Centralizing Monoids on a Three-Element Set
Related to Majority Functions

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Abstract

We consider commutation between multi-variable functions defined over a fixed non-empty set $A$. A centralizing monoid $M$ is a set of unary functions which commute with all members of some given set $F$ of functions, where $F$ is called a witness of $M$. In this paper, we focus on the case where $A$ is a three-element set and determine all centralizing monoids which have ternary majority functions as their witnesses.

Keywords: clone; centralizing monoid; majority function

1 Introduction

For a fixed non-empty set $A$ we consider multi-variable functions defined on $A$. A centralizer is the set of functions which commute with all members of some set $F$ of functions and a centralizing monoid $M$ is the unary part of some centralizer, that is, the set of unary functions of some centralizer. (Precise definitions will appear in Section 2.) The set $F$ is called a witness of the centralizing monoid $M$. Centralizers and centralizing monoids have been studied for many years, sometimes under different names, by various authors (e.g., [Da79], [Sza85], etc.).

As a continuation of our previous work ([MR11] and [MR12]), we restrict our attention to the case where $A$ is a three-element set. We choose the class of ternary majority functions and determine all centralizing monoids which have sets of ternary majority functions as their witnesses. It turns out that there exist considerably small number of centralizing monoids having majority functions as their witnesses. Note that the preliminary version of this paper appeared in [GMR15].

2 Definitions and Basic Facts

Let $A$ be a finite set with $|A| > 1$. By an $n$-variable function defined over $A$ we mean a map from $A^n$ into $A$. As usual, we denote by $O_A^{(n)}$, for $n \geq 1$, the set of $n$-variable functions defined over $A$, and by $O_A$ the set of functions defined over $A$, i.e., $O_A = \bigcup_{n=1}^{\infty} O_A^{(n)}$. A function $e_i^n \in O_A^{(n)}$, for $1 \leq i \leq n$, is called $n$-variable $i$-th projection if it satisfies $f(x_1, \ldots, x_i, \ldots, x_n) = x_i$ for any $(x_1, \ldots, x_n) \in A^n$. We denote by $J_A$ the set of projections over $A$.

Given $f \in O_A^{(n)}$ and $g \in O_A^{(m)}$, for $m, n \geq 1$, $f$ commutes with $g$, or $f$ and $g$ commute, if the following equation holds for all $x_{11}, \ldots, x_{1n}, \ldots, x_{mn} \in A$.

$$f(g(x_{11}, \ldots, x_{1m}), \ldots, g(x_{n1}, \ldots, x_{nm})) = g(f(x_{11}, \ldots, x_{1n}), \ldots, f(x_{m1}, \ldots, x_{mn}))$$
We write $f \perp g$ when $f$ commutes with $g$. The relation $\perp$ on $O_A$ is obviously symmetric.

In this paper we are concerned with commutation where one of the functions is unary, e.g., $f \in O_A^{(n)}$ and $g \in O_A^{(1)}$. In such case, $f$ and $g$ commute if $f(g(x_1), \ldots, g(x_n)) = g(f(x_1, \ldots, x_n))$ holds for all $x_1, \ldots, x_n \in A$.

For a subset $F$ of $O_A$ the set of functions in $O_A$ which commute with all members of $F$ is denoted by $F^*$, i.e.,

$$F^* = \{ g \in O_A \mid g \perp f \text{ for all } f \in F \}.$$ 

When $F = \{ f \}$ we often write $f^*$ instead of $F^*$. We also write $F^{**}$ for $(F^*)^*$. A subset $C$ of $O_A$ is a centralizer if there is a subset $F \subseteq O_A$ satisfying $C = F^*$. Evidently, for any $F \subseteq O_A$ the centralizer $F^*$ is a clone.

A non-empty subset $M$ of $O_A^{(1)}$ is a monoid if it is closed under composition and contains the identity. The unary part of any centralizer $F^*$, i.e., $F^* \cap O_A^{(1)}$, is a monoid.

For a subset $M$ of $O_A^{(1)}$, the following conditions (i) and (ii) are equivalent to each other.

