数理解析研究所講究録
第 1964 巻 2015 年 46-54

HNN-extensions, amalgamated free products and their group rings

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A ring R is (right) primitive provided it has a faithful irreducible (right) R-module, or equivalently, there exists a maximal right ideal in R which includes no non-trivial ideal of R. In the present note, we improve or generalize [1] and [6]; we give primitivity of group rings of amalgamated free products and HNN-extensions of groups.

1 Introduction

A ring R is (right) primitive provided it has a faithful irreducible (right) R-module. If a non-trivial group G is finite or abelian, then the group ring KG over a field K is never primitive. The first example of a primitive group ring was offered by Formanek and Snider [5] in 1972. After that, many examples of primitive group rings were constructed. In 1978, Domanov [2], Farkas-Passman [3], and Roseblade [10] gave the complete solution to primitivity of group rings of polycyclic-by-finite groups. Such groups belong to the class of noetherian groups. It is not easy to find a noetherian group which is not polycyclic-by-finite [9]. Therefore, almost all other known infinite groups belong to the class of non-noetherian groups. A group of the class of finitely generated non-noetherian groups has often non-abelian free subgroups; for instance, a free group, a locally free group, a free product, an amalgamated free product, an HNN-extension, a Fuchsian group, a one relator group, etc. Primitivity of group rings of non-noetherian groups have been obtained gradually (in 1973 [4], in 1989 [1], in 2007 [6], in 2011 [7]). However, much of them remains unknown.

In [8], we considered the following condition:

(*) For each subset M of G consisting of finite number of elements not equal to 1, there exist three distinct elements a, b, c in G such that whenever $x_i \in \{a, b, c\}$ and $(x_1^{-1}g_1x_1)\cdots(x_m^{-1}g_mx_m) = 1$ for some $g_i \in M$, $x_i = x_{i+1}$ holds for some i,

*Partially supported by Grants-in-Aid for Scientific Research under grant no. 26400055

and gave the following theorem:

Theorem 1.1. *Let G be a non-trivial group which has a free subgroup whose cardinality is the same as that of G. Suppose that G satisfies the condition $(*)$:*

If R is a domain with $|R| \leq |G|$, then the group ring RG of G over R is primitive.

In particular, the group algebra KG is primitive for any field K.

In order to prove the above theorem, we construct a maximal right ideal in KG which includes no non-trivial ideal of KG. We need then to show that the constructed right ideal is proper. To do this, we use an elementary graph-theoretic method. That is, we define an SR-graph and a SR-cycle. Then the proof of Theorem 1.1 can be reduced to finding an SR-cycle in a given SR-graph (See the next section for the details).

In the present note, as an application of the above theorem, we improve or generalize [1] and [6]; we give primitivity of group algebras of amalgamated free products and HNN-extensions of groups.

2 SR-graphs

Let $\mathcal{G} = (V, E)$ denote a simple graph; a finite undirected graph which has no multiple edges or loops, where V is the set of vertices and E is the set of edges. A finite sequence $v_0 e_1 v_1 \cdots e_p v_p$ whose terms are alternately elements e_q's in E and v_q's in V is called a path of length p in $\mathcal{G}$ if $v_q \neq v_{q'}$ for any $q, q' \in \{0, 1, \cdots, p\}$ with $q \neq q'$; it is often simply denoted by $v_0 v_1 \cdots v_p$. Two vertices v and w of $\mathcal{G}$ are said to be connected if there exists a path from v to w in $\mathcal{G}$. Connection is an equivalence relation on V, and so there exists a decomposition of V into subsets C_i's $(1 \leq i \leq m)$ for some $m > 0$ such that $v, w \in V$ are connected if and only if both v and w belong to the same set C_i. The subgraph (C_i, E_i) of $\mathcal{G}$ generated by C_i is called a (connected) component of $\mathcal{G}$. Any graph is a disjoint union of components. For $v \in V$, we denote by $C(v)$ the component of $\mathcal{G}$ which contains the vertex v.

