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The law βA on Trices

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We can consider various adaptations of laws in bisemilattice to in triple-semilattice. The character might change greatly even if the form is almost similar. We introduce "the law βA" of a such configuration type. This law means saying, "The result by three different operations of two different elements is different". The definition is simple. But, as it plays an important role on triple-semilattice. We prove that the distribution law must not coexist with it. The order from one of the operation of trice that satisfy the βA is not linearly order. And we discuss the relation between the triangle natural law and βA.

1 Preliminaries

A semilattice $(S,\ *)$ is a set S with a single binary, idempotent, commutative and associative operation $*$.

$$a*a=a \qquad (idempotent) \tag{1}$$
$$a*b=b*a \qquad (commutative) \tag{2}$$
$$a*(b*c)=(a*b)*c \qquad (associative) \tag{3}$$

Under the relation defined by $a\leq_{*}b\Longleftrightarrow a*b=b$, any semilattice $(S,\ *)$ is a partially ordered set $(S,\ \leq_{*})$. Let A be a set. A **bisemilattice** $(A,\ *_{1}, *_{2})$ is an algebra which has two semilattice operations, that is, $(A, *_{1})$ and $(A, *_{2})$ are semilattices, respectively. Hence, we can construct two ordered sets $(A, \leq_{1})$ and $(A, \leq_{2})$. A lattice is a bisemilattice which satisfy the following:

$[B]$ (absorption laws)

$$a*_{1}(a*_{2}b)=a \tag{4}$$
$$a*_{2}(a*_{1}b)=a \tag{5}$$

for every $a, b\in A$.

The following laws are satisfied from absorption laws. For the following laws, there may be a suitable name as well as absorption laws. But, it is shown with the sign ($[A]$ etc.) to avoid bringing preconception. We defined these in [5]. We know $[A]$ and $[E]$ are equivalent. Let $a, b \in A$.

$[A]$ if and only if the following condition hold:

$$a \leq_1 b \text{ and } a \leq_2 b \implies a = b. \tag{6}$$

$[E]$ if and only if the following condition hold:

$$a *_1 b = a *_2 b \implies a = b. \tag{7}$$

Let T be a set. A **triple-semilattice** $(T, *_1, *_2, *_3)$ is an algebra which has three semilattice operations. That is, $(T, *_1)$, $(T, *_2)$ and $(T, *_3)$ are semilattices, respectively. We construct three ordered sets $(T, \leq_1)$, $(T, \leq_2)$ and $(T, \leq_3)$. Next properties are adaptation from $[B]$, $[A]$ and $[E]$. Let $a, b \in T$.

$[*B]$ if and only if the following six conditions hold:

$$((a *_1 b) *_2 b) *_3 b = b \tag{8}$$
$$((a *_1 b) *_3 b) *_2 b = b \tag{9}$$
$$((a *_2 b) *_1 b) *_3 b = b \tag{10}$$
$$((a *_2 b) *_3 b) *_1 b = b \tag{11}$$
$$((a *_3 b) *_1 b) *_2 b = b \tag{12}$$
$$((a *_3 b) *_2 b) *_1 b = b \tag{13}$$

for every $a, b \in T$.

$[*A]$ if and only if the following condition hold:

$$a \leq_1 b \text{ and } a \leq_2 b \text{ and } a \leq_3 b \implies a = b. \tag{14}$$

$[*E]$ if and only if the following condition hold:

$$a *_1 b = a *_2 b = a *_3 b \implies a = b. \tag{15}$$

In [3], we proposed the algebra system with three semilattice operations with $[*B]$ and we call this $[*B]$ **roundabout-absorption laws**(*or* **r-absorption laws**). The algebraic structure is called a **trice**. We know $[*A]$ and $[*E]$ are equivalent. And these are satisfied from $[*B]$. In a word, on bisemilattice, the $[A]$ is one of a weak law that can be led from the absorption law. The $[*A]$ that adjusted to triple-semilattice was a weak law as for $[A]$ (in [5]).

