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A NEW SCALARIZATION APPROACH FOR SET OPTIMIZATION PROBLEMS
(集合最適化問題に対する新しいスカラー化手法)

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Abstract

In the paper, we introduce a scalarization function of sets, which is based on the Euclidean inner product and a base of ordering cone, and investigate their properties. Moreover, we consider an approximate efficient solution for set optimization problems based on a set-criterion, and show that this solution can be obtained by solving set optimization problems scalarized by our function. Furthermore, we prove that any sequence, which is defined by solution sets of scalarized set optimization problems, converges to an efficient solution of the original set optimization problems.

1 Introduction

Set optimization (or set-valued optimization) has been widely developed by many researchers. In these optimization problems, there are three types of solution concepts. First is based on a vector criterion, second is based on a set criterion, and last is a complete lattice approach. The second solution concept is presented by Kuroiwa [1]. The last solution concept is used in order to apply set optimization approach to mathematical finance (see [2]–[3]).

On the other hand, several scalarization approaches have been proposed as one of the important tools in vector optimization ([4]–[6] and references therein). Also, some researchers consider certain generalizations of those scalarization approaches and apply them in set optimization [7]–[9]. In 2006, Hamel and Löhne [10] proposed some new scalarization functions of sets based on two types of set-relations introduced in [11]. Some researchers investigate properties and applications of Hamel and Löhne type scalarization functions in set optimization ([12]–[16]). From the results in these papers, we obtain that solutions of scalarized optimization problems by these scalarization approaches are weak efficient solutions of the original set optimization problems. However, solutions of these scalarized optimization problems are not necessarily efficient solutions of the original set optimization problems. In particular, the error between these solutions and efficient solutions may be very large.

The aim of the paper is to propose a new scalarization function of sets and investigate its properties. Moreover, we define an approximate solution for set optimization problems based on Kuroiwa's approach, and show that this solution can be obtained by solving

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scalarized set optimization problems by our new function.

The organization of the paper is as follows. In Section 2, we introduce some basic concepts in set optimization. In Section 3, we introduce a new scalarization function of sets based on inner product, and investigate some properties of this function. In Section 4, we introduce an approximate solution for set optimization problems by using a set-relation in [11], and show that this solution can be obtained by solving optimization problems scalarized by a function introduced in Section 3.

2 Preliminary results

Firstly, we give the preliminary terminology and notation, which will be used in the paper. Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space where $n \in \mathbb{N}$, and let A, B be two nonempty sets in $\mathbb{R}^n$. We denote the origin of $\mathbb{R}^n$ by θ_n; the topological interior, topological closure, and complement of A by $\mathrm{int}A$, $\mathrm{cl}A$, and A^c, respectively; the product of $\alpha \in \mathbb{R}$ and A by $\alpha A := \{\alpha a | a \in A\}$; the algebraic sum, algebraic difference of A and B by $A + B := \{a + b | a \in A,\, b \in B\}$, $A - B := \{a - b | a \in A,\, b \in B\}$, respectively; the convex hull of A by $\mathrm{conv}A$. Moreover, we denote the set $\{(x_1, \ldots, x_n)^T \in \mathbb{R}^n | x_j \geq 0 \text{ for } j = 1, \ldots, n\}$ by $\mathbb{R}^n_+$; the family of all nonempty subsets of $\mathbb{R}^n$ by $\wp(\mathbb{R}^n)$; the family of all nonempty compact subsets of $\mathbb{R}^n$ by $C(\mathbb{R}^n)$.