We define a graph which has two distinct edge sets E and F on the same vertex set V. We call such a triple (V, E, F) an SR-graph provided that $(V, E \cup F)$ is a simple graph (i.e. a finite undirected graph which has no multiple edges or loops) and every component of the graph (V, E) is a complete graph (see Fig 1 and Fig 2). That is, we define an SR-graph as follows:

Definition 2.1. *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$ be simple graphs with the same vertex set V. For $v \in V$, let $U(v)$ be the set consisting of all neighbours of v in $\mathcal{H}$*

and v itself: $U(v) = \{w \in V \mid vw \in F\} \cup \{v\}$. A triple (V, E, F) is an SR-graph (for a sprint relay like graph) if it satisfies the following conditions:

(SR1) *For any $v \in V$, $C(v) \cap U(v) = \{v\}$.*
(SR2) *Every component of $\mathcal{G}$ is a complete graph.*

If $\mathcal{G}$ has no isolated vertices, that is, if $v \in V$ then $vw \in E$ for some $w \in V$, then SR-graph (V, E, F) is called a proper SR-graph.

We call $U(v)$ the SR-neighbour set of $v \in V$, and set $\mathfrak{U}(V) = \{U(v) \mid v \in V\}$. For $v, w \in V$ with $v \neq w$, it may happen that $U(v) = U(w)$, and so $|\mathfrak{U}(V)| \leq |V|$ generally. Let $\mathcal{S} = (V, E, F)$ be an SR-graph. We say $\mathcal{S}$ is connected if the graph $(V, E \cup F)$ is connected.

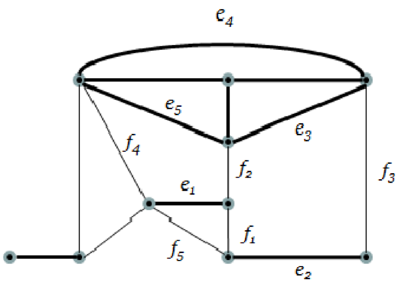

Fig 1. An example of an SR-graph: bold solid lines are edges in E and normal solid lines are edges in F. Sequences $(e_1, f_1, e_2, f_3, e_4, f_4)$, $(e_1, f_2, e_3, f_3, e_2, f_5)$ and (e_1, f_2, e_5, f_4) are SR-cycles.

Fig 2. Prohibits: It is not allowed to exist the above subgraph in an SR-graph.

Definition 2.2. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $p > 1$. Then a path $v_1 w_1 v_2 w_2, \cdots, v_p w_p v_{p+1}$ in the graph $(V, E \cup F)$ is called a SR-path of length p in $\mathcal{S}$ if either $e_q = v_q w_q \in E$ and $f_q = w_q v_{q+1} \in F$ or $f_q = v_q w_q \in F$ and $e_q = w_q v_{q+1} \in E$ for $1 \leq q \leq p$; simply denoted by $(e_1, f_1, \cdots, e_p, f_p)$ or $(f_1, e_1, \cdots, f_p, e_p)$, respectively. If, in addition, it is a cycle in $(V, E \cup F)$; namely, $v_{p+1} = v_1$, then it is an SR-cycle of length p in $\mathcal{S}$.*

To prove Theorem 1.1, we use two results for SR-graphs (Theorem 2.4 and Theorem 2.5) and apply them to the Formanek's method. We can give Formanek's method, as follows:

Proposition 2.3. (See [4]) *Let RG be the group ring of a group G over a ring R with identity. If for each non-zero $a \in RG$, there exists an element $\varepsilon(a)$ in the ideal $RGaRG$ generated by a such that the right ideal $\rho = \sum_{a \in RG \setminus \{0\}} (\varepsilon(a) + 1) RG$ is proper; namely, $\rho \neq RG$, then RG is primitive.*

