On SCT automorphism groups
of divisible designs

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In this talk we consider automorphism groups SCTs of divisible designs acting regularly on the set of point classes and determine the relations among SCTs, RDSs and $\lambda$-planar functions.

§1 Divisible Designs and class regularity

A divisible design $(m, u, k, \lambda)$-DD is an incidence structure $(\mathbb{P}, \mathbb{B})$, where

(i) $\mathbb{P}$ is a set of $mu$ points partitioned into $m$ classes $\mathcal{C}$ (called point classes), each of size $u$,

(ii) $\mathbb{B}$ is a collection of $k$-subsets of $\mathbb{P}$ (called blocks),

(iii) Any two distinct points in the same point class are incident with no blocks and any two points in distinct point classes are incident with exactly $\lambda$ blocks.

We can show the following : $|\mathbb{P}| = mu$, $|\mathbb{B}| = u^2m(m-1)\lambda/k(k-1)$.

An $(m, u, k, \lambda)$-DD with $k = m$ is called a transversal design and denoted by $TD_{\lambda}(k, u)$. A $TD_{\lambda}(k, u)$ is called a symmetric transversal design and denoted by $STD_{\lambda}(k, u)$ with $k = u\lambda$ if its dual is also a $TD_{\lambda}(k, u)$. We note that an $(m, 1, k, \lambda)$-DD is just a $2-(m, k, \lambda)$ design.

Partial difference matrices

Definition. (Jungnickel [2]) Let $U$ be a group of order $u$. An $m \times t$ matrix $D = [d_{ij}]$ with entries from $U \cup \{0\}$ is called an $(m, u, k, \lambda)$-partial difference matrix (PDM) over $U$ if the following conditions are satisfied:

(i) Each column of $D$ has exactly $k$ nonzero entries.

(ii) $\sum_{1 \leq j \leq t} d_{ij}d_{\ell j}^{-1} = \lambda U$, $\forall i \neq \ell$, where $0^{-1} = 0$, $0 \cdot g = g \cdot 0 = 0 \ \forall g \in U$

and $t = |\mathbb{B}|/|G| = m(m-1)u\lambda/k(k-1))$. 
An \((m, u, k, \lambda)\)-PDM with \(m = k\) over a group \(U\) of order \(u\) is called a \((u, k, \lambda)\)-difference matrix (DM). Moreover, a \((u, u, \lambda)\)-DM, denoted by \(GH(u, \lambda)\), is called a generalized Hadamard matrix.

Example. Set \(U = \langle a \rangle \simeq \mathbb{Z}_3\).

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 1 & a & 0 & a^2 \\
a & 1 & a & 0 & a^2 \\
1 & 0 & a^2 & 1 & a \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & a & a^2 & a^2 \\
1 & 1 & a^2 & a & a \\
1 & a & a^2 & 1 & a \\
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 1 \\
1 & a & a^2 \\
1 & a^2 & a \\
\end{bmatrix}
\]

\((5, 3, 4, 1)\)-PDM \((3, 3, 2)\)-DM \(GH(3, 1)\)

Class regularity

Following results are known.

Result. (D. Jungnickel [3]) The existence of an \((m, u, k, \lambda)\)-DD admitting a class regular automorphism group \(U\)

\[\iff\] The existence of a \((m, u, k, \lambda)\)-partial difference matrix over \(U\)

Result. (D.A. Drake [2]) Assume \(U\) is a group of even order \(u\) and \(2 \nmid \lambda\). If a Sylow 2-subgroup of \(U\) is cyclic then there exists no \((u, k, \lambda)\)-DM over \(U\) for \(k \geq 3\).

We now consider the regular action of a subgroup \(G\) of \(\text{Aut}(\mathcal{D})\) on the set of point classes \(\mathscr{C} = \{C_i \mid i \in I_m\}\), where \(I_m = \{1, 2, \cdots, m\}\).

\[\text{SCT groups and SCT matrices}\]

Let \((\mathbb{P}, \mathbb{B})\) be a \((m, u, k, \lambda)\)-DD and \(G \leq \text{Aut}(\mathbb{P}, \mathbb{B})\). We say \(G\) is an \(SCT(m, u, k, \lambda)\) group if \(G\) is semiregular on \(\mathbb{P} \cup \mathbb{B}\) and regular on the set of point classes \(\mathscr{C} = \{C_1, \cdots, C_m\}\). (Note that \(|G| = m\).)