Now we define the partial order on $\mathbb{R}^n$ by $\mathbb{R}^n_+$ as follows:

$$x \leq_{\mathbb{R}^n_+} y \text{ iff } y - x \in \mathbb{R}^n_+ \text{ for } x, y \in \mathbb{R}^n.$$

When $x \leq_{\mathbb{R}^n_+} y$ for $x, y \in \mathbb{R}^n$, we define the order interval with respect to $\mathbb{R}^n_+$ between x and y by $[x, y]_{\mathbb{R}^n_+} := \{z \in \mathbb{R}^n | x \leq_{\mathbb{R}^n_+} z \text{ and } z \leq_{\mathbb{R}^n_+} y\}$. When $x, y \in \mathbb{R}$, $[x, y]_{\mathbb{R}_+}$ is denoted by $[x, y]$. We say that $B \subset \mathbb{R}^n$ is a base of $\mathbb{R}^n_+$ iff B is convex and each $k \in \mathbb{R}^n_+ \setminus \{\theta_n\}$ has a unique representation of the form $k = \lambda b$ for some $\lambda > 0$ and $b \in B$.

Throughout the paper, $\mathbb{R}^n$ is the n-dimensional Euclidean space with the Euclidean norm $\|\cdot\|$, $D := \{x \in \mathbb{R}^m | F(x) \neq \emptyset\}$ is convex with nonempty topological interior, and $F : D \rightrightarrows \mathbb{R}^n$ is a set-valued map.

Definition 2.1 ([6]). Let $A \in \wp(\mathbb{R}^n)$. Then A is said to be $\mathbb{R}^n_+$-convex (resp., closed) iff $A + \mathbb{R}^n_+$ is convex (resp., closed). Also, we say that a set-valued map $F : D \rightrightarrows \mathbb{R}^n$ is $\mathbb{R}^n_+$-property valued on D if $F(x)$ has the $\mathbb{R}^n_+$-property for every $x \in D$.

Let $A \in \wp(\mathbb{R}^n)$. Then $a_0 \in A$ is said to be minimal element of A iff

$$(a_0 - \mathbb{R}^n_+) \cap A = \{a_0\}.$$

If $\mathbb{R}^n_+$ is replaced by $\mathrm{int}\mathbb{R}^n_+$, then it is called weak minimal element of A. We denote the set of all minimal (resp., weak minimal) elements of A by $\mathrm{Min}A$ (resp., $\mathrm{WMin}A$).

Now we consider two types of set-relation. Let $A_1, A_2 \in \wp(\mathbb{R}^n)$. Then we write

$$A_1 \leq^{(l)}_{\mathbb{R}^n_+} A_2 \quad \text{by} \quad A_2 \subseteq A_1 + \mathbb{R}^n_+.$$

$$A_1 \leq^{(u)}_{\mathbb{R}^n_+} A_2 \quad \text{by} \quad A_1 \subseteq A_2 - \mathbb{R}^n_+.$$

Based on these set-relations, Kuroiwa [1] proposes the following minimal element concepts of a family of sets. Let $\mathcal{A} \subseteq \wp(\mathbb{R}^n)$. Then $A_0 \in \mathcal{A}$ is said to be type $(*)$ minimal element of $\mathcal{A}$ iff for any $A \in \mathcal{A}$,

$$A \leq^{(*)}_{\mathbb{R}^n_+} A_0 \quad \text{implies} \quad A_0 \leq^{(*)}_{\mathbb{R}^n_+} A,$$

where $* = l, u$. From [17], it is enough to consider only the case of $\leq^{(l)}_{\mathbb{R}^n_+}$. Therefore, we call it minimal element and denote $\leq^{(l)}_{\mathbb{R}^n_+}$ by $\preceq_{\mathbb{R}^n_+}$ simply. If $\mathbb{R}^n_+$ is replaced by $\mathrm{int}\mathbb{R}^n_+$ then it is called weak minimal element of $\mathcal{A}$. We denote the family of all minimal (resp., weak minimal) elements of $\mathcal{A}$ by $\mathrm{Min}_l\mathcal{A}$ (resp., $\mathrm{WMin}_l\mathcal{A}$).

Next, let us recall convexity and continuity notions of set-valued map (see [6], [11]).