The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make ρ be proper. Now, ρ is proper if and only if $r \neq 1$ for all $r \in \rho$. Since ρ is generated by the elements of form $(\varepsilon(a)+1)$ with $a \neq 0$, r has the presentation, $r = \sum_{(a,b)\in\Pi}(\varepsilon(a)+1)b$, where Π is a subset which consists of finite number of elements of $RG \times RG$ both of whose components are non-zero. Moreover, $\varepsilon(a)$ and b are linear combinations of elements of G, and so we have

$$r = \sum_{(a,b)\in\Pi} \sum_{g\in S_a, h\in T_b} (\alpha_g \beta_h gh + \beta_h h), \tag{1}$$

where S_a and T_b are the support of $\varepsilon(a)$ and b respectively and both α_g and β_h are elements in K. In the above presentation (1), if there exists gh such that $gh \neq 1$ and does not coincide with the other $g'h'$'s and h''s, then $r \neq 1$ holds. Strictly speaking: Let $\Omega_{ab} = S_a \times T_b$. If there exist $(a,b) \in \Pi$ and (g,h) in Ω_{ab} with $gh \neq 1$ such that $gh \neq g'h'$ and $gh \neq h'$ for any $(c,d) \in \Pi$ and for any (g',h') in Ω_{cd} with $(g',h') \neq (g,h)$, then $r \neq 1$ holds.

On the contrary, if $r = 1$, then for each gh in (1) with $gh \neq 1$, there exists another $g'h'$ or h' in (1) such that either $gh = g'h'$ or $gh = h'$ holds. Suppose here that there exist (g_{2i-1}, h_i) and (g_{2i}, h_{i+1}) $(i = 1, \cdots, m)$ in $V = \bigcup_{(a,b)\in\Pi} \Omega_{ab} \cup T_b$ such that the following equations hold:

$$\begin{array}{l} g_1h_1 = g_2h_2, \\ \qquad g_3h_2 = g_4h_3, \\ \qquad\qquad \ddots \\ \qquad\qquad g_{2m-1}h_m = g_{2m}h_{m+1} \quad \text{and} \quad h_{m+1} = h_1. \end{array} \tag{2}$$

Eliminating h_i's in the above, we can see that these equations imply the equation $g_1^{-1}g_2 \cdots g_{2m-1}^{-1}g_{2m} = 1$. If we can choose $\varepsilon(a)$'s so that their supports g_i's never satisfy such an equation, then we can prove that $r \neq 1$ holds by contradiction. We need therefore only to see when supports g's of $\varepsilon(a)$'s satisfy equations as described in (2).

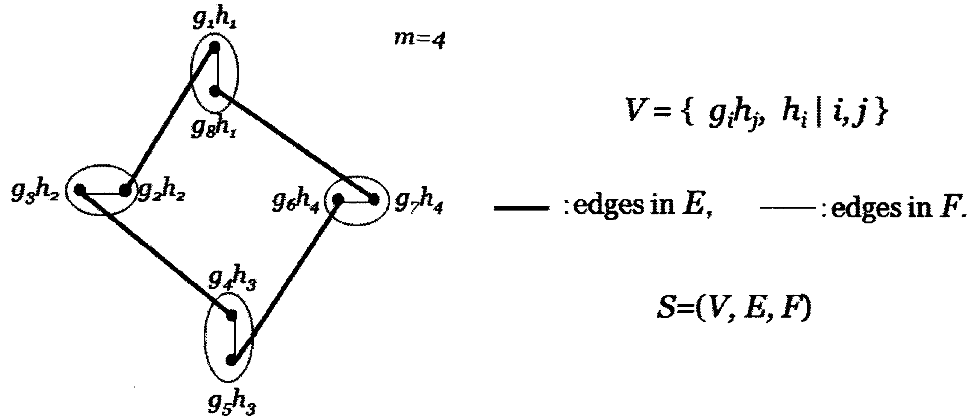

Fig 3. Equations as described in (2) for m=4.