1. $M = M^{**} \cap O_A^{(1)}$
2. There exists a subset $F$ of $O_A$ such that $M = F^* \cap O_A^{(1)}$

Remark For an algebra $A = (A; F)$, a unary function $g \in O_A^{(1)}$ is called an endomorphism of $A$ if $f \perp g$ holds for all $f \in F$. Denote by $\text{End}(A)$ the set of endomorphisms of $A$. The conditions above are equivalent to the condition that there exists an algebra $A = (A; F)$ such that $M = \text{End}(A)$.

A subset $M$ of $O_A^{(1)}$ is a centralizing monoid if $M$ satisfies the above conditions. (In this case, $M$ is obviously a monoid.) By Item (ii) above, a centralizing monoid is the unary part, i.e., the subset consisting of all unary functions, of some centralizer. Henceforth, $F$ in Item (ii) is called a witness of a centralizing monoid $M$. The centralizing monoid $M$ is denoted by $M(F)$ if $F$ is a witness of $M$. When $F$ is $\{ f \}$ we simply write $M(f)$ for $M(\{ f \})$.

A centralizing monoid $M$ is maximal if $M$ is not $O_A^{(1)}$ and there is no centralizing monoid $M'$ satisfying $M \subset M' \subset O_A^{(1)}$. (The symbol $\subset$ denotes proper inclusion.)

Some basic facts on centralizing monoids and witnesses follow. Recall that a function $f$ in $O_A$ is called a minimal function if (i) $f$ generates a minimal clone $C$ and (ii) $f$ has the minimum arity among functions generating $C$.

Proposition 2.1 ([MR11])

1. Every centralizing monoid has a finite subset $F$ of $O_A$ as its witness.
2. Every maximal centralizing monoid has a singleton set as its witness. Moreover, this singleton set can be a set consisting of a minimal function.

Now we review some known results on $E_3$. From now on, we concentrate on the case where the base set $A$ is $E_3 = \{0, 1, 2\}$. We write $O_A^{(n)}$ and $O_3$ for $O_A^{(n)}$ and $O_A$, respectively.

The set of all minimal clones on $E_3$ was described by B. Csákány [Cs83]. There are 84 such clones. Among them, 7 minimal clones are generated by ternary majority functions. As usual, a majority function is defined as follows: For a ternary function $f$, $f$ is a majority function if $f(x, x, y) = f(y, x, x) = x$ holds for all $x, y \in E_3$.

We have determined all maximal centralizing monoids on $E_3$ in [MR11]. There are 10 maximal centralizing monoids on $E_3$. More precisely:

1. For every constant function $c$ in $O_3^{(1)}$, $M(c)$ is a maximal centralizing monoid.
2. For every minimal majority function $m$ in $O_3^{(3)}$, $M(m)$ is a maximal centralizing monoid.

(The number of such maximal centralizing monoids is 7.)
Proof First, assume \( m \) and, hence, \( m \in E \). In this paper, we shall determine all centralizing monoids on \( E \) however, the questions still remain open. These facts motivate us to study centralizing monoids with majority functions as their witnesses. In this paper, we shall determine all centralizing monoids on \( E_3 \) with a set of ternary majority functions as their witnesses.

3 Majority Functions as Witness

In this section we shall determine all centralizing monoids on \( E_3 \) which have witnesses consisting only of ternary majority functions.

3.1 Notation

(1) The set of triples on \( E_3 \) consisting of mutually distinct components will be denoted by \( \sigma_3 \), i.e., \( \sigma_3 = \{(a, b, c) \in (E_3)^3 \mid \{a, b, c\} = E_3\} \).

(2) A unary function \( s \) on \( E_3 \) is denoted by \( s_{abc} \) if \( s(0) = a, s(1) = b, s(2) = c \) for \( a, b, c \in E_3 \).

(3) We divide the set of 2-valued unary functions in \( O_3^{(1)} \) into three groups, \( T_0, T_1 \) and \( T_2 \), as follows:

\[
T_0 = \{s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221}\}
\]

\[
T_1 = \{s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212}\}
\]

\[
T_2 = \{s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122}\}
\]

3.2 Properties for Commutation

3.2.1 Constants

It is evident that every constant function commutes with all majority functions.

