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STRONG CONVERGENCE OF ITERATIVE ALGORITHMS FOR SOLVING OPTIMIZATION PROBLEMS

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ABSTRACT. In this talk, we consider iterative algorithms for solving a certain optimization problem in Hilbert spaces, where the constraint set is the set of fixed points of strictly pseudocontractive mapping T. Under suitable conditions on control parameters, we establish strong convergence of the sequence generated by the proposed iterative algorithm to a fixed point of the mapping T, which is the unique solution of the optimization problem. As a direct consequence, we obtain the unique minimum-norm fixed point of T.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and let $T:C\to C$ be a self-mapping on C. We denote by F(T) the set of fixed points of T, that is, $F(T):=\{x\in C:Tx=x\}$.

We recall that a mapping $T:C\to H$ is said to be *k-strictly pseudocontractive* if there exists a constant $k\in[0,1)$such that

$$\|Tx-Ty\|^{2}\leq\|x-y\|^{2}+k\|(I-T)x-(I-T)y\|^{2}, \quad \forall x,\ y\in C.$$

Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is nonexpansive (*i.e.*, $\|Tx-Ty\|\leq\|x-y\|$, $\forall x,\ y\in C$) if and only if T is 0-strictly pseudocontractive. Recently, many authors have been devoting the studies on the problems of finding fixed points for pseudocontractive mappings, see, for example, [1, 3, 4, 5, 11, 16] and the references therein

Let A be a strongly positive bounded linear self-adjoint operator on H with a constant $\overline{\gamma}>0$, that is, there exists a constant $\overline{\gamma}>0$ such that

$$\langle Ax,x\rangle\geq\overline{\gamma}\|x\|^{2}, \quad \forall x\in H.$$

Let $f:C\to C$ be a contractive mapping with constant $\alpha\in(0,1)$, that is, there exists a constant $\alpha\in(0,1)$ such that $\|f(x)-f(y)\|\leq\alpha\|x-y\|$ for all $x,\ y\in C$.

The following optimization problem has been studied extensively by many authors:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax,x\rangle+\frac{1}{2}\|x-u\|^{2}-h(x),$$

where $\Omega=\bigcap_{i=1}^{\infty}C_{i}$, $C_{1},C_{2},\cdots$, are infinitely many closed convex subsets of H such that $\bigcap_{i=1}^{\infty}C_{i}\neq\emptyset$, $u\in H$, $\mu\geq 0$ is a real number, A is a strongly positive bounded linear self-adjoint operator on H and h is a potential function for γf (*i.e.*, $h'(x)=\gamma f(x)$ for all $x\in H$). For this kind of minimization problems, see, for example, Bauschke and Borwein [2], Combettes [7], Deutsch and Yamada [8], Jung [10] and Xu [18] when $h(x)=\langle x,b\rangle$ for b is a given point in H.

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Iterative algorithms for nonexpansive mappings and strictly pseudocontractive mappings have recently been applied to solve the optimization problem, where the constraint set is the set of fixed points of the mapping, see, e.q., [5, 8, 11, 15, 19, 20] and the references therein. Some iterative algorithms for equilibrium problems, variational inequality problems and fixed point problems to solve optimization problem, where the constraint set is the common set of the set of solutions of the problems and the set of fixed points of the mappings, were also investigated by many authors recently, see, e.q., [12, 21, 22] and the references therein.

Inspired and motivated by the recent works in this direction, in this paper, we consider the following optimization problem

$$\min_{x\in F(T)} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{1.1}$$

where $F(T)$ is the set of fixed points of a k-strictly pseudocontractive mapping T. We introduce new implicit and explicit iterative algorithms for a k-strictly pseudocontractive mapping T related to the optimization problem (1.1), and then prove that the sequences generated by the proposed iterative algorithms converge strongly to a fixed point of the mapping T, which solves the optimization problem (1.1). In particular, in order to establish strong convergence of explicit iterative algorithm, we utilize weak and different control conditions in comparison with previous ones. As a direct consequence, we obtain the unique minimum-norm point in the set $F(T)$.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. In the following, when $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

We need some facts and tools in a real Hilbert space which are listed as lemmas below. We will use them in the proofs for the main results in next section.

Recall that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \le \|x - y\|$$

for all $y \in C$. P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive.

Lemma 2.1 ([9]). *Let H a real Hilbert space, let C be a nonempty closed convex subset of H, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

The following Lemmas 2.2 and 2.3 are not hard to prove (see also Lemmas 2.3 and 2.5 in [15]).