Roughly speaking, we regard V above as the set of vertices and for $v = (g, h)$ and $w = (g', h')$ in V, we take an element vw as an edge in E provided $gh = g'h'$ in G, and take vw as an edge in F provided $g \neq g'$ and $h = h'$ in G (see Fig 3). In this situation, if there exists an SR-cycle $v_1w_1v_2w_2, \cdots, v_pw_pv_1$ in the SR-graph (V, E, F) whose adjacent terms are alternately elements v_iw_i in E and w_iv_{i+1} in F, then there exist (g_i, h_j)'s in V satisfying the desired equations as described in (2). Thus the problem can be reduced to find an SR-cycle in a given SR-graph.

By making use of graph theoretic considerations, we can prove the following theorems:

Theorem 2.4. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and let ω_E and ω_F be, respectively, the number of components of $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V, F)$. Suppose that every component of $\mathcal{H} = (V, F)$ is a complete graph and $\mathcal{S}$ is connected. Then $\mathcal{S}$ has an SR-cycle if and only if $\omega_E + \omega_F < |V| + 1$.*

In particular, if $\mathcal{S}$ is proper and $\alpha \leq \gamma$ then $\mathcal{S}$ has an SR-cycle.

Theorem 2.5. *Let $\mathcal{S} = (V, E, F)$ be an SR-graph and $\mathfrak{C}(V) = \{V_1, \cdots, V_n\}$ with $n > 0$. Suppose that every component $\mathcal{H}_i = (V_i, F_i)$ of $\mathcal{H}$ is a complete k-partite graph with $k > 1$, where k is depend on $\mathcal{H}_i$. If $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in \{1, \cdots, n\}$ and $|I_{\mathcal{G}}(V)| \leq n$ then $\mathcal{S}$ has an SR-cycle.*

3 Amalgamated free products

In what follows in this section, let $A *_H B$ be the free product of A and B with H amalgamated, and suppose that $A \neq H \neq B$.

For $x \in A *_H B$ with $x \notin H$ and for $u_i \in (A \cup B) \setminus H$ $(i = 1, \cdots, n)$, $x = u_1 \cdots u_n$ is a normal form for x provided u_i and u_{i+1} are not both in A or not both in B. Although a normal form $x = u_1 \cdots u_n$ is not unique, the length n of x is well defined and it is denoted here by $l(x)$. If $x \in H$, we define $l(x) = 0$. For $x, V_1, \cdots, V_m \in A *_H B$, we write $x \equiv V_1 \cdots V_m$ and say that the product $V_1 \cdots V_m$ is a reduced form provided that $x = V_1 \cdots V_m$ and $l(x) = l(V_1) + \cdots + l(V_m)$.

Let KG be the group algebra of a group G over a field K. In 1973, Formanek gave the primitivity of KG of the free product $G = A * B$:

Theorem 3.1. *(Formanek[4]) Let A and B be non-trivial groups, and $G = A * B$ the free product of A and B. If $G \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$, then KG is primitive for any field K.*

In 1989, Balogn [1] showed the following result for amalgamated free products:

Theorem 3.2. *(Balogn [1]) Let A and B be non-trivial groups, and $G = A *_H B$ the free product of A and B with H amalgamated. If there exist $a \in A$ and $b \in B$ with $a^2 \notin A$ and $b^2 \notin B$ such that $\langle ab\acute{a}, bab \rangle$ is free, $a^{-1}Ha \cap H = 1$ and $b^{-1}Hb \cap H = 1$, then KG is primitive for any field K.*

If $H = 1$ in the above then Theorem 3.2 needs the condition $A \neq \mathbb{Z}_2$ and $B \neq \mathbb{Z}_2$ for KG to be primitive, and so the above result is not complete generalization of Theorem 3.1. As a complete generalization of Theorem 3.1, we can get the following theorem:

Theorem 3.3. *Let A and B be non-trivial groups, and $G = A *_H B$ the free product of A and B with H amalgamated. If $B \neq H$ and there exist $a \in A$ with $a^2 \notin A$ such that $a^{-1}Ha \cap H = 1$, then KG is primitive for any field K.*