3.2.2 2-Valued Unary Function

**Lemma 3.1** Let \( m \in O_3^{(3)} \) be a ternary majority function and \( s \in O_3^{(1)} \) be a 2-valued unary function with range \( \{\alpha, \beta\} \subset E_3 \) satisfying \( |s^{-1}(\alpha)| = 2 \) (and hence \( |s^{-1}(\beta)| = 1 \)). Then, \( s \) and \( m \) commute if and only if \( m(a, b, c) \in s^{-1}(\alpha) \) for every \( (a, b, c) \in \sigma_3 \).

**Proof** First, assume \( s \perp m \) and let \( (a, b, c) \) be a triple in \( \sigma_3 \). Commutation of \( s \) and \( m \) implies \( s(m(a, b, c)) = m(s(a), s(b), s(c)) \). By assumption on \( \alpha \), two of \( s(a), s(b) \) and \( s(c) \) are equal to \( \alpha \) and, hence, \( m(s(a), s(b), s(c)) = \alpha \). Thus, we have \( s(m(a, b, c)) = \alpha \) as desired.

Conversely, assume \( s(m(a, b, c)) = \alpha \) for any \( (a, b, c) \in \sigma_3 \). Due to the assumption on \( \alpha \) we have \( m(s(a), s(b), s(c)) = \alpha \), showing the commutation of \( s \) and \( m \) on \( \sigma_3 \). On the other hand, \( s(m(a, b, c)) = m(s(a), s(b), s(c)) \) for any \( (a, b, c) \in (E_3)^3 \setminus \sigma_3 \). Hence we have \( s \perp m \). \( \square \)

For ternary majority functions \( m \in O_3^{(3)} \), we define three kinds of conditions (A0), (A1), (A2) regarding the values of \( m \) on \( \sigma_3 \). Note that the values of \( m \) on \( (E_3)^3 \setminus \sigma_3 \) are determined by definition.
(A0) \( m(x, y, z) \in \{1, 2\} \) for all \((x, y, z) \in \sigma_3\)

(A1) \( m(x, y, z) \in \{0, 2\} \) for all \((x, y, z) \in \sigma_3\)

(A2) \( m(x, y, z) \in \{0, 1\} \) for all \((x, y, z) \in \sigma_3\)

Then, a condition for commutation between ternary majority functions and 2-valued unary functions is obtained as an immediate consequence of Lemma 3.1.

**Corollary 3.2** For a ternary majority function \( m \in \mathcal{O}_3^{(3)} \) the following holds.

(0) For \( s \in T_0 \), \( s \perp m \iff m \) satisfies (A0)

(1) For \( s \in T_1 \), \( s \perp m \iff m \) satisfies (A1)

(2) For \( s \in T_2 \), \( s \perp m \iff m \) satisfies (A2)

### 3.2.3 Permutation

We consider commutation between permutations and ternary functions of some kind.

**Lemma 3.3** Let \( f \in \mathcal{O}_3^{(3)} \) be a ternary function on \( E_3 \) satisfying \( f(x, y, x) = f(x, x, z) = x \) for all \( x, y, z \in E_3 \). Then, for each permutation \( s \in \mathcal{O}_3^{(1)} \) on \( E_3 \) the following holds.

(1) Let \( \{a, b, c\} = E_3 \). For a transposition \( s = (a b) \), \( s \perp f \) if and only if each of the following sets is either \( \{a, b\} \) or \( \{c\} \):

\[
\{f(a, b, b), f(b, a, a)\}, \quad \{f(a, c, c), f(b, c, c)\}, \quad \{f(c, a, a), f(c, b, b)\}, \\
\{f(a, c, b), f(b, a, c)\}, \quad \{f(a, c, b), f(c, b, a)\}, \quad \{f(c, a, b), f(c, c, b)\}, \\
\{f(a, b, c), f(b, a, c)\}, \quad \{f(a, a, b), f(b, c, c)\}, \quad \{f(c, a, b), f(c, b, a)\}
\]