Definition 2.2. Let $F : D \rightrightarrows \mathbb{R}^n$ be a set-valued map. Then F is called

(i) $\mathbb{R}^n_+$-convex on D iff for any $x, y \in D$ and $\lambda \in [0, 1]$,

$$F(\lambda x + (1-\lambda)y) \preceq_{\mathbb{R}^n_+} \lambda F(x) + (1-\lambda)F(y);$$

(ii) upper continuous at $x \in D$ iff for any $V \in \wp(\mathbb{R}^n)$, which is an open set with $F(x) \subseteq V$, there exists an open neighborhood U_x of x such that $F(y) \subseteq V$ for any $y \in U_x$;

(iii) lower continuous at $x \in D$ iff for any $V \in \wp(\mathbb{R}^n)$, which is an open set with $F(x) \cap V \neq \emptyset$, there exists an open neighborhood U_x of x such that $F(y) \cap V \neq \emptyset$ for any $y \in U_x$;

(iv) continuous at $x \in D$ iff it is lower and upper continuous at $x \in D$.

Moreover, we say that F is upper continuous (resp., lower continuous, continuous) on D iff F is upper continuous (resp., lower continuous, continuous) at every $x \in D$.

It is easy to check that the following propositions hold.

Proposition 2.1. *Let $F : D \rightrightarrows \mathbb{R}^n$ be a set-valued map. Then the following statements hold:*

(i) *If F is $\mathbb{R}^n_+$-convex on D then F is $\mathbb{R}^n_+$-convex valued on D;*

(ii) *If F is $\mathbb{R}^n_+$-convex on D then $\bigcup_{x \in D} F(x)$ is $\mathbb{R}^n_+$-convex.*

Proposition 2.2 ([6]). *Let $A \subset D$ be a nonempty compact set and $F : A \rightrightarrows \mathbb{R}^n$. If F is compact-valued and upper continuous on A, then $\bigcup_{x \in A} F(x)$ is compact.*

3 Scalarization of sets

Let $i = 1, \ldots, n$, $k_j := (k_j^1, \ldots, k_j^n)^T \in \mathbb{R}^n$ a vector such that $k_j^j = 1$ and $k_j^i = 0$ for each $j \neq i$, and let $B := \mathrm{conv}\{k_j\}_{j=1}^n$. Then it is clear that B is a base of $\mathbb{R}^n_+$. At first, we recall the scalarization function of sets $\phi(\cdot, \cdot) : C(\mathbb{R}^n) \times B \to \mathbb{R}$ defined as follows:

$$\phi(A, k) := \inf_{a \in A} \langle a, k \rangle$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^n$.

Then we give the following lemmas.

Lemma 3.1 ([18]). *Let $A_1, A_2 \in C(\mathbb{R}^n)$. Then, the following statements hold:*

(i) *For any* $k \in B$,
$$\phi(A_1, k) := \min_{a \in A_1} \langle a, k \rangle.$$
(ii) *If* $A_1 \preceq_{\mathbb{R}^n_+} A_2$*, then* $\phi(A_1, k) \leq \phi(A_2, k)$ *for any* $k \in B$.
(iii) *If* $A_1 \preceq_{\mathrm{int}\mathbb{R}^n_+} A_2$*, then* $\phi(A_1, k) < \phi(A_2, k)$ *for any* $k \in B$.
(iv) *If the following two conditions are satisfied:*
  (i) A_2 *is* $\mathbb{R}^n_+$*-convex;*
  (ii) $A_2 \not\preceq_{\mathbb{R}^n_+} A_1$*;*
  then there exists a $k_0 \in B$ *such that* $\phi(A_1, k_0) < \phi(A_2, k_0)$.

Lemma 3.2 ([18]). *Let* $A \in C(\mathbb{R}^n)$*. Then* $\phi(A, \cdot)$ *is continuous on* B.

