extend the methods obtained in [5] to "complex time".

In Section 2, we develop an abstract theory of complex time-ordered exponential. In Section 3, we state and prove the Gell-Mann – Low formula in an abstract form under some assumptions.

2 Abstract construction of time-ordered exponential on the complex plane and its properties

Let $\mathcal{H}$ be a complex Hilbert space. The inner product and the norm of $\mathcal{H}$ are denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ (anti-linear in the first variable) and $\|\cdot\|_{\mathcal{H}}$ respectively. When there can be no danger of confusion, then the subscript $\mathcal{H}$ in $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ is omitted. For a linear operator T in $\mathcal{H}$, we denote its domain (resp. range) by $D(T)$ (resp. $R(T)$). We also denote the adjoint of T by T^* and the closure by $\bar{T}$ if these exist. For a self-adjoint operator T, $E_T(\cdot)$ denotes the spectral measure of T. The symbol $T|_D$ denotes the restriction of a linear operator T to the subspace D. For a linear operators S and T on a Hilbert space, $D(S+T) := D(S)\cap D(T)$, $D(ST) := \{\Psi \in D(T) \,|\, T\Psi \in D(S)\}$ unless otherwise stated.

We begin by defining a time-ordered product of operator-valued functions and the time-ordered exponential of an operator-valued function in an unambiguous way. Let $z, z' \in \mathbb{C}$ and Γ be a piecewisely continuously differentiable simple curve in $\mathbb{C}$ from z' to z. That is, Γ is a map from a closed interval $I = [\alpha, \beta]$ in $\mathbb{R}$ into $\mathbb{C}$, which is piecewisely continuously differentiable and injective, satisfying

$$\Gamma(\alpha) = z', \quad \Gamma(\beta) = z. \tag{2.1}$$

We define a linear order $\succ$ on $\Gamma(I) = \{\Gamma(t) \mid t \in I\} \subset \mathbb{C}$ as follows. For $\zeta_1, \zeta_2 \in \Gamma(I)$, there exist $t_1, t_2 \in I$ with $\Gamma(t_1) = \zeta_1$ and $\Gamma(t_2) = \zeta_2$. Then, $\zeta_1 \succ \zeta_2$ if and only if $t_1 > t_2$.

In what follows, we denote $\Gamma(I)$ simply by Γ. Let $\mathfrak{S}_n$ be the symmetric group of order $n \in \mathbb{N}$ and $L(\mathcal{H})$ be (not necessarily bounded) linear operators in $\mathcal{H}$. For mappings $F_1, F_2, \ldots, F_k$ ($k \in \mathbb{N}$) from Γ into $L(\mathcal{H})$, we define a map $T[F_1 \ldots F_k]$ from Γ^k into $L(\mathcal{H})$ by

$$\begin{aligned} &D\left(T[F_1 \ldots F_k](\zeta_1, \ldots, \zeta_k)\right) \\ &\quad := \bigcap_{\sigma\in\mathfrak{S}_k} \bigcap_{(\zeta_1,\ldots,\zeta_k)\in\Gamma^k} D\left(F_{\sigma(1)}(\zeta_{\sigma(1)}) \ldots F_{\sigma(k)}(\zeta_{\sigma(k)})\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &T[F_1 \ldots F_k](\zeta_1, \ldots, \zeta_k)\Psi \\ &\quad := \sum_{\sigma\in\mathfrak{S}_k} \chi_{P_\sigma}(\zeta_1, \ldots, \zeta_k) F_{\sigma(1)}(\zeta_{\sigma(1)}) \ldots F_{\sigma(k)}(\zeta_{\sigma(k)})\Psi, \end{aligned} \tag{2.3}$$

for $\Psi \in D\left(T[F_1 \ldots F_k](\zeta_1, \ldots, \zeta_k)\right)$, where χ_J denotes the characteristic function of the set J, and

$$P_\sigma = \{(\zeta_1, \ldots, \zeta_k) \in \Gamma^k \,|\, \zeta_{\sigma(1)} \succ \cdots \succ \zeta_{\sigma(k)}\}, \quad \sigma \in \mathfrak{S}_k. \tag{2.4}$$

In what follows, we sometimes adopt a little bit confusing notation

$$T\left(F_1(\zeta_1) \ldots F_k(\zeta_k)\right) := T[F_1 \ldots F_k](\zeta_1, \ldots, \zeta_k), \tag{2.5}$$

and call it the *time-ordered product* of $F_1(\zeta_1), \ldots, F_k(\zeta_k)$ along the curve Γ, even though the operation T does *not* act on the product of operators $F_1(\zeta_1), \ldots, F_k(\zeta_k)$ but on the product of mappings $F_1, \ldots, F_k$.