(2) For \( s = (012) \) or \( (021) \), \( s \perp f \) if and only if each of the following triples belongs to \( \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \):

\[
( f(0, 1, 1), f(1, 2, 2), f(2, 0, 0) ), \\
( f(0, 1, 2), f(1, 2, 0), f(2, 0, 1) ), \\
( f(0, 2, 1), f(1, 0, 2), f(2, 1, 0) ), \\
( f(0, 2, 2), f(1, 0, 0), f(2, 1, 1) )
\]

The proof is straightforward from the definition of commutation \( s \perp f \). Observe that ternary majority functions satisfy the assumption imposed on \( f \) in Lemma 3.3.

For ternary majority functions \( m \in \mathcal{O}_3^{(3)} \), we define four kinds of conditions, \( (C_i) \) \( 0 \leq i \leq 3 \), determined by the values of \( m \) on \( \sigma_3 \). First, for every \( i \in \{0, 1, 2\} \), \( (C_i) \) is given as follows.

\( (C_i) \) Each of the following sets is \( \{i\} \) or \( \{a, b\} \) where \( \{i, a, b\} = E_3 \):

\[
\{m(i, a, b), m(i, b, a)\}, \quad \{m(a, i, b), m(b, i, a)\}, \quad \{m(a, b, i), m(b, a, i)\}.
\]

Then, \( (C_3) \) is the following.

\( (C_3) \) Each of the following triples belongs to \( \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\} \):

\[
(m(0, 1, 2), m(1, 2, 0), m(2, 0, 1)), \quad (m(0, 2, 1), m(1, 0, 2), m(2, 1, 0))
\]

We have the conditions connecting permutations and properties \( (C_i) \)’s.

**Corollary 3.4** For a ternary majority function \( m \in \mathcal{O}_3^{(3)} \) the following holds.

(0) For \( s = (12) \), \( s \perp m \iff m \) satisfies (C0)

(1) For \( s = (02) \), \( s \perp m \iff m \) satisfies (C1)

(2) For \( s = (01) \), \( s \perp m \iff m \) satisfies (C2)

(3) For \( s \in \{(012), (021)\} \), \( s \perp m \iff m \) satisfies (C3)
3.2.4 Combined Conditions

We shall investigate combinations of conditions from \((Ai)\)’s \((0 \leq i \leq 2)\) and \((Cj)\)’s \((0 \leq j \leq 3)\).

(i) A-group

Combined conditions of \((Ai)\)’s are the following. For \(i, j \in E_3\) such that \(0 \leq i < j \leq 2\),

\[
(Ai) \land (Aj) \quad m(\sigma_3) = E_3 \setminus \{i, j\}.
\]

Obviously, no majority functions satisfy all three conditions \((A0) \land (A1) \land (A2)\).

The collection of conditions \((Ai)\), \(0 \leq i \leq 2\), and \((Ai) \land (Aj)\), \(0 \leq i < j \leq 2\), will be called A-group.

(ii) C-group

Concerning combined conditions of \((Ci)\) \((0 \leq i \leq 3)\) for majority functions, any combination of two, three or four conditions of \((Ci)\) yields the same condition, which we denote by \((C^*)\).

\[
(C^*) \quad m_{|\sigma_3} = c^i_\ell \quad \text{for some } \ell \in \{1, 2, 3\}
\]

(Note that, in general, \(f_S\) denotes the restriction of \(f\) to a subset \(S\) of the domain of \(f\).)

The collection of conditions \((Ci)\), \(1 \leq i \leq 4\), and \((C^*)\) will be called C-group.

(iii) Compounds of A-group and C-group

What remains to be considered is the collection of compound cases: one from A-group and one from C-group.

For mutually distinct \(i, j, \ell \in E_3\) (that is, \(\{i, j, \ell\} = E_3\)), assuming \(i < j\), the compound condition of \((Ai) \land (Aj)\) and \((C\ell)\) is as follows.

\[
((Ai) \land (Aj)) \land (C\ell) \quad m(\sigma_3) = \{\ell\}
\]

This is the same as \(m(\sigma_3) = E_3 \setminus \{i, j\}\), which appeared above as \((Ai) \land (Aj)\). Moreover, combined conditions \((Ai) \land (C\ell)\) for \(i, \ell \in E_3\) with \(i \neq \ell\) also result in the same condition as above.