Let T be a triple-semilattice. We say that an ordered triple $(a,\ b,\ c)$ is in a **triangular situation** if $(a,\ b,\ c)$ have the following properties:

$$a *_3 b = c \text{ and } a *_2 c = b \text{ and } b *_1 c = a. \tag{16}$$

We draw three figures and show the example of triple-semilattices. The set $\{a, b, c\}$ of triangular situation is one of trice. When it is shown in figure, it is Fig.1.
We say that T has the **triangle constructive law** if T has the following properties:

$$(d *_1 e) *_3 (d *_2 e) = d *_3 e \tag{17}$$
$$(d *_1 e) *_2 (d *_3 e) = d *_2 e \tag{18}$$
$$(d *_3 e) *_1 (d *_2 e) = d *_1 e \tag{19}$$

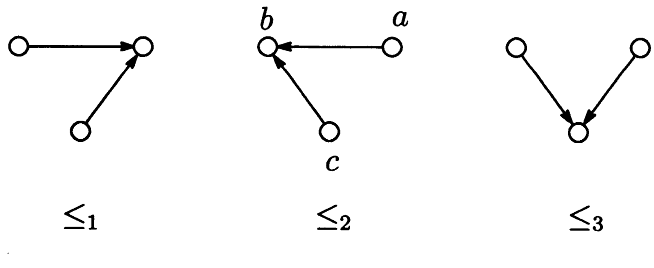
for all $d, e \in T$. That is, the ordered triple $(d *_1 e,\ d *_2 e,\ d *_3 e)$ is a triangular situation.
We say that T has the **triangle natural law** if T has the following properties:

$$b *_1 c = a \text{ and } a *_2 c = b \implies a *_3 b = c \tag{20}$$
$$c *_2 a = b \text{ and } b *_3 a = c \implies b *_1 c = a \tag{21}$$
$$a *_3 b = c \text{ and } c *_1 b = a \implies c *_2 a = b. \tag{22}$$

for $a, b, c \in T$.
We say that T has the **distributive law** if T has the following six properties:

$$a *_1 (b *_2 c) = (a *_1 b) *_2 (a *_1 c) \tag{23}$$
$$a *_2 (b *_1 c) = (a *_2 b) *_1 (a *_2 c) \tag{24}$$
$$a *_1 (b *_3 c) = (a *_1 b) *_3 (a *_1 c) \tag{25}$$
$$a *_3 (b *_1 c) = (a *_3 b) *_1 (a *_3 c) \tag{26}$$
$$a *_2 (b *_3 c) = (a *_2 b) *_3 (a *_2 c) \tag{27}$$
$$a *_3 (b *_2 c) = (a *_3 b) *_2 (a *_3 c) \tag{28}$$

for every $a, b, c \in T$. We investigated distributive trices in [4].



Three figures show the same set. A little circle expresses an element. An element in the same position in figures is the same. We depict three orders in the set by arrows of figures. We suppose that arrowhead is larger than the other end.

Figure 1: triangular situation

2 Definition of βA

Now, we can think about different law $[\beta A]$ on triple-semilattice. This $[\beta A]$ is related to the $[A]$ as well as $[*A]$. But, it is a strong law and important. Let T be a triple-semilattice and $a, b \in T$.

$[\beta A]$ if and only if the following three conditions hold:

$$a \leq_1 b \text{ and } a \leq_2 b \implies a = b \tag{29}$$
$$a \leq_2 b \text{ and } a \leq_3 b \implies a = b \tag{30}$$
$$a \leq_3 b \text{ and } a \leq_1 b \implies a = b. \tag{31}$$

Next $[\beta E]$ means saying, "The result of three different operations is different". The $[\beta A]$ and the $[\beta E]$ are equivalent.

$[\beta E]$ if and only if the following three conditions hold:

$$a *_1 b = a *_2 b \implies a = b \tag{32}$$
$$a *_2 b = a *_3 b \implies a = b \tag{33}$$
$$a *_3 b = a *_1 b \implies a = b. \tag{34}$$

Theorem 1 $[\beta A]$ and $[\beta E]$ are equivalent. Accurately, (29) and (32) are equivalent, (30) and (33) are equivalent and (31) and (34) are equivalent.