Next, we define a concept of time-ordered exponential of an operator-valued function. Let $F : \Gamma \to L(\mathcal{H})$ and let $C(F) \subset \mathcal{H}$ be a subspace spanned by all the vectors $\Psi \in \mathcal{H}$ such that the mapping

$$(\zeta_1, \ldots, \zeta_n) \mapsto F(\zeta_1) \ldots F(\zeta_n)\Psi \tag{2.6}$$

is strongly continuous in the variables $(\zeta_1, \dots, \zeta_n) \in \Gamma^n$. We define a time-ordered exponential operator by

$$D\left(T\exp\left(\int_\Gamma d\zeta\, F(\zeta)\right)\right)$$
$$:= \left\{\Psi \in C(F) \,\middle|\, \sum_{n=0}^{\infty} \frac{1}{n!} \left\| \int_{\Gamma^n} d\zeta_1 \dots d\zeta_n\, T\left(F(\zeta_1)\dots F(\zeta_n)\right)\Psi \right\| < \infty \right\}, \tag{2.7}$$
$$T\exp\left(\int_\Gamma d\zeta\, F(\zeta)\right)\Psi := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma^n} d\zeta_1 \dots d\zeta_n\, T\left(F(\zeta_1)\dots F(\zeta_n)\right)\Psi, \tag{2.8}$$

where the integration is understood in the strong sense.

We also define a more general time-ordered exponential operator. Let $F_1, F_2, \dots, F_k, \dots, F_{k+n}$ be the mappings from Γ into $L(\mathcal{H})$. We define a map from Γ^n into $L(\mathcal{H})$, which is labeled by $(\zeta_1, \dots, \zeta_k) \in \Gamma^k$,

$$T[F_1(\zeta_1)F_2(\zeta_2)\dots F_k(\zeta_k)F_{k+1}\dots F_{k+n}] : \Gamma^n \to L(\mathcal{H}) \tag{2.9}$$

by the relations

$$D\left(T[F_1(\zeta_1)F_2(\zeta_2)\dots F_k(\zeta_k)F_{k+1}\dots F_{k+n}](\zeta_{k+1}, \dots, \zeta_{k+n})\right)$$
$$:= \bigcap_{\sigma\in\mathfrak{S}_{k+n}} \bigcap_{(\zeta_{k+1},\dots,\zeta_{k+n})\in\Gamma^n} D(F_{\sigma(1)}(\zeta_{\sigma(1)})\dots F_{\sigma(k+n)}(\zeta_{\sigma(k+n)})), \tag{2.10}$$
$$T[F_1(\zeta_1)F_2(\zeta_2)\dots F_k(\zeta_k)F_{k+1}\dots F_{k+n}](\zeta_{k+1}, \dots, \zeta_{k+n})\Psi$$
$$:= \sum_{\sigma\in\mathfrak{S}_{k+n}} \chi_{P'_{n,\sigma}}(\zeta_{k+1}, \dots, \zeta_{k+n}) F_{\sigma(1)}(\zeta_{\sigma(1)})\dots F_{\sigma(k+n)}(\zeta_{\sigma(k+n)})\Psi, \tag{2.11}$$

for $\Psi \in D\left(T[F_1(\zeta_1)F_2(\zeta_2)\dots F_k(\zeta_k)F_{k+1}\dots F_{k+n}](\zeta_{k+1}, \dots, \zeta_{k+n})\right)$. Here, we denote

$$P'_{n,\sigma} := \{(\zeta_{k+1}, \dots, \zeta_{k+n}) \in \Gamma^n \mid \zeta_{\sigma(1)} \succ \dots \succ \zeta_{\sigma(k+n)}\} \tag{2.12}$$

for $\sigma \in \mathfrak{S}_{k+n}$. In this case, we also employ a confusing notation (really confusing in the case)

$$T\left(F_1(\zeta_1)\dots F_{k+n}(\zeta_{k+n})\right)$$
$$:= T[F_1(\zeta_1)F_2(\zeta_2)\dots F_k(\zeta_k)F_{k+1}\dots F_{k+n}](\zeta_{k+1}, \dots, \zeta_{k+n}), \tag{2.13}$$

and call it a *time-ordered product* of $F_1(\zeta_1), \dots, F_{k+n}(\zeta_{k+n})$ along the curve Γ, following physics literatures. We never use this notation unless it can be clearly understood from the context which variables of $(\zeta_1, \dots, \zeta_{k+n})$ are fixed and which variables are function argument.