Assume that \(G\) is an \(SCT(m, u, k, \lambda)\) group of a \((m, u, k, \lambda)\)-DD \(\mathcal{D} = (\mathbb{P}, \mathbb{B})\).

Choose a point class \(C = \{p_1, \cdots, p_u\} \in \mathscr{C}\). Then \(\mathbb{P} = \bigcup_{i \in I_u} p_i^G\) and \(\mathbb{B} = \bigcup_{j \in I_s} B_j^G\), where \(s = |\mathbb{B}|/|G|\).

A \(u \times s\) matrix \(M_D = [D_{ij}] (D_{ij} \subset G)\) over \(G\) is defined by

\[D_{ij} = \{g \in G \mid p_i^g \in B_j\} \quad (i \in I_u, j \in I_s)\]

Theorem 1. The following holds.

\[
\sum_{j \in I_s} D_{ij} D_{i' j}^{(-1)} = \begin{cases} 
\rho + \lambda(G-1) & \text{if } i = \ell, \\
\lambda(G-1) & \text{otherwise}, 
\end{cases}
\]

where \(\rho = (m-1)u\lambda/(k-1)\).

\[
\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_s
\]

Definition. Let \(G\) be a group of order \(m\). Let \(u, s \in \mathbb{N}\). For subsets \(D_{ij} \subset G\) \((i \in I_u, j \in I_s)\) we call a \(u \times s\) matrix

\[
\begin{bmatrix}
D_{11} & \cdots & D_{1s} \\
\vdots & \vdots & \vdots \\
D_{u1} & \cdots & D_{us}
\end{bmatrix}
\]

an \(SCT(m, u, k, \lambda)\)-
matrix over $G$ if it satisfies the following for some $\rho \in \mathbb{N}$.

$$
\sum_{j \in I_s} D_{ij} D_{\ell j}^{(-1)} = \begin{cases} 
\rho + \lambda (G - 1) & \text{if } i = \ell, \\
\lambda (G - 1) & \text{otherwise,}
\end{cases}
$$

$$
\sum_{i \in I_u} |D_{ij}| = k \quad \forall j \in I_s
$$

**Remark.** (i) $s = (m - 1)u^2 \lambda / k(k - 1)$, $\rho = (m - 1)u\lambda / (k - 1)$

(ii) An SCT$(m, 1, k, \lambda)$-matrix is just an $(m, k, \lambda)$-difference family.

An incidence structure $\mathcal{D}(\mathbb{P}, \mathbb{B})$ defined by the following is an $(m, u, k, \lambda)$-DD admitting $G$ as an SCT group under the action $(i, w)g = (i, wg)$ for $i \in \{1, \cdots, u\}$ and $w, g \in G$.

$\mathbb{P} = \{1, 2, \cdots, u\} \times G$

$\mathbb{B} = \{B_{j,g} | j \in I_s, g \in G\}$, where $B_{j,g} = \bigcup_{i \in I_u} (i, D_{ij}g)$

$(m, u, k, \lambda)$-DD with SCT-group $\iff$ SCT$(m, u, k, \lambda)$-matrix

**Example.** (i) The following is an SCT$(9, 2, 9, 9)$ matrix over $G := \langle a, b \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$:

$$
\begin{pmatrix}
\langle a \rangle & \langle b \rangle & G - \langle ab \rangle & G - \langle ab^2 \rangle \\
G - \langle a \rangle & G - \langle b \rangle & \langle ab \rangle & \langle ab^2 \rangle
\end{pmatrix}
$$

This matrix gives a TD$_9(9, 2)$, which is not obtained from any difference matrix by Drake's result.

(ii) The following is an SCT$(12, 5, 11, 2)$ matrix over $Alt(4) = N \rtimes H$, $N = \{1, a, b, c\} \simeq E_4$, $H = \{1, d, d^2\} \simeq \mathbb{Z}_3$:

$$
\begin{pmatrix}
0 & \alpha & \beta & \gamma & \delta \\
\alpha & \beta & \gamma & \delta & 0 \\
\beta & \gamma & \delta & 0 & \alpha \\
\gamma & \delta & 0 & \alpha & \beta \\
\delta & 0 & \alpha & \beta & \gamma
\end{pmatrix}, \text{ where } \begin{cases} 
\alpha = ad + cd^2 \\
\beta = d + bd^2 + d^2 + cd \\
\gamma = b + c \\
\delta = ad^2 + bd + a
\end{cases}
$$

From this we obtain a $(12, 5, 11, 2)$-DD with the full automorphism group isomorphic to $Alt(5)$ ($\simeq Alt(4) \simeq N \rtimes H$). This DD is not class regular, hence not obtained from any partial difference matrix.