Let $K_j^m(\lambda) := \left\{ y \in \mathbb{R}^n \,\middle|\, \langle y, k_j \rangle \in \left[\frac{m-1}{\lambda}, \frac{m}{\lambda}\right] \right\}$ where $j = 1, \dots, n$, $\lambda \in \mathbb{N}$, and $m = 1, \dots, \lambda$. Then we define the sets $B_i(\lambda)$ as follows where $i = 1, \dots, \lambda^n$:

$$\begin{aligned}
B_1(\lambda) &:= K_1^1(\lambda) \cap \ldots \cap K_n^1(\lambda) \cap B \\
B_2(\lambda) &:= K_1^1(\lambda) \cap \ldots \cap K_n^2(\lambda) \cap B \\
&\vdots \\
B_\lambda(\lambda) &:= K_1^1(\lambda) \cap \ldots \cap K_n^\lambda(\lambda) \cap B \\
B_{\lambda+1}(\lambda) &:= K_1^1(\lambda) \cap \ldots \cap K_{n-1}^2(\lambda) \cap K_n^1(\lambda) \cap B \\
&\vdots \\
B_{2\lambda}(\lambda) &:= K_1^1(\lambda) \cap \ldots \cap K_{n-1}^2(\lambda) \cap K_n^\lambda(\lambda) \cap B \\
&\vdots \\
B_{\lambda^n}(\lambda) &:= K_1^\lambda(\lambda) \cap \ldots \cap K_{n-1}^\lambda(\lambda) \cap K_n^\lambda(\lambda) \cap B
\end{aligned}$$

Let A be a nonempty compact and $\mathbb{R}^n_+$-convex subset of $\mathbb{R}^n$. Based on these sets and Lemma 3.2, a scalarization function of sets Φ_λ is defined by

$$\Phi_\lambda(A) := \frac{1}{\lambda^n} \sum_{i=1}^{\lambda^n} \max_{k \in B_i(\lambda)} \phi(A, k)$$

where

$$\max_{k \in B_i(\lambda)} \phi(A, k) = \begin{cases} \max\limits_{k \in B_i(\lambda)} \min\limits_{a \in A} \langle k, a \rangle & (B_i(\lambda) \neq \emptyset), \\ 0 & (B_i(\lambda) = \emptyset). \end{cases}$$

Then, we give several properties of this function.

Lemma 3.3 ([18]). *Let* $A_1, A_2 \in C(\mathbb{R}^n)$*. Then the following statements hold:*

(i) $\Phi_\lambda(A_1) \in \mathbb{R}$ *for any* $\lambda \in \mathbb{N}$.
(ii) *If* $A_1 \preceq_{\mathbb{R}^n_+} A_2$ *then* $\Phi_\lambda(A_1) \leq \Phi_\lambda(A_2)$ *for any* $\lambda \in \mathbb{N}$.
(iii) *If* $A_1 \preceq_{\mathrm{int}\mathbb{R}^n_+} A_2$ *then* $\Phi_\lambda(A_1) < \Phi_\lambda(A_2)$ *for any* $\lambda \in \mathbb{N}$.
(iv) *If the following two conditions are satisfied:*
  (i) A_2 *is* $\mathbb{R}^n_+$*-convex;*

(ii) $A_1 \preceq_{\mathbb{R}^n_+} A_2$ *and* $A_2 \not\preceq_{\mathbb{R}^n_+} A_1$;
then there exists a $\bar{\lambda} \in \mathbb{N}$ *such that* $\Phi_\lambda(A_1) < \Phi_\lambda(A_2)$ *for all* $\lambda \geq \bar{\lambda}$.

(v) *There exists a* $r_{A_1} \in \mathbb{R}$ *such that*

$$\lim_{\lambda \to \infty} \Phi_\lambda(A_1) = r_{A_1}.$$

4 Main results

Let $F : D \rightrightarrows \mathbb{R}^n$ be a set-valued map. We consider the following set optimization problem:

$$(\text{SOP}) \begin{cases} \text{Minimize} & F(x) \\ \text{subject to} & x \in D. \end{cases}$$

We say that $\bar{x} \in D$ is an efficient (resp., weak efficient) solution of (SOP) iff $F(\bar{x})$ is a minimal (resp., weak minimal) element of the family of sets $F(D) := \{F(x) | x \in D\}$. Now we define an approximate solution of (SOP).