Proof This proof is the simulation of Proposition 2 in [5]. At first, we prove that (29) imply (32). Let $a *_1 b = a *_2 b = c$. Then, $a \leq_1 c$, $b \leq_1 c$, $a \leq_2 c$ and $b \leq_2 c$. From (29), $a \leq_1 c$ and $a \leq_2 c$ imply $a = c$. Similarly, $b \leq_1 c$ and $b \leq_2 c$ imply $b = c$. Hence $a = c = b$. Conversely, we prove that (32) imply (29). let $a \leq_1 b$ and $a \leq_2 b$. Then, $a *_1 b = b$ and $a *_2 b = b$. Hence, $a *_1 b = a *_2 b$. If (32) is satisfied, $a = b$. Therefore, (29) and (32) are equivalent. Similarly, (30) and (33) are equivalent and (31) and (34) are equivalent.

After this, we omit square brackets of [βA]. We will use "the law βA" or " βA".

The $[*B]$ imply the $[*A]$, that is, the $[*A]$ that adjusted to triple-semilattice was a weak law as for $[A]$. However, the situation of βA is different. There are trices which does not have the βA. We show that some typical trices doesn't have the βA in the following two sections.

3 The βA and the distributive law

Lemma 1 Let T be a trice and $a,\ b,\ c \in T$. If $a \leq_1 b$, $b \leq_2 a$ and $a *_3 b = c$, then $c \leq_1 b$ and $c \leq_2 a$.

Proof From r-absorption law, $((a *_2 b) *_3 b) *_1 b = b$. From assumption $b \leq_2 a$, $((a *_2 b) *_3 b) *_1 b = (a *_3 b) *_1 b = c *_1 b$. Hence $c *_1 b = b$, that is, $c \leq_1 b$. Similarly, $c \leq_2 a$.

Lemma 2 Let T be a non-trivial trice (not one point trice). There exist $a, b \in T$ $(a \neq b)$ and $m, n \in \{1, 2, 3\}$ $(m \neq n)$ such that $a \leq_m b$, $b \leq_n a$.

Proof Let $c \neq d$ $(c, d \in T)$. From $c \leq_1 c *_1 d$ and $d \leq_1 c *_1 d$, $c \neq c *_1 d$ or $d \neq c *_1 d$. Suppose that $d \neq c *_1 d$ and $d \leq_1 c *_1 d$, that is, $d < c *_1 d$. From r-absorption law, $((c *_1 d) *_2 d) *_3 d = d$. Hence $(c *_1 d) *_2 d \leq_3 d$. It is clear that $d \leq_2 (c *_1 d) *_2 d$. If $(c *_1 d) *_2 d \neq d$, let $a = (c *_1 d) *_2 d$, $b = d$, $m = 3$ and $n = 2$. On the other hand, if $(c *_1 d) *_2 d = d$, $c *_1 d \leq_2 d$. From $d \neq c *_1 d$ and $d \leq_1 c *_1 d$, let $a = c *_1 d$, $b = d$, $m = 2$ and $n = 1$.

Theorem 2 The non-trivial trice cannot satisfy either βA or distributive law.

Proof Let T be a trice with βA. We prove T doesn't satisfy distributive law. Suppose that $a \neq b$, $a \leq_1 b$ and $b \leq_2 a$, from Lemma 2. And let $c = a *_3 b$. If $c = a *_3 b = b$, then $a \leq_3 b$. From $a \leq_1 b$ and βA, we obtain $a = b$. This is contradiction. Hence, $c \neq b$. Similarly, $c \neq a$. Using Lemma 1, $c \leq_1 b$ and $c \leq_2 a$. If T satisfy the distributive law,

$$\begin{aligned} c &= a *_3 b \\ &= (a *_2 b) *_3 (b *_2 b) \quad (\text{from } \ b \leq_2 a) \\ &= (a *_3 b) *_2 b \qquad\qquad (\text{distributive law}) \\ &= c *_2 b. \end{aligned}$$

Hence, $b \leq_2 c$. From $c = a *_3 b$, $b \leq_3 c$. We obtain $c = b$ by βA. This is contradiction.