Lemma 2.2. *Let $\mu > 0$, and let $A : H \to H$ be a strongly positive bounded linear self-adjoint operator on a Hilbert space H with a constant $\overline{\gamma} \in (0,1)$ such that $(1+\mu)\overline{\gamma} < 1$. Let $0 < \rho \le (1+\mu\|A\|)^{-1}$. Then $\|I - \rho(I+\mu A)\| < 1 - \rho(1+\mu)\overline{\gamma}$.*

Lemma 2.3. *Let H be a real Hilbert space, and let C be a nonempty closed subspace of H. Let $f : C \to C$ be a contractive mapping with constant $\alpha \in (0,1)$, and let $A : C \to C$ be a strongly positive bounded linear self-adjoint operator with a constant $\overline{\gamma} \in (0,1)$. Let $\mu > 0$ and $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$ with $(1+\mu)\overline{\gamma} < 1$. Then for all $x,\ y \in C$,*

$$\langle x - y, ((I+\mu A) - \gamma f)x - ((I+\mu A) - \gamma f)y\rangle \ge ((1+\mu)\overline{\gamma} - \gamma\alpha)\|x-y\|^2.$$

That is, $(I+\mu A) - \gamma f$ is strongly monotone with a constant $(1+\mu)\overline{\gamma} - \gamma\alpha$.

Lemma 2.4 ([23]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let $T : C \to H$ be a k-strictly pseudocontractive mapping. Then the following hold:*

(i) *The fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined,*

(ii) $F(P_C T) = F(T)$,
(iii) *If we define a mapping* $S : C \to H$ *by* $Sx = \lambda x + (1-\lambda)Tx$ *for all* $x \in C$. *Then, as* $\lambda \in [k, 1)$, S *is a nonexpansive mapping such that* $F(S) = F(T)$.

Lemma 2.5 ([14, 18]). *Let* $\{s_n\}$ *be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n + r_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:
(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$,
(iii) $r_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} r_n < \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6. *In a Hilbert space* H, *the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,\ y \in H.$$

Let LIM be a Banach limit. According to time and circumstances, we use $LIM_n(a_n)$ instead of $LIM(a)$ Then the following are well-known:
(i) for all $n \geq 1, a_n \leq c_n$ implies $LIM_n(a_n) \leq LIM_n(c_n)$,
(ii) $LIM_n(a_{n+N}) = LIM_n(a_n)$ for any fixed positive integer N,
(iii) $\liminf_{n\to\infty} a_n \leq LIM_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $\{a_n\} \in l^\infty$

The following lemma was given in Proposition 2 in [17].

Lemma 2.7. *Let* $a \in \mathbb{R}$ *be a real number, and let a sequence* $\{a_n\} \in \ell^\infty$ *satisfy the condition* $LIM_n(a_n) \leq a$ *for all Banach limit* LIM. *If* $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, *then* $\limsup_{n\to\infty} a_n \leq a$.

The following lemma can be found in [21](see also Lemma 2.1 in [10]).

Lemma 2.8. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *and let* $g : C \to \mathbb{R} \cup \{\infty\}$ *be a proper lower semicontiunous differentiable convex function. If* x^* *is a solution to the minimization problem*

$$g(x^*) = \inf_{x\in C} g(x),$$

then

$$\langle g'(x^*), x - x^*\rangle \geq 0, \quad x \in C.$$

In particular, if x^* *solves the optimization problem*

$$\min_{x\in C} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x),$$

then

$$\langle u + (\gamma f - (I+\mu A))x^*, x - x^*\rangle \leq 0, \quad x \in C,$$

where h *is a potential function for* γf.

Finally, we recall that the sequence $\{x_n\}$ in H is said to be *weakly asymptotically regular* if

$$w - \lim_{n\to\infty}(x_{n+1} - x_n) = 0, \text{ that is, } x_{n+1} - x_n \rightharpoonup 0$$

and *asymptotically regular* if

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$$

respectively.

3. Main results

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space;
- C is a nonempty closed subspace of H;
- $T : C \to H$ is a k-strictly pseudocontractive mapping with $F(T) \neq \emptyset$ for some $0 \le k < 1$;
- $S : C \to H$ is a mapping defined by $Sx = kx + (1-k)Tx$;
- $A : C \to C$ is a strongly positive bounded linear self-adjoint operator with a constant $\overline{\gamma} \in (0,1)$;
- $f : C \to C$ is a contractive mapping with a constant $\alpha \in (0,1)$;
- $\mu > 0$ and $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$ with $(1+\mu)\overline{\gamma} < 1$;
- $u \in C$ is a fixed element;
- P_C is a metric projection of H onto C.