Definition 4.1. Let $F : D \rightrightarrows \mathbb{R}^n$ be a set-valued map and $\epsilon > 0$. Then $\bar{x} \in D$ is called ϵ-approximate solution of (SOP) iff for any $x \in D$,

$$F(x) \preceq_{\mathbb{R}^n_+} F(\bar{x}) \quad \text{implies} \quad F(\bar{x}) - \epsilon B \preceq_{\mathbb{R}^n_+} F(x).$$

We denote the family of sets $\{F(x) | x$ is an ϵ-approximate solution of (SOP)$\}$ by $\epsilon\text{Min}_l F(D)$; the set of all ϵ-approximate solution of (SOP) by $S_\epsilon(F)$.

To illustrate the notion above, we give the following simple examples.

Example 4.1. Let $X := [-1, 1]$. Then we consider the set-valued map $F : X \rightrightarrows \mathbb{R}^2$ defined by

$$F(x) := \text{conv}\left\{ \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} -x \\ -1 \end{pmatrix} \right\}.$$

Then

$$\text{Min}_l F(X) = \{F(1)\},$$

$$\text{WMin}_l F(X) = F(X),$$

$$\epsilon\text{Min}_l F(X) = \{F(x) | x \in [1 - \epsilon, 1]\}.$$

Example 4.2. Let $X := [0, 1]$. Then we consider the set-valued map $F : X \rightrightarrows \mathbb{R}$ defined by

$$F(x) := [x, x + 1].$$

Then

$$\text{Min}_l F(X) = \text{WMin}_l F(X) = \{0\},$$

$$\epsilon\text{Min}_l F(X) = \{F(x) | x \leq \epsilon\}.$$

Examples 4.1 and 4.2 show that any weak efficient solution (resp., ϵ-approximate solution) is not necessary ϵ-approximate solution (resp., weak efficient solution) for some $\epsilon > 0$.

In this section, we consider a scalarized optimization problem by Φ_λ, and show that an approximate solution of the original set optimization problem can be obtained by solving this problem. For this end, we give the following lemmas.

Lemma 4.1 ([17]). *Let A be a nonempty compact subset of D and $F : A \rightrightarrows \mathbb{R}^n$ a compact set-valued map. If F is upper continuous on A, then $\mathrm{Min}_l F(A) \neq \emptyset$. In particular, for each $a \in A$ there exists an $a' \in A$ satisfying with $F(a') \in \mathrm{Min}_l F(A)$ such that $F(a') \preceq_{\mathbb{R}^n_+} F(a)$.*

Lemma 4.2 ([18]). *Let A be a nonempty compact convex subset of D and $F : A \rightrightarrows \mathbb{R}^n$ a compact set-valued map. Then the following statements hold:*

(i) *If F is $\mathbb{R}^n_+$-convex on A, then $\Phi_\lambda(F)$ is convex on A for any $\lambda \in \mathbb{N}$.*
(ii) *If F is continuous on A, then $\Phi_\lambda(F)$ is continuous on A for any $\lambda \in \mathbb{N}$.*

By the proof of Theorem 3.2 in [19], we obtain the following lemma.

Lemma 4.3. *Let $A \subseteq \mathrm{int}D$ and $F : A \rightrightarrows \mathbb{R}^n$ a set-valued map. If F is compact and $\mathbb{R}^n_+$-convex on A, then for any $x \in A$ and $\epsilon > 0$ there exist a $k \in \mathrm{int}\mathbb{R}^n_+ \cap B$ and $M > 0$ such that*

$$F(y) \subset F(x) - M\|y - x\|k + \mathbb{R}^n_+$$

for any $y \in V_\epsilon(x) := \{y \in \mathbb{R}^m \mid \|y - x\| < \epsilon\}$.