Using this notation, we can define a more general time-ordered exponential operator. Let $F_1, \dots, F_k, F$ be operator-valued functions from Γ into $L(\mathcal{H})$ and $F_{k+1} = \dots = F_{k+n} = F$. Let $C(F_1, \dots, F_k, F)$ be a linear subspace spanned by all the vectors Ψ for which the mappings

$$(\zeta_{k+1}, \dots, \zeta_{k+n}) \mapsto F_{\sigma(1)}(\zeta_{\sigma(1)})\dots F_{\sigma(k+n)}(\zeta_{\sigma(k+n)})\Psi \tag{2.14}$$

are continuous for all fixed $(\zeta_1, \dots, \zeta_k)$ and all $\sigma \in \mathfrak{S}_{n+k}$. Then, on the domain

$$D\left(TF_1(\zeta_1)\dots F_k(\zeta_k)\exp\left(\int_\Gamma d\zeta\, F(\zeta)\right)\right) := \{\Psi \in C(F_1, \dots, F_k, F) \mid$$
$$\sum_{n=0}^{\infty} \frac{1}{n!}\left\| \int_{\Gamma^n} d\zeta_{k+1}\dots d\zeta_{k+n}\, T\left(F_1(\zeta_1)\dots F_k(\zeta_k)F(\zeta_{k+1})\dots F(\zeta_{k+n})\right)\Psi \right\| < \infty \right\}, \tag{2.15}$$

We define

$$TF_1(\zeta_1)\dots F_k(\zeta_k)\exp\left(\int_\Gamma d\zeta\, F(\zeta)\right)\Psi$$
$$:= \sum_{n=0}^{\infty} \frac{1}{n!}\int_{\Gamma^n} d\zeta_{k+1}\dots d\zeta_{k+n}\, T\left(F_1(\zeta_1)\dots F_k(\zeta_k)F(\zeta_{k+1})\dots F(\zeta_{k+n})\right)\Psi. \tag{2.16}$$

We remark that for all $\sigma \in \mathfrak{S}_k$,

$$TF_1(\zeta_1)\dots F_k(\zeta_k)\exp\left(\int_\Gamma d\zeta\, F(\zeta)\right)$$
$$= TF_{\sigma(1)}(\zeta_{\sigma(1)})\dots F_{\sigma(k)}(\zeta_{\sigma(k)})\exp\left(\int_\Gamma d\zeta\, F(\zeta)\right). \tag{2.17}$$

We introduce a class of operators which plays a crucial role in the following analyses. Let H_0 be a non-negative self-adjoint operator in $\mathcal{H}$.

Definition 2.1 ($\mathcal{C}_0$-class). We say that a linear operator T is in $\mathcal{C}_0$-*class* if T satisfies the following (I)-(III):

(I) T is densely defined and closed.

(II) T and T^* are $H_0^{1/2}$-bounded.

(III) There exists a constant $b \geq 0$ such that, for all $E \geq 0$, T and T^* map $R(E_{H_0}([0, E]))$ into $R(E_{H_0}([0, E + b]))$.

We define

$$V_E := R(E_{H_0}([0, E])), \tag{2.18}$$
$$D_{\text{fin}} := \bigcup_{E \geq 0} V_E, \tag{2.19}$$

and denote the set consisting of all the $\mathcal{C}_0$-class operators also by $\mathcal{C}_0$. Note that the subspace D_{fin} is dense in $\mathcal{H}$ since H_0 is self-adjoint. For $A \in \mathcal{C}_0$, we define

$$A(z) := e^{izH_0}Ae^{-izH_0}, \quad z \in \mathbb{C}. \tag{2.20}$$

Note that $A(z)$ is closable since its adjoint includes the operator $e^{iz^*H_0}A^*e^{-iz^*H_0}$ which is densely defined. We denote the closure of $A(z)$ by the same symbol. In this notation, one obtains

$$A(z)^* \supset A^*(z^*). \tag{2.21}$$

The basic properties of the time-ordered exponential is summarized in the following Theorems 2.1-2.5.

Theorem 2.1. *Let A be in C_0 and $z, z' \in \mathbb{C}$.*

(i) Take a piecewisely continuously differentiable simple curve $\Gamma_{z,z'}$ which starts at z' and ends at z with $\operatorname{Im} z' \leq \operatorname{Im} z$. Then,

$$D_{\text{fin}} \subset D\left(T\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta A(\zeta)\right)\right) \tag{2.22}$$

and the restriction

$$T\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta A(\zeta)\right)\Big|_{D_{\text{fin}}} \tag{2.23}$$

does not depend upon the simple curve from z' to z and depends only on z and z', justifying the notation

$$U(A; z, z') := T\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta A(\zeta)\right)\Big|_{D_{\text{fin}}}. \tag{2.24}$$

(ii) $U(A; z, z')$ is closable, and satisfies the following inclusion relation:

$$U(A; z, z')^* \supset \overline{U(A^*; z'^*, z^*)}. \tag{2.25}$$

Theorem 2.2. *Let T_k, A_k $(k = 1, ..., m,\ m \geq 1)$ be C_0-class operators. Then, for all $z_k, z_k' \in \mathbb{C}$ $(k = 1, ..., m)$ with $\operatorname{Im} z_k \leq \operatorname{Im} z_k'$ and $\zeta_k, \zeta_k' \in \mathbb{C}$, it follows that*

$$D_{\text{fin}} \subset D(T_m(\zeta_m, \zeta_m')\overline{U(A_m; z_m, z_m')} \cdots T_1(\zeta_1, \zeta_1')\overline{U(A_1; z_1, z_1')}). \tag{2.26}$$