First, in order to find a solution of the optimization problem (1.1), we construct the following iterative algorithm which generates a net $\{x_t\}$ in an implicit way:

$$x_t = t(u + \gamma f(x_t)) + (I - t(I + \mu A))P_C S x_t, \quad \forall t \in \left(0, \frac{1}{1+\mu\|A\|}\right). \tag{3.1}$$

To this end, for $t \in (0,1)$ such that $t < (1+\mu\|A\|)^{-1}$, consider a mapping $Q_t : C \to C$ by

$$Q_t x = t(u + \gamma f(x)) + (I - t(I+\mu A))P_C S x, \quad \forall x \in C.$$

It is easy to see that Q_t is a contraction with constant $1 - t((1+\mu)\overline{\gamma} - \gamma\alpha)$. Indeed, by Lemma 2.2, we have

$$\begin{aligned}\|Q_t x - Q_t y\| &\le t\gamma\|f(x) - f(y)\| + \|(I - t(I+\mu A))(P_C S x - P_C S y)\| \\ &\le t\gamma\alpha\|x - y\| + (1 - t(1+\mu)\overline{\gamma})\|x - y\| \\ &= (1 - t((1+\mu)\overline{\gamma} - \gamma\alpha))\|x - y\|.\end{aligned}$$

Hence Q_t has a unique fixed point, denoted x_t, which uniquely solve the fixed point equation

$$x_t = t(u + \gamma f(x_t)) + (I - t(I+\mu A))P_C S x_t.$$

If we take $\mu = 0$, $u = 0$ and $f = 0$ in (3.1), then we have

$$x_t = (1-t)P_C S x_t, \quad \forall t \in (0,1). \tag{3.2}$$

We summary the basic properties of the net $\{x_t\}$, which can be proved by the same method in [15].

Proposition 3.1. *Let $\{x_t\}$ be defined by the implicit algorithm* (3.1). *Then*

(i) *$\{x_t\}$ is bounded for $t \in (0, (1+\mu\|A\|)^{-1})$;*
(ii) *$\lim_{t\to 0}\|x_t - P_C S x_t\| = 0$;*
(iii) *x_t defines a continuous path from $(0, (1+\mu\|A\|)^{-1})$ in C.*

We provide the following result for the existence of solutions of the optimization problem (1.1).

Theorem 3.2. *The net $\{x_t\}$ defined by the implicit algorithm* (3.1) *converges strongly to a fixed point $\widetilde{x}$ of T as $t \to 0$, which solves the following variational inequality:*

$$\langle u + (\gamma f - (I+\mu A))\widetilde{x}, p - \widetilde{x}\rangle \le 0, \quad p \in F(T).$$

This $\widetilde{x}$ is a solution of the optimization problem (1.1).

From Theorem 3.2, we can deduce the following result.

Corollary 3.3. *The net* $\{x_t\}$ *defined by the implicit algorithm* (3.2) *converges strongly to a fixed point* $\widetilde{x}$ *of* T *as* $t \to 0$, *which solves the following minimization problem: find* $x^* \in F(T)$ *such that*

$$\|x^*\| = \min_{x \in F(T)} \|x\|.$$

Now, we propose the following iterative algorithm which generates a sequence $\{x_n\}$ in an explicit way:

$$x_{n+1} = \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu A))P_C S x_n, \quad n \geq 0, \tag{3.3}$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $x_0 \in C$ is selected arbitrarily.

First, we prove the following main result.

Theorem 3.4. *Let* $\{x_n\}$ *be a sequence in* C *generated by the iterative algorithm* (3.3), *and let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *which satisfies condition:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$.

Let LIM be a Banach limit. Then

$$LIM_n(\langle u + \gamma f(q) - (I + \mu A)q, x_n - q\rangle) \leq 0,$$

where $q = \lim_{t\to 0^+} x_t$ *with* x_t *being defined by the implicit algorithm* (3.1).

Now, using Theorem 3.4, we establish the strong convergence of the explicit algorithm (3.3) for finding a solution of the optimization problem (1.1).

Theorem 3.5. *Let* $\{x_n\}$ *be a sequence in* C *generated by the iterative algorithm* (3.3), *and let* $\{\alpha_n\}$ *be a sequence in* $(0, 1)$ *which satisfies conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $\{x_n\}$ *is weakly asymptotically regular, then* $\{x_n\}$ *converges strongly to* $q \in F(T)$, *which solves the optimization problem* (1.1).