Next, the combined condition \((Ai) \land (Ci)\) for \(i \in E_3\) is as follows.

\[
(Ai) \land (Ci) \quad \text{Each of the following sets is } \{a, b\}:
\]

\[
\{m(i, a, b), m(i, b, a)\}, \{m(a, i, b), m(b, i, a)\},
\] \[
\{m(a, b, i), m(b, a, i)\}
\]

For all other combined conditions such as \((A0) \land (C3)\), \((A1) \land (C3)\), etc., no majority function satisfies such sets of conditions.

3.2.5 Results

We present all centralizing monoids which have witnesses of ternary majority functions. It is readily, and interestingly, verified that every centralizing monoid which has a non-empty set of majority functions as its witness has a singleton set (of a majority function) as its witness.

Centralizing monoids on \(E_3\) having majority functions as their witnesses

1. \(L_0 = \{c_0, c_1, c_2\} \cup \{\text{id}, s_{021}\}\)

   Number of elements: 5

   Property of witness: (C0)

   Example of witness: \(m(0, 1, 2) = m(0, 2, 1) = m(1, 0, 2) = m(2, 0, 1) = 0,\)

   \(m(1, 2, 0) = 1, m(2, 1, 0) = 2\)
(2) \( L_1 = \{ c_0, c_1, c_2 \} \cup \{ \text{id}, s_{210} \} \)
Number of elements: 5
Property of witness: (C1)
Example of witness: \( m(1,0,2) = m(1,2,0) = m(0,1,2) = m(2,1,0) = 1, \\
 m(0,2,1) = 0, m(2,0,1) = 2 \)

(3) \( L_2 = \{ c_0, c_1, c_2 \} \cup \{ \text{id}, s_{102} \} \)
Number of elements: 5
Property of witness: (C2)
Example of witness: \( m(2,0,1) = m(2,1,0) = m(0,2,1) = m(1,2,0) = 2, \\
 m(0,1,2) = 0, m(1,0,2) = 1 \)

(4) \( L_3 = \{ c_0, c_1, c_2 \} \cup \{ \text{id}, s_{120}, s_{201} \} \)
Number of elements: 6
Property of witness: (C3)
Example of witness: \( m(0,1,2) = 0, m(1,2,0) = 1, m(2,0,1) = 2, \\
 m(0,2,1) = 1, m(1,0,2) = 2, m(2,1,0) = 0 \)

(5) \( L_4 = \{ c_0, c_1, c_2 \} \cup \{ \text{id}, s_{021}, s_{102}, s_{120}, s_{201}, s_{210} \} \)
Number of elements: 9
Property of witness: (C*)
Example of witness: \( m(0,1,2) = 0, m(1,2,0) = 1, m(2,0,1) = 2, \\
 m(0,2,1) = 1, m(1,0,2) = 1, m(2,1,0) = 2 \)

(6) \( M_0 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221} \} \cup \{ \text{id} \} \)
Number of elements: 10
Property of witness: (A0)
Example of witness: \( m(0,1,2) = 1, m(1,2,0) = 1, m(2,0,1) = 2, \\
 m(0,2,1) = 1, m(1,0,2) = 1, m(2,1,0) = 2 \)

(7) \( M_1 = \{ c_0, c_1, c_2 \} \cup \{ s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212} \} \cup \{ \text{id} \} \)
Number of elements: 10
Property of witness: (A1)
Example of witness: \( m(0,1,2) = 0, m(1,2,0) = 0, m(2,0,1) = 2, \\
 m(0,2,1) = 0, m(1,0,2) = 0, m(2,1,0) = 2 \)