Theorem 3 The non-trivial triple-semilattice cannot satisfy either βA or distributive law.

Proof Suppose that a triple-semilattice satisfy either βA and distributive law. Let $a\ b \in T$. It is clear that $b \leq_2 (a *_1 b) *_2 b$ and $b \leq_1 (a *_2 b) *_1 b$. From distributive law, $(a *_1 b) *_2 b = (a *_1 b) *_2 (b *_1 b) = (a *_2 b) *_1 b$. By βA, $b = (a *_1 b) *_2 b = (a *_1 b) *_2 (b *_1 b) = (a *_2 b) *_1 b$. This indicate absorption law. We can obtain r-absorption law $b = ((a *_1 b) *_2 b) *_3 b = b$ etc. Hence, T is a trice. This contradicts Theorem 2.

Example 1 Let T be a two points set $\{a, b\}$. We introduce three orders in T by :

$$a \leq_1 b, \quad a \leq_1 b, \quad a \leq_3 b. \tag{35}$$

See Fig. 2. This T does not have the βA. And this is not a trice. But this has distributive law.

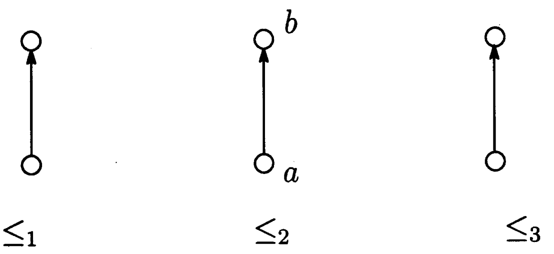


Figure 2: two elements triple-semilattice

Example 2 Let T be a three points set $\{a, b, c\}$. We introduce three orders in T by :

$$c \leq_1 b, \quad b \leq_1 a \tag{36}$$
$$a \leq_2 c, \quad c \leq_2 b \tag{37}$$
$$b \leq_3 a, \quad a \leq_3 c. \tag{38}$$

See Fig. 3. This T does not have distributive law. And this does not have the βA. But this is a trice.

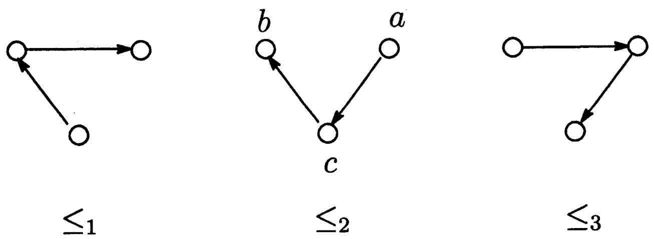


Figure 3: triangular circuit

4 The βA and linearly order

Let $(T, *_1, *_2, *_3)$ be a triple-semilattice. Then, we can construct three ordered sets $(T, \leq_1)$, $(T, \leq_2)$ and $(T, \leq_3)$. One of these ordered sets might be a linearly ordered set. (See Fig. 2 and Fig. 3. In these examples, each order of three is linearly.)

Theorem 4 If $(T, *_1, *_2, *_3)$ is a non-trivial trice with the βA, then neither $(T, \leq_1)$, $(T, \leq_2)$ nor $(T, \leq_3)$ are linearly ordered sets.

Proof Suppose that $(T, \leq_1)$ is a linearly ordered set. Let $a, b \in T$ such that $a \neq b$. Here, let $c = a *_2 b$. From $a \leq_2 c$ and $b \leq_2 c$, $a \neq c$ or $b \neq c$. Now, we assume $a \neq c$, that is, $a <_2 c$. If $a \leq_1 c$, then $a = c$ from the βA. It is impossible. As $\leq_1$ is linearly order, we claim that $c \leq_1 a$.
Let $d = a *_3 c$ and we do the situation division by d.