Moreover, for all $\Psi \in D_{\text{fin}}$,

$$\begin{aligned} &T_m(\zeta_m, \zeta_m')\overline{U(A_m; z_m, z_m')} \cdots T_1(\zeta_1, \zeta_1')\overline{U(A_1; z_1, z_1')}\Psi \\ &= \sum_{n_1,\ldots,n_m=0}^{\infty} T_m(\zeta_m, \zeta_m')V_{n_m}(A_m; z_m, z_m') \cdots \\ &\qquad \cdots T_1(\zeta_1, \zeta_1')V_{n_1}(A_1; z_1, z_1')\Psi, \end{aligned} \tag{2.27}$$

where the right-hand side converges absolutely, and does not depend upon the summation order. Furthermore, this convergence is locally uniform in the complex variables

$$z_1, z_1', \zeta_1, \zeta_1', \ldots, z_m, z_m', \zeta_m, \zeta_m'.$$

By Theorem 2.2, it is natural to introduce the set of all the polynomials $\mathfrak{A}$ generated by

$$\left\{T, \overline{U(A;z,z')}, e^{i\zeta H_0} \,|T, A \in \mathcal{C}_0,\ z,z',\zeta \in \mathbb{C},\ \operatorname{Im} z \leq \operatorname{Im} z'\right\}. \tag{2.28}$$

It is clear that all $a \in \mathfrak{A}$ are closable, since they have densely defined adjoints and the subspace D_{fin} is a common domain of $\mathfrak{A}$. We define a dense subspace $\mathcal{D}$ by

$$\mathcal{D} := \mathfrak{A} D_{\text{fin}}. \tag{2.29}$$

Theorem 2.2 shows that $\mathcal{D}$ is also a common domain of $\mathfrak{A}$. Moreover, for all $\Psi \in \mathcal{D}$, there exists a sequence $\{\Psi_N\}_N \subset D_{\text{fin}}$ such that

$$\Psi_N \to \Psi, \quad a\Psi_N \to a\Psi \quad (a \in \mathfrak{A}) \tag{2.30}$$

as N tends to infinity. This implies that if an equality $a = b$ $(a, b \in \mathfrak{A})$ holds on D_{fin}, then $a = b$ on $\mathcal{D}$ and the convergence is locally uniform in all the complex variables included in a and b. From this observation, we immediately have

Corollary 2.1. *Let A be in $\mathcal{C}_0$ and $z, z' \in \mathbb{C}$ with $\operatorname{Im} z \leq \operatorname{Im} z'$. Then,*

$$\mathcal{D} \subset D\left(T\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta A(\zeta)\right)\right) \tag{2.31}$$

and for all $\Psi \in \mathcal{D}$,

$$T\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta A(\zeta)\right)\Psi = \overline{U(A;z,z')}\Psi. \tag{2.32}$$

In particular,

$$T\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta A(\zeta)\right)\Psi \tag{2.33}$$

is independent of the simple curve $\Gamma_{z,z'}$ and depends only on z, z' if $\Psi \in \mathcal{D}$.

Theorem 2.3. *Let A be in $\mathcal{C}_0$ and $z, z' \in \mathbb{C}$.*

(i) For all $\Psi \in \mathcal{D}$, the vector-valued function

$$\{(z,z') \,|\, \operatorname{Im} z \leq \operatorname{Im} z'\} \ni (z,z') \mapsto \overline{U(A;z,z')}\Psi \in \mathcal{H}$$

is analytic on the region $\{\operatorname{Im} z < \operatorname{Im} z'\}$ and continuous on $\{\operatorname{Im} z \leq \operatorname{Im} z'\}$. Moreover, it is a solution of differential equations

$$\frac{\partial}{\partial z}\overline{U(A;z,z')}\Psi = -iA(z)\,\overline{U(A;z,z')}\Psi, \tag{2.34}$$

$$\frac{\partial}{\partial z'}\overline{U(A;z,z')}\Psi = i\overline{U(A;z,z')}A(z')\Psi, \tag{2.35}$$

on $\{\operatorname{Im} z < \operatorname{Im} z'\}$.