In order to prove above theorem, we need the following lemma:

Lemma 3.4. *Let $G = A *_H B$ the free product of A and B with H amalgamated. If $B \neq H$ and there exist $a \in A$ with $a^2 \notin A$ such that $a^{-1}Ha \cap H = 1$, then G satisfies the condition $(*)$.*

Proof. Let $1 \neq f \in G$ with $l(f) = l$. If a normal form for f begins with an element in $A \setminus H$ and ends with an element in $B \setminus H$, then we say that f is of type AB. Similarly, we define the types BA, AA and BB. If $l > 0$ then f is of type one of the above four types.

Let a be an element in A with $a^2 \notin A$ such that $a^{-1}Ha \cap H = 1$. For finite number of elements $f_1, \cdots .f_n$ with $f_i \neq f_j$ for $i \neq j$ in G, we set

$$x_i = (b^{-1}a)^{\omega_i} ab^{-1}a^{-1}(b^{-1}a)^{\omega_i},$$

where $\omega_i = l + i$ for $i \in \{1, 2, 3\}$ and l is the maximum number in the set $\{l(f_i) \mid 1 \leq i \leq n\}$.

Let $g_{ip} = x_i^{-1} f_p x_i$ $(p = 1, \cdots, n)$. We see then that for each $i \in \{1, 2, 3\}$ and each $p \in \{1, 2, \cdots, n\}$, a reduced form of $W_{ip} = (a^{-1}b)^{\omega_i} f_p (b^{-1}a)^{\omega_i}$ has the form either $W_{ip} \equiv (b^{-1}a)^{\pm k}$ for some $k > 0$ or $W_{ip} \equiv (a^{-1}b)V_{ip}(b^{-1}a)$ for some non-empty word V_{ip}. In either case, since $a^2 \in A \setminus H$, a normal form of $a^{-1}W_{ip}a$ is of type AA. We have then that

$$g_{ip} \equiv X_i^{-1} A_{ip} X_i, \tag{3}$$

where $X_i = b^{-1}a^{-1}(b^{-1}a)^{\omega_i}$ and $A_{ip} = a^{-1}W_{ip}a$. If $i \neq j$, say $i > j$, then a normal form of $X_i X_j^{-1}$ is $b^{-1}a^{-1}(b^{-1}a)^{\omega_i - \omega_j - 1} b^{-1}a^2 b$ which is of type BB. Therefore we have

$$g_{ip} g_{jq} \equiv X_i^{-1} A_{ip} B_{ij} A_{jq} X_j, \tag{4}$$

where $B_{ij} = b^{-1}a^{-1}(b^{-1}a)^{\omega_i - \omega_j - 1} b^{-1}a^2 b$.

Now, let $g = g_1 \cdots g_k$ be the product of any finite number of elements g_i's in $\bigcup_{j=1}^{3} M^{x_j}$, where $M^{x_j} = \{x_j^{-1} f_i x_j \mid 1 \leq i \leq n\}$. Since a reduced form of g_i has the form (3), if both of g_i and g_{i+1} are not in the same M^{x_j} for any i, then by noting that a reduced form of $g_i g_{i+1}$ has the form (4), it can be easily seen by induction on k that $g \equiv X_1^{-1} U X_k$ holds for some non-empty word U in G. Hence, in particular, $g \neq 1$. This completes the proof of the lemma. □