i. $d \leq_1 c \leq_1 a$

$$\begin{array}{rcll} c & = & ((a *_1 c) *_3 c) *_2 c & \text{(roundabout absorption law)} \\ & = & (a *_3 c) *_2 c & \text{(from } \quad c \leq_1 a) \\ & = & d *_2 c. & \end{array}$$

Hence $d \leq_2 c$. From the βA law and $d \leq_1 c$, we obtain $d = c$. From $d = a *_3 c$, $a \leq_3 d = c$. Hence, from the βA and $a \leq_2 c$, we obtain $a = c$. This contradicts assumption.

ii. $c \leq_1 a \leq_1 d$

$$\begin{array}{rcll} a & = & ((c *_2 a) *_3 a) *_1 a & \text{(roundabout absorption law)} \\ & = & (c *_3 a) *_1 a & \text{(from } \quad c \leq_1 a) \\ & = & d *_1 a. & \end{array}$$

Hence $d \leq_1 a$. From $a \leq_1 d$, we obtain $a = d$. From $d = a *_3 c$, $c \leq_3 d = a$. Hence, from the βA and $c \leq_1 a$, we obtain $a = c$. This contradicts assumption.

iii. $c <_1 d <_1 a$
From $d = a *_3 c$, $c \leq_3 d$. From the βA and $c \leq_1 d$, we obtain $c = d$. This contradicts $c <_1 d$.

Therefore, $(T, \leq_1)$ is not a linearly ordered set. Similarly, $(T, \leq_2)$ and $(T, \leq_3)$ are not linearly ordered sets.

Theorem 5 If $(T, *_1, *_2, *_3)$ is a non-trivial triple-semilattice (not one point triple-semilattice) with the βA, then neither $(T, \leq_1)$, $(T, \leq_2)$ nor $(T, \leq_3)$ are linearly ordered sets with smallest element.

Proof Suppose that $(T, \leq_1)$ is a linearly ordered set with smallest element m of $\leq_1$. For every $b \in T$, let $c = b *_2 m$. Then, $m \leq_2 c$. As m is the smallest element of $\leq_1$, $m \leq_1 c$. By the βA, $m = c$. As $m = b *_2 m$ for every $b \in T$, m is the largest element of $\leq_2$. Similarly, m is the largest element of $\leq_3$. For every $a \in T$, we obtain $a \leq_2 m$ and $a \leq_3 m$. By the βA , $a = m$. Hence, T is a one point set. This is contradiction.

Corollary 1 If $(T, *_1, *_2, *_3)$ is a non-trivial finite triple-semilattice with the βA , then neither $(T, \leq_1)$, $(T, \leq_2)$ nor $(T, \leq_3)$ are linearly ordered sets.

Example 3 Let T be a infinite points set $\{a_n : n \in Z\ (Integers)\}$. We introduce three orders in T by :

$$a_n \leq_1 a_{n+1} \tag{39}$$

$$a_{2n} \leq_2 a_{2n-2}, \quad a_{2n-1} \leq_2 a_{2n-2} \tag{40}$$

$$a_{2n+1} \leq_3 a_{2n-1}, \quad a_{2n} \leq_3 a_{2n-1} \tag{41}$$

for every $n \in Z$.
See Fig. 4. This T is infinite triple-semilattice with the βA s. The $\leq_1$ is a linearly order. The T is not a trice. The T don't have triangle natural law, The T don't have triangle constructive law.