(ii) *For all* $\Psi \in \mathcal{D}$, *the vector valued function* $\mathbb{R}^2 \ni (t,t') \mapsto \overline{U(A;t,t')}\Psi$ *is continuously differentiable on the region* $\mathbb{R}^2$, *satisfying the differential equations*

$$\frac{\partial}{\partial t}\overline{U(A;t,t')}\Psi = -iA(t)\,\overline{U(A;t,t')}\Psi, \tag{2.36}$$

$$\frac{\partial}{\partial t'}\overline{U(A;t,t')}\Psi = i\overline{U(A;t,t')}A(t')\Psi. \tag{2.37}$$

Theorem 2.4. *Let* $A \in \mathcal{C}_0$ *and* $z, z', z'' \in \mathbb{C}$. *Then, the following properties hold.*

(i) *If* $\operatorname{Im} z \le \operatorname{Im} z' \le \operatorname{Im} z''$, *the equalities*

$$\overline{U(A;z,z)} = I, \quad \overline{U(A;z,z')}\,\overline{U(A;z',z'')} = \overline{U(A;z,z'')} \tag{2.38}$$

hold on the subspace $\mathcal{D}$, *where* I *is the identity operator.*

(ii) *Let* $\operatorname{Im} z \le \operatorname{Im} z'$. *Then,* $U(A;z,z')$ *is translationally invariant in the sense that the equality*

$$e^{izH_0}\overline{U(A;z',z'')}e^{-izH_0}\Psi = \overline{U(A;z'+z,z''+z)} \tag{2.39}$$

holds on the subspace $\mathcal{D}$.

(iii) *For all* $t, t' \in \mathbb{R}$, $\overline{U(A;t,t')}$ *is unitary. Moreover, for all* $t, t', t'' \in \mathbb{R}$, *the operator equality*

$$\overline{U(A;t,t')}\,\overline{U(A;t',t'')} = \overline{U(A;t,t'')} \tag{2.40}$$

holds.

Theorem 2.5. *Let* $A_1, \dots A_k, B \in \mathcal{C}_0$, *and* $z, z' \in \mathbb{C}$ *with* $\operatorname{Im} z \le \operatorname{Im} z'$. *Let* $\zeta_1, \dots, \zeta_k \in \mathbb{C}$ *and suppose that there exists a permutation* $\sigma \in \mathfrak{S}_k$ *satisfying*

$$\operatorname{Im} z \le \operatorname{Im} \zeta_{\sigma(1)} \le \cdots \le \operatorname{Im} \zeta_{\sigma(k)} \le \operatorname{Im} z'. \tag{2.41}$$

Take a simple curve $\Gamma_{z,z'}$ *from* z' *to* z *on which* $\zeta_{\sigma(1)} \succ \cdots \succ \zeta_{\sigma(k)}$. *Then, we have*

$$\mathcal{D} \subset D\left(TA_1(\zeta_1)\dots A_k(\zeta_k)\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta\, B(\zeta)\right)\right) \tag{2.42}$$

and

$$\begin{aligned} &TA_1(\zeta_1)\dots A_k(\zeta_k)\exp\left(-i\int_{\Gamma_{z,z'}} d\zeta\, B(\zeta)\right)\Psi \\ &\quad = \overline{U(B;z,\zeta_{\sigma(1)})}A_{\sigma(1)}(\zeta_{\sigma(1)})\overline{U(B;\zeta_{\sigma(1)},\zeta_{\sigma(2)})}\dots \\ &\quad\quad \dots\overline{U(B;\zeta_{\sigma(k-1)},\zeta_{\sigma(k)})}A_{\sigma(k)}(\zeta_{\sigma(k)})\overline{U(B;\zeta_{\sigma(k)},z')}\Psi \end{aligned} \tag{2.43}$$

for all $\Psi \in \mathcal{D}$.

3 Complex time evolution and Gell-Man – Low formula

In this section, we consider the operator

$$H = H_0 + H_1 \tag{3.1}$$

with $H_1 \in \mathcal{C}_0$, and we state and derive the Gell-Mann – Low formula. In what follows, we shortly denote

$$V_n(z,z') := V_n(H_1; z, z'), \quad U(z,z') := U(H_1; z, z'). \tag{3.2}$$

We define complex time evolution operator

$$W(z) := e^{-izH_0}\overline{U(z,0)} \tag{3.3}$$

for $z \in \mathbb{C}$ with $\operatorname{Im} z \le 0$. The operator $W(z)$ generates the "complex time evolution" in the following sense:

Theorem 3.1. *For all $\Psi \in \mathcal{D}$, the mapping $z \mapsto W(z)\Psi$ is analytic on the lower half plane and satisfies the "complex Schrödinger equation"*

$$\frac{d}{dz}W(z)\Psi = -iHW(z)\Psi. \tag{3.4}$$

Proof. We first remark that $\mathcal{D} \subset D(H_0)$. This can be seen by noting that $\mathcal{D} \subset D(e^{H_0}) \subset D(H_0)$. By Theorem 2.1, one can easily estimate

$$\left\| \frac{W(z+h)\Psi - W(z)\Psi}{h} - (-iH)W(z)\Psi \right\| \tag{3.5}$$

to know that this vanishes in the limit $h \to 0$. □

Theorem 3.2. *Suppose that H_1 is a $\mathcal{C}_0$-class symmetric operator. Then, H is self-adjoint and bounded below. Moreover, it follows that*

$$\overline{W(z)} = e^{-izH}, \tag{3.6}$$

for all $z \in \mathbb{C}$ with $\operatorname{Im} z \le 0$. In particular, it follows that

$$\overline{U(z,z')} = e^{izH_0}e^{-i(z-z')H}e^{-iz'H_0}, \quad \operatorname{Im} z \le \operatorname{Im} z'. \tag{3.7}$$