**Relations among SCT aut. , Class regular aut. and RDS**

$\exists$ SCT aut. $\iff$ $\exists$ SCT mat.

$$
\Downarrow
$$

Divisible design $\supset$ Transversal design

$$
\Uparrow
$$

$\exists$ class regular aut. $\iff$ $\exists$ partial DM $\supset$ DM $\supset$ GH mat.

$\exists$ SCT aut. $\&$ $\exists$ class regular aut. $\iff$ $\exists$ splitting relative difference set
Difference families and SCT matrices

A family of $k$-subsets $\{D_1, \cdots, D_n\}$ of a group $G$ of order $v$ is called an $n$-$(v, k, \lambda)$ difference family if

$$D_1D_1^{(-1)} + \cdots + D_nD_n^{(-1)} = kn + \lambda(G - 1).$$

From an $n$-$(v, k, \lambda)$ difference family in a group $G$ we obtain a $2$-$(v, k, \lambda)$ design $(\mathbb{P}, \mathbb{B})$:

$$\mathbb{P} = G, \quad \mathbb{B} = \{D_i x \mid i \in I_n, \ x \in G\}.$$

In the following we give a relation between difference families and SCT matrices with $u = 2$.

**Theorem 2.** Let $\{D_1, \cdots, D_{4d}\}$ be a $4d$-$(m, k, d(4k - m))$ difference family in a group $G$ of order $m$. Set $C_i = G - D_i$ for $i \in I_{4d}$. Then the following is an SCT$(m, 2, m, dm)$ matrix corresponding to a TD$_{dm}(m, 2)$.

$$M = \begin{bmatrix} D_1 & \cdots & D_{2d} & C_{2d+1} & \cdots & C_{4d} \\ C_1 & \cdots & C_{2d} & D_{2d+1} & \cdots & D_{4d} \end{bmatrix}$$

$$C_iC_i^{(-1)} = D_iD_i^{(-1)} + (m - 2k)G$$

$$D_iC_i^{(-1)} = C_iD_i^{(-1)} = kG - D_iD_i^{(-1)}$$

Some theorems on difference families

The following results on difference families are known.

**Result.** (Leung-Ma-Schmidt [4]) Let $q$ be a prime power and $d > 0$ an integer. Suppose, either (i) $q \equiv 2d - 1 \pmod{4d}$ and $2 \nmid d$ or (ii) $q \equiv 4d - 1 \pmod{8d}$. Then there exists a $4d$-$(q^2, (q^2 - q)/2, dq^2 - 2dq)$ difference family in $(GF(q^2), +)$.

**Result.** (Q. Xiang [6]) Let $q$ be a power of a prime and $b, c$ positive integers such that $q + 1 = 2^cb$ and $c \geq 2$ with $2 \nmid b$. Then there exists a $2c$-$(q^2, (q^2 - q)/2, 2^{c-2}(q^2 - 2q))$ difference family in $(GF(q^2), +)$.

**Remark.** Set $d = 2^{c-2}$ in the above result. Then $2^{c-2}(q^2, (q^2 - q)/2, 2^{c-2}(q^2 - 2q))$ is identical with $4d$-$(q^2, (q^2 - q)/2, dq^2 - 2dq)$.

We now apply Theorem 2 to the above results for $m = q^2, k = (q^2 - q)/2$.

TD$_{dq^2}(q^2, 2)s$ admitting SCT groups

**Proposition.** Let $q$ be a power of a prime and $d$ a positive integer satisfying one of the following:

(i) $q \equiv 2d - 1 \pmod{4d}$.

(ii) $q \equiv 4d - 1 \pmod{8d}$.

(iii) $4d \mid q + 1, 8d \nmid q + 1$ with $d$ a power of 2.

Then, there exists an SCT$(q^2, 2, q^2, dq^2)$ matrix over $(GF(q^2), +)$ and the resulting TD$_{dq^2}(q^2, 2)$ admits an SCT automorphism group of order $q^2$.

**Remark.** If $2 \nmid dq$, then no TD$_{dq^2}(q^2, 2)s$ are obtained from difference matrices by Drake’s result.
§3 Direct product RDSs and SCTs

Let $\mathcal{G}$ be a group of order $um$ and $U$ its (not necessarily normal) subgroup of order $u$. A $k$-subset $D$ of $\mathcal{G}$ is called an $(m, u, k, \lambda)$-relative difference set (or, RDS for short) relative to $U$ if $DD^{(-1)} = k + \lambda(\mathcal{G} - U)$. Usually $U$ is called the forbidden subgroup.