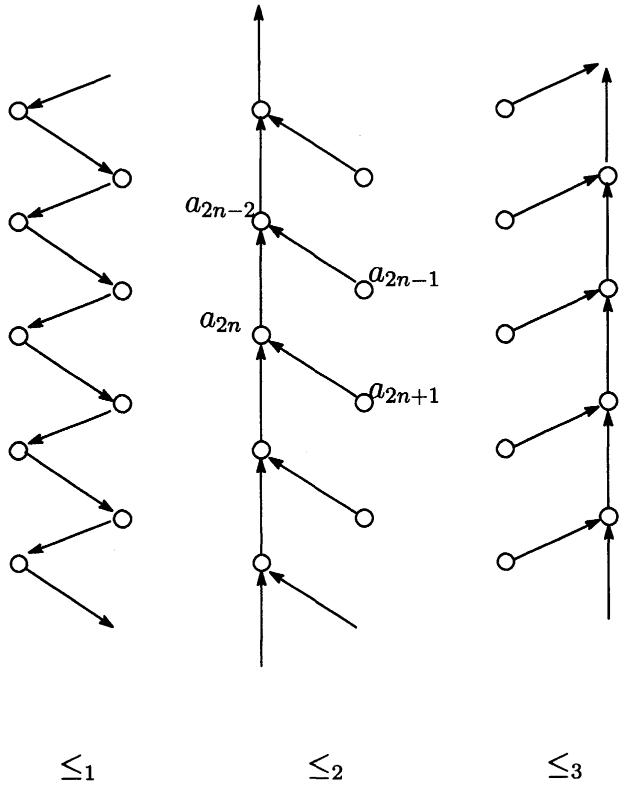


Figure 4: one linear ordered infinite triple-semilattice

5 The βA and the triangle natural law

The theorem 1 in [7] means the next.

Theorem 6 Triangle constructive triple-semilattice with r-absorption laws has the βA .

We will try to extend it. At first, please see the following example.

Example 4 Let T be a six points set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. We introduce three orders in T by :

$$a_2 \leq_1 a_1, \quad a_3 \leq_1 a_1, \quad a_1 \leq_1 b_2, \quad a_1 \leq_1 b_3, \quad b_2 \leq_1 b_1, \quad b_3 \leq_1 b_1 \tag{42}$$

$$a_1 \leq_2 a_2, \quad a_3 \leq_2 a_2, \quad a_2 \leq_2 b_1, \quad a_2 \leq_2 b_3, \quad b_1 \leq_2 b_2, \quad b_3 \leq_2 b_2 \tag{43}$$

$$a_1 \leq_3 a_3, \quad a_2 \leq_3 a_3, \quad a_3 \leq_3 b_1, \quad a_3 \leq_3 b_2, \quad b_1 \leq_3 b_3, \quad b_2 \leq_3 b_3. \tag{44}$$

See Fig. 5. The T has the triangle constructive law. The (a_1, a_2, a_3) and (b_1, b_2, b_3) are triangular situations. The T is not a trice. The T does not have the βA .

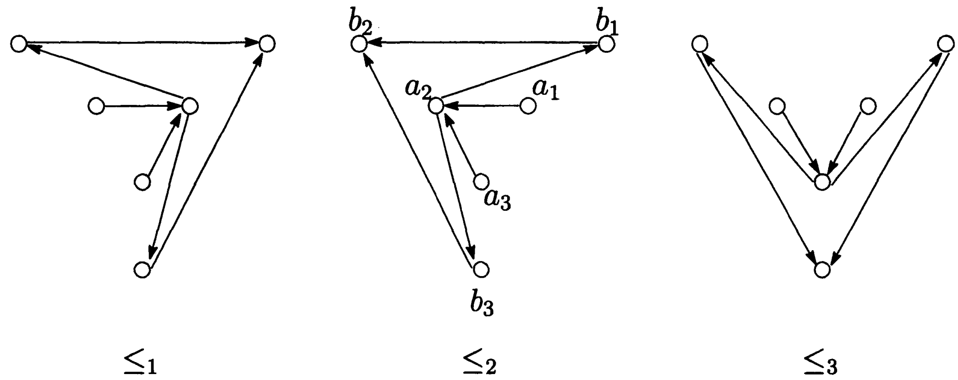


Figure 5: Example 4

We cannot exclude roundabout absorption laws from Theorem 6. By Proposition 7 in [3], we proved triangle constructive trice has the triangle natural law. However, triangle constructive law and triangle natural law are independent if roundabout absorption law is not assumed. Example 4 does not have the triangle natural law. Because $a_1 *_1 b_1 = b_1$, $a_1 *_2 b_1 = b_1$ but $b_1 *_3 b_1 = b_1 \neq a_1$. Essentially, Example 1 is not triangle natural but triangle constructive. We obtain next theorem. This is easy. It is essential and important.