Proof. By the present assumption, H_1 is $H_0^{1/2}$-bounded. This implies that H_1 is infinitesimally small with respect to H_0 and thus H is self-adjoint with $D(H) = D(H_0)$, and bounded below by the Kato-Rellich Theorem.

By Theorem 3.1, the function $z \mapsto \langle e^{-iz^*H}\Phi, W(z)\Psi\rangle$ is differentiable in z with $\operatorname{Im} z < 0$ for all $\Psi \in \mathcal{D}$, $\Phi \in D_0(H) := \cup_{L \in \mathbb{R}} R(E_H([-L, L]))$, and we have

$$\begin{aligned} \frac{d}{dz}\langle e^{-iz^*H}\Phi, W(z)\Psi\rangle &= \langle -iHe^{-iz^*H}\Phi, W(z)\Psi\rangle + \\ &\quad + \langle e^{-iz^*H}\Phi, -iHW(z)\Psi\rangle \\ &= 0. \end{aligned} \tag{3.8}$$

Thus, one finds

$$\langle \Phi, \Psi \rangle = \langle e^{-iz^* H}\Phi, W(z)\Psi \rangle, \tag{3.9}$$

for all $\Psi \in \mathcal{D}$ and $\Phi \in D_0(H)$. Since $D_0(H)$ is a core of $e^{-iz^* H}$, we obtain from (3.9) $W(z)\Psi \in D(e^{izH})$ and

$$e^{izH} W(z)\Psi = \Psi. \tag{3.10}$$

Hence, we arrive at

$$W(z)\Psi = e^{-izH}\Psi, \tag{3.11}$$

for all $z \in \mathbb{C}$ with $\operatorname{Im} z < 0$. But, since both sides of (3.11) are continuous on the region $\operatorname{Im} z \leq 0$, (3.11) must hold on $\operatorname{Im} z \leq 0$. Since the both sides are bounded, one has

$$\overline{W(z)} = e^{-izH}, \quad \operatorname{Im} z \leq 0. \tag{3.12}$$

For z, z' satisfying $\operatorname{Im} z \leq \operatorname{Im} z'$, we have from (2.39)

$$\begin{aligned} W(z - z')\Psi &= e^{-i(z-z')H_0}\overline{U(z - z', 0)}\Psi \\ &= e^{-izH_0}\overline{U(z, z')}\, e^{iz' H_0}\Psi, \quad \Psi \in \mathcal{D}. \end{aligned} \tag{3.13}$$

This implies

$$\overline{U(z, z')}\Psi = e^{izH_0} e^{-i(z-z')H} e^{-iz' H_0}\Psi. \tag{3.14}$$

□

We introduce assumptions needed to derive the Gell-Mann – Low formula. For a linear operator T, we denote the spectrum of T by $\sigma(T)$. If T is self-adjoint and bounded from below, then we define

$$E_0(T) := \inf \sigma(T). \tag{3.15}$$

We say that T has a ground state if $E_0(T)$ is an eigenvalue of T. In that case, $E_0(T)$ is called the *ground energy* of T, and each non-zero vector in $\ker(T - E_0(T))$ is called a *ground state* of T. If $\dim \ker(T - E_0(T)) = 1$, we say that T has a unique ground state. The following assumption are used to prove the Gell-Mann – Low formula.

Assumption 3.1. *(I) H_0 has a unique ground state Ω_0 ($\|\Omega_0\| = 1$), and the ground energy is zero: $E_0(H_0) = 0$.*

(II) H_1 is a $\mathcal{C}_0$-class symmetric operator, and H has a unique ground state Ω ($\|\Omega\| = 1$).

(III) $\langle \Omega, \Omega_0 \rangle \neq 0$.

Under Assumption 3.1, we define the m-point Green's function $G_m(z_1, \dots, z_m)$ by

$$G_m(z_1, \dots, z_m) := e^{i(z_1 - z_m)E_0(H)} \langle \Omega, A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) \Omega \rangle , \tag{3.16}$$

for $\operatorname{Im} z_1 \le \dots \le \operatorname{Im} z_m$, provided that the right-hand-side is well-defined. The Gell-Mann – Low formula is given by:

Theorem 3.3. *Suppose that Assumption 3.1 holds. Let A_k ($k = 1, ..., m,\ m \ge 1$) be linear operators having the following properties:*

(I) Each A_k is in C_0-class.