Proof. First we note that from condition (C1), without loss of generality, we assume that $\alpha_n \leq (1 + \mu\|A\|)^{-1}$ and $\frac{2((1+\mu)\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}\alpha_n < 1$ for $n \geq 0$. Let $q = \lim_{t\to 0} x_t$ with x_t being defined by (3.1). Then we know from Theorem 3.2 that $q \in F(T)$, and q is unique solution of the optimization problem (1.1).

We divide the proof into three steps:

Step 1. We show that $\{x_n\}$ is bounded. Indeed, we know that $\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|u\| + \|\gamma f(p) - (I+\mu A)p\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}\right\}$ for all $n \geq 0$ and all $p \in F(T)$ in the proof of Theorem 3.2. Hence $\{x_n\}$ is bounded and so are $\{f(x_n)\}$, $\{P_C S x_n\}$ and $\{(I + \mu A)P_C S x_n\}$.

Step 2. We show that $\limsup_{n\to\infty}\langle u + \gamma f(q) - (I + \mu A)q, x_n - q\rangle \leq 0$, where $q = \lim_{t\to 0} x_t$ with x_t being defined by (3.1). To this end, put

$$a_n := \langle u + \gamma f(q) - (I + \mu A)q, x_n - q\rangle, \quad n \geq 1.$$

Then Theorem 3.2 implies that $LIM_n(a_n) \leq 0$ for any Banach limit LIM. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}(a_{n+1} - a_n) = \lim_{j\to\infty}(a_{n_j+1} - a_{n_j})$$

and $x_{n_j} \rightharpoonup v \in H$. This implies that $x_{n_j+1} \rightharpoonup v$ since $\{x_n\}$ is weakly asymptotically regular. Therefore, we have

$$w - \lim_{j\to\infty}(q - x_{n_j+1}) = w - \lim_{j\to\infty}(q - x_{n_j}) = (q - v),$$

and so

$$\limsup_{n\to\infty}(a_{n+1}-a_n)=\lim_{j\to\infty}\langle u+\gamma f(q)-(I+\mu A)q,(q-x_{n_j+1})-(q-x_{n_j})\rangle=0.$$

Then Lemma 2.7 implies that $\limsup_{n\to\infty} a_n \le 0$, that is,

$$\limsup_{n\to\infty}\langle u+\gamma f(q)-(I+\mu A)q, x_n-q\rangle \le 0.$$

Step 3. We show that $\lim_{n\to\infty}\|x_n-q\|=0$. To do this, set $\overline{A}=I+\mu A$. Indeed, from Lemma 2.2 and Lemma 2.6, we derive

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\alpha_n(u+\gamma f(x_n)-\overline{A}q)+(I-\alpha_n\overline{A})(P_CSx_n-q)\|\\ &\le \|(I-\alpha_n\overline{A})(P_CSx_n-q)\|^2+2\alpha_n\langle u+\gamma f(x_n)-\overline{A}q,x_{n+1}-q\rangle\\ &\le (I-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n-q\|^2\\ &\quad +2\alpha_n\gamma\langle f(x_n)-f(q),x_{n+1}-q\rangle+2\alpha_n\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle\\ &\le (1-(1+\mu)\overline{\gamma}\alpha_n)^2\|x_n-q\|^2\\ &\quad +2\alpha_n\gamma\alpha\|x_n-q\|\|x_{n+1}-q\|+2\alpha_n\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle\\ &\le (1-(1+\mu)\overline{\gamma})\alpha_n)^2\|x_n-q\|^2+\alpha_n\gamma\alpha[\|x_n-q\|^2+\|x_{n+1}-q\|^2]\\ &\quad +2\alpha_n\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle,\end{aligned}$$

that is,

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \frac{1-2(1+\mu)\overline{\gamma}\alpha_n+((1+\mu)\overline{\gamma})^2\alpha_n^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2\\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle\\ &= \left(1-\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right)\|x_n-q\|^2+\frac{((1+\mu)\overline{\gamma})^2\alpha_n^2}{1-\alpha_n\gamma\alpha}\|x_n-q\|^2\\ &\quad +\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle\\ &\le \left(1-\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\alpha_n\right)\|x_n-q\|^2+\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\times\\ &\quad \left(\frac{((1+\mu)\overline{\gamma})^2\alpha_n}{2((1+\mu)\overline{\gamma}-\gamma\alpha)}M_1+\frac{1}{(1+\mu)\overline{\gamma}-\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle\right)\\ &= (1-\lambda_n)\|x_n-q\|^2+\lambda_n\delta_n,\end{aligned}$$

where $M_1=\sup\{\|x_n-q\|^2:n\ge 0\}$, $\lambda_n=\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\alpha_n$ and

$$\delta_n=\frac{((1+\mu)\overline{\gamma})^2\alpha_n}{2((1+\mu)\overline{\gamma}-\gamma\alpha)}M_1+\frac{1}{(1+\mu)\overline{\gamma}-\gamma\alpha}\langle u+\gamma f(q)-\overline{A}q,x_{n+1}-q\rangle.$$