Proof of Theorem 3.3. By virtue of Lemma 3.4, we need only to show that G has a free subgroup whose cardinality is the same as that of G. Let I be a set with $|I| = |G|$, and let $a \in A \setminus H$ such that $a^{-1} H a \cap H = 1$ and $b \in B \setminus H$. If $|A \setminus H| = |G|$ (resp. $|B \setminus H| = |G|$), then for each $i \in I$, there exists $a_i \in A \setminus H$ (resp. $b_i \in B \setminus H$) such that $a_i \neq a_j$ (resp. $b_i \neq b_j$) for $i \neq j$. We have then that the subgroup of G generated by $a_i b (ab)^2 a_i b$ (resp. $(ab_i)^3$) $(i \in I)$ is freely generated by them. On the other hand, if $|H| = |G|$, then for each $i \in I$, there exists $h_i \in H$ such that $h_i \neq h_j$ for $i \neq j$. We set $M_1 = \{\, x_i^{\pm 1}, x_i^{-1} x_j \mid i, j \in I, i \neq j\}$ and $M_2 = \{\, y_i^{\pm 1}, y_i^{-1} y_j \mid i, j \in I, i \neq j\}$ where $x_i = a^{-1} h_i a$ and $y_i = b^{-1} a^{-1} h_i a b$. It is obvious that for each finite number of elements $g_1, \cdots, g_m$ in $M_1 \cup M_2$, whenever $g_1 \cdots g_m = 1$, both g_i and g_{i+1} are in the same M_j for some i and j. Hence, it easily follows that the subgroup of G generated by $z_i = x_i y_i^{-1}$ $(i \in I)$, is freely generated by them. □

4 HNN-extensions of groups

Let G be a group, and let A and B be subgroups of G with an isomorphism $\varphi : A \longrightarrow B$. Then HNN extension of G relative to A, B and φ is the group

$$G^* = \langle G, t \mid t^{-1} a t = \varphi(a), a \in A \rangle.$$

The group G is called the base of G^*, t is called the stable letter, and A and B are called the associated subgroups. If $A = G$ then $G^* = G_\varphi$ is called the ascending HNN extension of G determined by φ.

In [6], the present author showed the following result, which has been generalized to arbitrary cardinal case in [7]:

Theorem 4.1. *Let F be a nonabelian free group, and F_φ the ascending HNN extension of F determined by φ.*
(i) *In case $\varphi(F) = F$, the group ring KF_φ is primitive for a field K if and only if either $|K| \leq |F|$ or F_φ is not virtually the direct product $F \times \mathbb{Z}$.*
(ii) *In case $\varphi(F) \neq F$, if the rank of F is at most countably infinite, then the group ring KF_φ is primitive for any field K.*

To prove the above theorem, the main difficulty was to prove (ii), and it can be easily done by Theorem 1.1 as follows.

Let F_i be the subgroup of F_φ generated by $\{t^i f t^{-i} \mid f \in F\}$, and $F_\infty = \bigcup_{i=1}^{\infty} F_i$. Since F_∞ is a normal subgroup of F_φ, it suffices to show KF_∞ is primitive. Let $f_1, \cdots, f_n$ be finite number of elements in F_∞ with $f_i \neq f_j$ for $i \neq j$. Then there exists $k > 0$ such that $f_i \in F_k$ for all $i = 1, \cdots, n$. Since F_k is a non-abelian free group, there exists a base X with $|X| > 1$ such that $F_k = \langle X \rangle$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, and let m be the maximum length of the words in $\{f_1, \cdots, f_n\}$ on X, where the length of a word v is defined for the reduced word equivalent to v on X. We set $z_l = x_1^{2m+l} x_2 x_1^{2m+l}$, where $l = 1, 2, 3$. Then it is easily verified that the above z_1, z_2, z_3 satisfy $(*)$ for $f_i \neq f_j$. Hence the conclusion follows from Theorem 1.1.

Now, as we saw just above, a non-abelian free group always satisfies $(*)$. In the same way as above, we can have the following result generally:

Theorem 4.2. *Let G be a group, and let $G^* = \langle G, t \mid t^{-1}at = \varphi(a), a \in A \rangle$ be the HNN extension of G relative to A, B and φ, where A and B are subgroups of G with $B \neq G$ and φ is an isomorphism $\varphi : A \longrightarrow B$.*
(i) *If $A \neq G$ then KG^* is primitive for any field K.*
(ii) *If $A = G$ and G satisfies the condition $(*)$, then KG^* is primitive for any field K.*

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