(II) For each k, there exists an integer $r_k \ge 0$ such that, for all $n \in \mathbb{N}$, A_k maps $D(H^{n+r_k})$ into $D(H^n)$.

Let $z_1, ..., z_m \in \mathbb{C}$ with $\operatorname{Im} z_1 \le \dots \le \operatorname{Im} z_m$. Choose a simple curve Γ_T^ε from $-T(1 - i\varepsilon)$ to $T(1 - i\varepsilon)$ ($T, \varepsilon > 0$) on which $z_1 \succ \dots \succ z_m$. Then, the m-point Green's function $G_m(z_1, \dots, z_m)$ is well-defined and satisfies the formula

$$G_m(z_1, \dots, z_m) = \lim_{T \to \infty} \frac{\left\langle \Omega_0, T A_1(z_1) \dots A_m(z_m) \exp\left(-i \int_{\Gamma_T^\varepsilon} d\zeta\, H_1(\zeta)\right) \Omega_0 \right\rangle}{\left\langle \Omega_0, T \exp\left(-i \int_{\Gamma_T^\varepsilon} d\zeta\, H_1(\zeta)\right) \Omega_0 \right\rangle}. \tag{3.17}$$

To prove the Gell-Mann – Low formula (3.17), we prepare some lemmas. We denote $E_0(H)$ simply by E_0.

Lemma 3.1. *For any $\varepsilon > 0$ and all Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$, we have*

$$\lim_{T \to \infty} f(H) e^{iT(\pm 1 - i\varepsilon)E_0} W(T(\pm 1 - i\varepsilon)) \Psi = f(E_0) P_0 \Psi, \quad \Psi \in D(f(H)), \tag{3.18}$$

where P_0 is the orthogonal projection onto the closed subspace $\ker(H - E_0)$.

Proof. By the functional calculus and Lebesgue's convergence Theorem, we have

$$\begin{aligned} &\left\| f(H) e^{iT(\pm 1 - i\varepsilon)E_0} W(T(\pm 1 - i\varepsilon)) \Psi - f(E_0) P_0 \Psi \right\|^2 \\ &= \left\| f(H) e^{\mp iT(H - E_0)} e^{-T\varepsilon(H - E_0)} \Psi - f(E_0) E_H(\{E_0\}) \Psi \right\|^2 \\ &= \int_{[E_0, \infty)} d \left\| E_H(\lambda) \Psi \right\|^2 \left| f(\lambda) (e^{-T\varepsilon(\lambda - E_0)} \Psi - \chi_{\{E_0\}}(\lambda)) \right|^2 \\ &= \int_{(E_0, \infty)} d \left\| E_H(\lambda) \Psi \right\|^2 \left| f(\lambda) e^{-T\varepsilon(\lambda - E_0)} \Psi \right|^2 \\ &\to 0, \end{aligned} \tag{3.19}$$

as T tends to infinity. □

Lemma 3.2. *Under the same assumption as in Theorem 3.3, the operators*

$$\widetilde{A_k} := (H-\zeta)^{\sum_{j=1}^{k-1} r_j} A_k (H-\zeta)^{-\sum_{j=1}^{k} r_j}, \quad k = 1, ..., m, \tag{3.20}$$

are bounded.

Proof. From the assumptions,

$$A_k (H-\zeta)^{-\sum_{j=1}^{k} r_j} \Psi \in D(H^{\sum_{j=1}^{k-1} r_j}), \tag{3.21}$$

for all $\Psi \in \mathcal{H}$. Thus,

$$D(\widetilde{A_k}) = \mathcal{H}.$$

On the other hand, it is easy to check that $\widetilde{A_k}$'s are closed. Hence, by the closed graph theorem, each $\widetilde{A_k}$'s are bounded. □

Lemma 3.3. *Under the same assumption as in Theorem 3.3, it follows that*

$$\begin{aligned} \lim_{T\to\infty} & A_1 W(z_1 - z_2) A_2 \dots \\ & \dots A_{m-1} W(z_{m-1} - z_m) A_m f(H) e^{iT(\pm 1 - i\varepsilon)} W(T(\pm 1 - i\varepsilon))\Psi \\ & \quad = A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m f(E_0) P_0 \Psi, \end{aligned} \tag{3.22}$$

for all Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ and $\Psi \in \bigcap_{n\in\mathbb{N}} D(H^n f(H))$.