Theorem 7 If a triple-semilattice T satisfies the triangle natural law, then T satisfies the βA .

Proof We prove the contraposition. Suppose that T doesn't satisfy βA. There exist $a, b \in T$ such that $a \leq_1 b$, $a \leq_2 b$ and $a \neq b$. Then, $a *_1 b = b$ and $a *_2 b = b$. From $b *_3 b = b \neq a$, T doesn't satisfy the triangle natural law.

We consider Theorem 7 is an extension of Theorem 6 (the theorem 1 in [7]). The triangle natural law is deeply related to the βA . To clarify the relation between the βA and the triangle natural law, we thought about the examples. See Example 5 and Example 6. Example 6 is a trice, but Example 5 is not a trice. We do not have an example that is not the triangle natural but the βA under the trice.

Example 5 Let T be a five points set $\{a, b, c, d, e\}$. We introduce three orders in T by :

$$b \leq_1 a, \quad c \leq_1 a, \quad d \leq_1 a, \quad e \leq_1 a \tag{45}$$
$$a \leq_2 b, \quad c \leq_2 b, \quad d \leq_2 b, \quad e \leq_2 b \tag{46}$$
$$a \leq_3 c, \quad b \leq_3 c, \quad d \leq_3 c, \quad e \leq_3 c. \tag{47}$$

See Fig. 6. This T does not have the triangle natural law. This is not a trice. But, this T has the triangle constructive and the βA s.

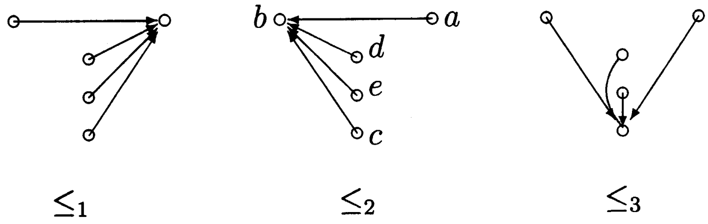


Figure 6: Example 5

Example 6 Let T be a six points set $\{a,b,c,d,e,f\}$. We introduce three orders in T by :

$$a \leq_1 b, \quad b \leq_1 c, \quad d \leq_1 b, \quad a \leq_1 e, \quad e \leq_1 c, \quad f \leq_1 e \tag{48}$$

$$b \leq_2 a, \quad c \leq_2 b, \quad d \leq_2 a, \quad e \leq_2 b, \quad c \leq_2 d, \quad f \leq_2 d \tag{49}$$

$$a \leq_3 d, \quad b \leq_3 d, \quad b \leq_3 e, \quad c \leq_3 e, \quad d \leq_3 f, \quad e \leq_3 f. \tag{50}$$

See Fig. 7. This T does not have the triangle constructive law. But, this has the triangle natural and the βA s. And this is a trice.

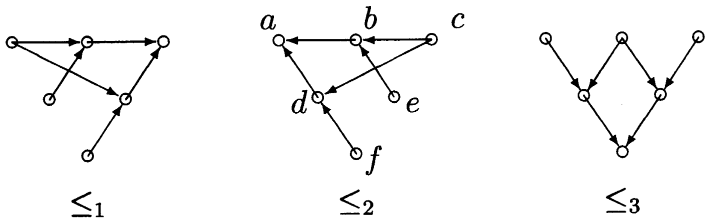


Figure 7: Example 6

6 Conclusion and Question

The triangular situation, the triangle constructive law and the triangle natural law are important concepts in triple-semilattices. These are concepts not considered in bisemilattice. The βA is a concept of relating deeply to these. It is very simple and it is a natural adaptation. We expect that it becomes the clue of the research.
We showed the typical example as simply as possible in this paper. However, we felt that the triangle natural law was difficult to manage. Particularly, there is a close relationship between the βA and the triangle natural law. Then, we submit the following question.

Question Is there a trice that is not the triangle natural but the βA ? Or, are the triangle natural law and the βA equivalent under the trice ?

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