From conditions (C1) and (C2) and Step 2, it is easy to see that $\lambda_n\to 0$, $\sum_{n=0}^{\infty}\lambda_n=\infty$ and $\limsup_{n\to\infty}\delta_n\le 0$. Hence, by Lemma 2.5, we conclude $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 3.6. *Let $\{x_n\}$ be a sequence in C generated by the iterative algorithm* (3.8), *and let $\{\alpha_n\}$ be a sequence in $(0,1)$ which satisfies conditions:*

(C1) $\lim_{n\to\infty}\alpha_n=0$;
(C2) $\sum_{n=0}^{\infty}\alpha_n=\infty$.

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the optimization problem (1.1).

Remark 3.7. If $\{\alpha_n\}$ in Corollary 3.6 satisfies conditions (C1), (C2) and

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$; or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$; or

(C4) $|\alpha_{n+1} - \alpha_n| \le o(\alpha_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty} \sigma_n < \infty$ (the perturbed control condition),

then the sequence $\{x_n\}$ generated by the iterative algorithm (3.8) is asymptotically regular. Now, we give only the proof in case when $\{\alpha_n\}$ satisfies conditions (C1), (C2) and (C4). By Step 1 in the proof of Theorem 3.3, there exists a constant $L > 0$ such that for all $n \ge 0$,

$$\|\overline{A}P_C Sx_n\| + \gamma\|f(x_n)\| \le L.$$

So,we obtain, for all $n \ge 0$,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(I - \alpha_n\overline{A})(P_C Sx_n - P_C Sx_{n-1}) + (\alpha_n - \alpha_{n-1})\overline{A}P_C Sx_{n-1} \\
&\quad + \gamma[\alpha_n(f(x_n) - f(x_{n-1})) + f(x_{n-1})(\alpha_n - \alpha_{n-1})]\| \\
&\le (1 - \alpha_n(1+\mu)\overline{\gamma})\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\overline{A}P_C Sx_{n-1}\| \\
&\quad + \gamma[\alpha_n\alpha\|x_n - x_{n-1}\| + \|f(x_{n-1})\||\alpha_n - \alpha_{n-1}] \\
&\le (1 - \alpha_n((1+\mu)\overline{\gamma} - \gamma\alpha))\|x_n - x_{n-1}\| + L|\alpha_n - \alpha_{n-1}| \\
&\le (1 - \alpha_n((1+\mu)\overline{\gamma} - \gamma\alpha))\|x_n - x_{n-1}\| + (o(\alpha_n) + \sigma_{n-1})L.
\end{aligned} \tag{3.16}$$

By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\lambda_n = \alpha_n((1+\mu)\overline{\gamma} - \gamma\alpha)$, $\lambda_n\delta_n = o(\alpha_n)L$ and $r_n = \sigma_{n-1}L$, from (3.16) we have

$$s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n\delta_n + r_n.$$

Hence, by (C1), (C2), (C4) and Lemma 2.5, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

In view of this observation, we have the following:

Corollary 3.8. *Let $\{x_n\}$ be a sequence in C generated by the iterative algorithm* (3.8), *and let $\{\alpha_n\}$ be a sequence in* $(0,1)$ *which satisfies conditions* (C1), (C2) *and* (C4) *(or conditions* (C1), (C2) *and* (C3)*). Then $\{x_n\}$ converges strongly to $q \in F(T)$, which solves the optimization problem* (1.1).

From Theorem 3.5, we can also deduce the following result.

Corollary 3.9. *Let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = (1 - \alpha_n)P_C Sx_n, \quad \forall n \ge 0,$$

and let $\{\alpha_n\} \subset (0,1)$ be a sequence satisfying conditions (C1) *and* (C2). *If $\{x_n\}$ is weakly asymptotically regular, then $\{x_n\}$ converges strongly to a fixed point q of T as $n \to \infty$, which solves the following minimization problem: find $x^* \in F(T)$ such that*

$$\|x^*\| = \min_{x\in F(T)} \|x\|.$$

Remark 3.10. (1) In Remark 3.1, condition (C4) on $\{\alpha_n\}$ is independent of condition (C3), which was imposed by Cho *et al.* [5], Marino and Xu [15] and others. For this fact, see [6, 13].

(2) We point out the our iterative algorithms (3.1) and (3.8) are different from those in the recent works in this direction.

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