Theorem 4.1 ([18]). *Let A be a nonempty compact convex subset of D with $\mathrm{int}A \neq \emptyset$ and $F : A \rightrightarrows \mathbb{R}^n$ a compact set-valued map. Assume that F is continuous and $\mathbb{R}^n_+$-convex on A, and $\bigcup \mathrm{Min}_l(F(X)) \subset \mathrm{int}A$. If $x_\lambda \in A$ is a solution to the following optimization problem:*

$$(\mathrm{SOP})_{\Phi_\lambda} \begin{cases} \mathrm{Min} & \Phi_\lambda(F(x)) \\ \text{subject to} & x \in A, \end{cases}$$

then x_λ is a weak efficient solution of (SOP). *In particular, if $x_\lambda \in \mathrm{int}A$ then there exists a $\epsilon_\lambda > 0$ such that the following statements hold:*

(i) *x_λ is an ϵ_λ-approximate solution of* (SOP),
(ii) *$\epsilon_\lambda \to 0$ as $\lambda \to \infty$.*

In [20], we consider a common solution of parametric vector optimization problems and give sufficient conditions for the existence of this solution by set optimization approach. Based on these results and Theorem 4.1, we obtain the following theorem.

Theorem 4.2. *Let A be a nonempty compact convex subset of D with $\mathrm{int}A \neq \emptyset$, T a compact subset of $\mathbb{R}_+$, $f : A \to \mathbb{R}^n$, $\mu : T \to \mathbb{R}^n$, and $g : A \times T \to \mathbb{R}^n$ defined by*

$$g(x,t) := f(x) + \mu(t).$$

Moreover, let $F : A \rightrightarrows \mathbb{R}^n$ be a set-valued map defined by

$$F(x) := g(x,T) = f(x) + \{\mu(t) : t \in T\}.$$

Assume that f is continuous and $\mathbb{R}^n_+$-convex on A, $\{\mu(t) : t \in T\}$ is compact and convex, and $\bigcup \mathrm{Min}_l(F(X)) \subset \mathrm{int}A$. Then there exists a $x_0 \in A$ with

$$\lim_{\lambda\to\infty} \Phi_\lambda(F(x_0)) = \min_{x\in A} \lim_{\lambda\to\infty} \Phi_\lambda(F(x)) \tag{4.1}$$

such that x_0 is a common solution of g, that is, for any $x \in A$ and $t \in T$,

$$g(x,t) \not\leq_{\mathbb{R}^n_+} g(x_0,t).$$

Proof. By Theorem 4.1, there exists a $x_0 \in A$ satisfying with (4.1). To use contradiction, we assume that x_0 is not a common solution of g. Then there exist a $t_0 \in T$ and $\bar{x} \in A$ such that

$$g(\bar{x},t_0) \leq_{\mathbb{R}^n_+} g(x_0,t_0) \quad \text{and} \quad g(x_0,t_0) \not\leq_{\mathbb{R}^n_+} g(\bar{x},t_0).$$

Since $\{\mu(t) : t \in T\}$ is compact, we have

$$F(\bar{x}) \preceq_{\mathbb{R}^n_+} F(x_0) \quad \text{and} \quad F(x_0) \not\preceq_{\mathbb{R}^n_+} F(\bar{x}).$$

By Lemma 3.3, there exists a $\lambda_0 \in \mathbb{N}$ such that $\Phi_\lambda(F(\bar{x})) < \Phi_\lambda(F(x_0))$ for any $\lambda \geq \lambda_0$. This contradicts (4.1). Therefore, x_0 is a common solution of g. □

5 Conclusion

In the paper, by using ϕ, which is a scalarization function of sets based on the inner product, we propose a new scalarization function of sets Φ_λ where $\lambda \in \mathbb{N}$, and investigate their properties. Moreover, we consider convex set optimization problems with a set-relation $\preceq_{\mathbb{R}^n_+}$, and introduce an approximate solution for (SOP). In Theorem 4.1, we prove that some convex set optimization problems can be replaced by scalar optimization problems by Φ_λ. Also, Theorem 4.2 shows that our scalarization function is useful to find a common solution of parametric convex vector optimization problems.

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