An $(m, u, k, \lambda)$-divisible design $\mathcal{D} = (\mathbb{P}, \mathbb{B})$ is obtained from $(m, u, k, \lambda)$-RDS in the following way: the set $\mathbb{P}$ of points are elements of $\mathcal{G}$ and the set of blocks $\mathbb{B}$ are subsets $Dx(x \in \mathcal{G})$. We note that the set of point classes are $\{Ug \mid g \in \mathcal{G}\}$.

We say $\mathcal{G}$ is splitting (over $U$) if there exists a subgroup $G$ of $\mathcal{G}$ of order $m$ such that $\mathcal{G} = GU$ and $G \cap U = 1$. In this case $G$ is an SCT$(m, u, k, \lambda)$ group of $\mathcal{D}$.

From now on we consider an SCT matrix obtained from a splitting abelian RDS; $\mathcal{G} = G \times U$.

Hypothesis 3. Let $G = \{g_1, \cdots, g_m\}$ and $U = \{w_1, \cdots, w_u\}$ be abelian groups of order $m$ and $u$, respectively. Suppose $D$ is an $(m, u, k, \lambda)$-RDS in the group $\mathcal{G} = G \times U$ relative to $U$. Set $\mathbb{P} = \mathcal{G} = \{w_ig_j \mid i \in I_u, j \in I_u\}$ and $\mathbb{B} = \{Dw_ig_j \mid i \in I_u, j \in I_u\}$. Then $\mathcal{D}_{D, \mathcal{G}} := (\mathbb{P}, \mathbb{B})$ is a $(m, u, k, \lambda)$-DD with the set $\mathbb{C} = \{Ug_1, \cdots, Ug_m\}$ of point classes.

We now consider the action of $G$ on $(\mathbb{P}, \mathbb{B})$ as an SCT group.

\begin{align*}
\{w_iG \mid i \in I_u\} & : \text{the set of } G\text{-orbits on } \mathbb{P}, \\
\{Dw_iG \mid i \in I_u\} & : \text{the set of } G\text{-orbits on } \mathbb{B}, \\
D &= G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \quad (\exists G_{w_1}, \cdots, \exists G_{w_u} \subset G).
\end{align*}

We choose a point class $\mathcal{C} = \{w_1, \cdots, w_u\} \in \mathbb{C}$ as a set of representatives of $G$-orbits on $\mathbb{P}$ and $\{Dw_1, \cdots, Dw_u\} \in \mathbb{B}$ as a set of representatives of $G$-orbits on $\mathbb{B}$.

Direct product RDSs and SCTs

Under Hypothesis 3, the corresponding $u \times u$ SCT matrix $[D_{ij}]$ is given by

\[D_{ij} = \{g \in G \mid (w_i)g \in Dw_j\} = G \cap Dw_i^{-1}w_j.\]

As $D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u \quad (G_{w_1}, \cdots, G_{w_u} \subset G)$, we have $[D_{ij}] = [G_{w_i}^{-1}w_j^{-1}]$, which we call an SCT matrix of standard form with respect to $\{D, G \times U\}$.

Similarly, if we choose a point class $\mathcal{C} = \{w_1g, \cdots, w_ng\} \in \mathbb{C}$ ($g \in G$) and $\{Dw_1g_{n_1}, \cdots, Dw_ng_{n_u}\} \subset \mathbb{B}$ ($n_1, \cdots, n_u \in I_u$) as sets of representatives of $G$-orbits on $\mathbb{P}$ and $\mathbb{B}$, respectively, then we have the following.

Lemma 4. Under Hypothesis 3, set $D = G_{w_1}w_1 \cup \cdots \cup G_{w_u}w_u$, where $G_{w_1}, \cdots, G_{w_u} \subset G$. Then a $u \times u$ matrix $[G_{w_i}^{-1}w_j^{-1}g_{n_j}]$ is an SCT$(m, u, k, \lambda)$ matrix.
Let notations be as in Lemma 4. Then we have the following.

**Proposition 5.** Set $M = [G_{w_iw_j^{-1}}]$, the SCT matrix of standard form with respect to $\{D, G \times U\}$. Then,

(i) any SCT matrix is obtained from $M$ by multiplication of any column by an element of $G$ and any permutation of rows and columns;

(ii) $M$ is circulant if $u$ is a prime and $w_i = w^{i-1}$ for $i \in I_u$, where $U = \langle w \rangle$.