(8) \( M_2 = \{ c_0, c_1, c_2 \} \cup \{ s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \} \cup \{ \text{id} \} \)
Number of elements: 10
Property of witness: (A2)
Example of witness: \( m(0,1,2) = 0, m(1,2,0) = 0, m(2,0,1) = 1, \\
 m(0,2,1) = 0, m(1,0,2) = 0, m(2,1,0) = 1 \)

(9) \( LM_0 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221} \} \cup \{ \text{id}, s_{021} \} \)
Number of elements: 11
Property of witness: (A0) + (C0)
Example of witness: \( m(0,1,2) = 1, m(0,2,1) = 2, m(1,0,2) = 2, m(2,0,1) = 1, \\
 m(1,2,0) = 1, m(2,1,0) = 2 \)

(10) \( LM_1 = \{ c_0, c_1, c_2 \} \cup \{ s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212} \} \cup \{ \text{id}, s_{210} \} \)
Number of elements: 11
Property of witness: (A1) + (C1)
Example of witness: \( m(1,0,2) = 0, m(1,2,0) = 2, m(0,1,2) = 2, m(2,1,0) = 0, \\
 m(0,2,1) = 0, m(2,0,1) = 2 \)
(11) \( LM_2 = \{ c_0, c_1, c_2 \} \cup \{ s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \} \cup \{ \text{id}, s_{102} \} \)

Number of elements: 11
Property of witness: (A2) + (C2)
Example of witness:
\[
m(2,0,1) = 0, \ m(2,1,0) = 1, \ m(0,2,1) = 1, \ m(1,2,0) = 0 \\
m(0,1,2) = 0, \ m(1,0,2) = 1
\]

(12) \( LM_3 = \{ c_0, c_1, c_2 \} \cup \{ s_{010}, s_{020}, s_{101}, s_{121}, s_{202}, s_{212}, s_{100}, s_{200}, s_{011}, s_{211}, s_{022}, s_{122} \} \)

\[ \cup \{ \text{id}, s_{021} \} \]
Number of elements: 17
Property of witness: (A1) + (A2) + (C0)
Example of witness:
\[
m(0,1,2) = 0, \ m(0,2,1) = 0, \ m(1,0,2) = 0, \ m(1,2,0) = 0, \\
m(2,0,1) = 0, \ m(2,1,0) = 0
\]

(13) \( LM_4 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221}, s_{100}, s_{000}, s_{011}, s_{111}, s_{022}, s_{122} \} \)

\[ \cup \{ \text{id}, s_{210} \} \]
Number of elements: 17
Property of witness: (A0) + (A2) + (C1)
Example of witness:
\[
m(0,1,2) = 1, \ m(0,2,1) = 1, \ m(1,0,2) = 1, \ m(1,2,0) = 1, \\
m(2,0,1) = 1, \ m(2,1,0) = 1
\]

(14) \( LM_5 = \{ c_0, c_1, c_2 \} \cup \{ s_{001}, s_{002}, s_{110}, s_{112}, s_{220}, s_{221}, s_{100}, s_{000}, s_{011}, s_{111}, s_{022}, s_{122} \} \)

\[ \cup \{ \text{id}, s_{102} \} \]
Number of elements: 17
Property of witness: (A0) + (A1) + (C2)
Example of witness:
\[
m(0,1,2) = 2, \ m(0,2,1) = 2, \ m(1,0,2) = 2, \ m(1,2,0) = 2, \\
m(2,0,1) = 2, \ m(2,1,0) = 2
\]

(15) \( K = \{ c_0, c_1, c_2 \} \cup \{ \text{id} \} \)

Number of elements: 4
Property of witness: None in A-group and C-group
Example of witness:
\[
m(0,1,2) = 0, \ m(0,2,1) = 1, \ m(1,0,2) = 1, \ m(1,2,0) = 2, \\
m(2,0,1) = 2, \ m(2,1,0) = 2
\]

To summarize briefly:

**Proposition 3.5** The number of centralizing monoids on \( E_3 \) which have sets of majority functions as their witnesses is 15.

We observe that the number 15 is remarkably small as compared with the number of ternary majority functions, that is, \( 3^6 = 729 \).

**Remark** Similar results were obtained for semiprojections in place of majority functions (See [GMR15]).

**References**