Proof. Under the present assumption, we see that each A_k leaves the subspace $\bigcap_{n=1}^{\infty} D(H^n)$ invariant, and thus Ψ belongs to the domain of the operator

$$A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m f(H) e^{iT(\pm 1 - i\varepsilon)} W(T(\pm 1 - i\varepsilon)).$$

Now let $\zeta \in \mathbb{C}\backslash\mathbb{R}$. Then, we can rewrite

$$\begin{aligned} & A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m \\ & \quad = \widetilde{A_1} W(z_1 - z_2) \cdots \widetilde{A_m} W(z_{m-1} - z_m)(H-\zeta)^{\sum_{k=1}^{m} r_k} \end{aligned} \tag{3.23}$$

with

$$\widetilde{A_k} := (H-\zeta)^{\sum_{j=1}^{k-1} r_j} A_k (H-\zeta)^{-\sum_{j=1}^{k} r_j}, \quad k = 1, ..., m. \tag{3.24}$$

Note that each of $\widetilde{A_k}$'s and $W(z_{k-1} - z_k)$'s are bounded operators by Theorem 3.2 and Lemma 3.2. Then, by Lemma 3.1, one sees that for all $n \geq 1$,

$$\begin{aligned} \lim_{T\to\infty} & (H-\zeta)^n e^{iT(\pm 1 - i\varepsilon)} W(T(\pm 1 - i\varepsilon))\Psi \\ & \quad = (E_0 - \zeta)^n P_0 \Psi = (H-\zeta)^n P_0 \Psi, \end{aligned} \tag{3.25}$$

which implies the desired result. □

Proof of Theorem 3.3. Put

$$\mathcal{O}_{z_1,\dots,z_m} := A_1 W(z_1 - z_2) A_2 \dots A_{m-1} W(z_{m-1} - z_m) A_m. \tag{3.26}$$

From Assumption 3.1, one finds

$$\Omega = \frac{P_0 \Omega_0}{\|P_0 \Omega_0\|} \tag{3.27}$$

to obtain

$$G_m(z_1, \dots, z_m) = e^{i(z_1 - z_m)E_0} \frac{\langle P_0\Omega_0, \mathcal{O}_{z_1,\dots,z_m} P_0 \Omega_0\rangle}{\langle P_0\Omega_0, P_0\Omega_0\rangle}. \tag{3.28}$$

By Lemmas 3.1 and 3.3, we have

$$\frac{\langle P_0\Omega_0, \mathcal{O}_{z_1,\dots,z_m} P_0 \Omega_0\rangle}{\langle P_0\Omega_0, P_0\Omega_0\rangle} = \\ \lim_{T\to\infty} \frac{\left\langle e^{-iz_1^*(H-E_0)} W(T(-1-i\varepsilon))\Omega_0, \mathcal{O}_{z_1,\dots,z_m} e^{-iz_m(H-E_0)} W(T(1-i\varepsilon))\Omega_0\right\rangle}{\langle W(T(-1-i\varepsilon))\Omega_0, W(T(1-i\varepsilon))\Omega_0\rangle}. \tag{3.29}$$

Using Theorem 3.2, we find

$$e^{-iz_1^*(H-E_0)} W(T(-1-i\varepsilon)) \\ = e^{iz_1^* E_0} e^{-iz_1^* H_0} \overline{U(z_1^*, T(1+i\varepsilon))} e^{iT(1+i\varepsilon)H_0} \tag{3.30}$$

$$e^{-iz_m(H-E_0)} W(T(1-i\varepsilon)) \\ = e^{iz_m E_0} e^{-iz_m H_0} \overline{U(z_m, -T(1-i\varepsilon))} e^{-iT(1-i\varepsilon)H_0} \tag{3.31}$$

on $\mathcal{D}$. Therefore, by Theorem 2.5, the numerator on the right-hand-side of (3.29) can be rewritten as

$$e^{-i(z_1-z_m)E_0} \langle \Omega_0, \overline{U(T(1-i\varepsilon), z_1)} A_1(z_1) \overline{U(z_1, z_2)} \dots \\ \dots \overline{U(z_{m-1}, z_m)} A_m(z_m) \overline{U(z_m, -T(1-i\varepsilon))} \Omega_0\rangle \\ = e^{-i(z_1-z_m)E_0} \langle \Omega_0, T A_1(z_1) \dots \\ \dots A_m(z_m) \exp\left(-i \int_{\Gamma_T^\varepsilon} d\zeta\, H_1(\zeta)\right) \Omega_0\rangle \tag{3.32}$$

and the denominater as

$$\langle \Omega_0, U(T(1-i\varepsilon), -T(1-i\varepsilon))\Omega_0\rangle \\ = \left\langle \Omega_0, T\exp\left(-i \int_{\Gamma_T^\varepsilon} d\zeta\, H_1(\zeta)\right) \Omega_0 \right\rangle. \tag{3.33}$$

Finally, inserting (3.29), (3.32), and (3.33) into (3.28), we arrive at the Gell-Mann – Low formula (3.17). □

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Department of Mathematics
Hokkaido University
060-0810, Sapporo
Japan
kouta@math.sci.hokudai.ac.jp

北海道大学・大学院理学院数学専攻　臼井　耕太