§ 4 **Spreads and SCTs**

**Theorem 6.** Let $q = p^e$ be a power of a prime $p$ and let $G$ be an elementary abelian $p$-group of order $q^2$. Let $\{H_1, \ldots, H_{q+1}\}$ be a spread of $G$ (i.e. $|H_i| = q, |H_i \cap H_j| = 1, \forall i \neq j$). Set $q_0 = q/p^m (= p^{e-m})$ and

\[
\begin{align*}
A_i &= H_{iq_0+1}^* + H_{iq_0+2}^* + \cdots + H_{(i+1)q_0}^* \quad (0 \leq i \leq p^m - 2), \\
A_{p^m-1} &= H_{1q_0+1}^* + H_{2q_0+1}^* + \cdots + H_{q_0+1}^* + H_{p^mq_0+1}^* + 1
\end{align*}
\]

Let $L = [n_{ij}]$ be a Latin square of order $p^m$ with entries from $\{0, 1, \ldots, p^m - 1\}$. Then the following is an $SCT(p^{2e}, p^m, p^{2e}, p^{2e-m})$ matrix, which gives an $STD_{q^2/p^m}(p^{2e}, p^m)$.

\[
\begin{bmatrix}
A_{n_{1,1}} & A_{n_{1,2}} & \cdots & A_{n_{1,p^m}} \\
A_{n_{2,1}} & A_{n_{2,2}} & \cdots & A_{n_{2,p^m}} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n_{p^m,1}} & A_{n_{p^m,2}} & \cdots & A_{n_{p^m,p^m}}
\end{bmatrix}
\]

Sketch of proof: (1) $\sum_{i \in I_{p^m}} A_i A_i^{-1} = q^2 + qq_0(G-1)$ ($\forall i \in I_{p^m}$).

(2) If $\{n_{i1}, \ldots, n_{ip^m}\} = \{n_{\ell 1}, \ldots, n_{\ell p^m}\} = I_{p^m}$ and $n_{i1} \neq n_{\ell 1}, \ldots, n_{ip^m} \neq n_{\ell p^m}$, then

\[
A_{i1} A_{\ell 1}^{-1} + \cdots + A_{ip^m} A_{\ell p^m}^{-1} = q_0 q(G-1)
\]

**An equivalence class in Latin squares of order $n$**

We show that some of the STDs obtained in Theorem 6 admit no class regular automorphism groups. This implies that these STDs are never obtained from generalized Hadamard matrices. In order to prove this we need a lemma on the set of Latin squares.

**Definition.** Let $e_1 = (1, 0, 0, \cdots, 0)$, $e_2 = (0, 1, 0, \cdots, 0)$, $\cdots$ be vectors of $V(n, \mathbb{C})$. For a permutation $\sigma = \left( \begin{array}{cccc}
1 & 2 & \cdots & n \\
r_1 & r_2 & \cdots & r_n
\end{array} \right)$ of $\Omega := \{1, 2, \ldots, n\}$, a permutation matrix $P_\sigma$ is defined by $e_i P_\sigma = e_{r_i}$ for each $i \in I_n$. Let $N$ be the group of permutation matrices of order $n$ and $\mathscr{L}$ the set of Latin squares on $\Omega$. We say Latin squares $L_1$ and $L_2$ in $\mathscr{L}$ are equivalent if $L_2 = PL_1Q$ for some $P, Q \in N$. Let $H := N \times N$ be the direct product and define the action of $H$ on $\mathscr{L}$ by $L(P, Q) = P^T L Q$ for $L \in \mathscr{L}$. Then $H$ is a permutation group on $\mathscr{L}$.
The number of Latin squares of order $n$

Let $\mathcal{L}_n$ be the set of Latin squares of order $n$ on $\{1, \ldots, n\}$. By Theorem III.1.19 of [1],

$$|\mathcal{L}_n| > f(n) := (n!)^{2n}/n^{n^2} \text{ for } n > 1.$$ 

$|\mathcal{L}_2| = (2-1)!2! > \lfloor f(2) \rfloor = 1,$

$|\mathcal{L}_3| = (3-1)!3! > \lfloor f(3) \rfloor = 2,$

$|\mathcal{L}_4| = 4(4-1)!4! > \lfloor f(4) \rfloor = 25,$

$|\mathcal{L}_5| = 56(5-1)!5! = 161280 > \lceil f(5) \rceil = 2077$.

Latin squares equivalent to a circulant one

$\mathcal{L} = \text{the set of Latin squares on } \Omega := \{1, 2, \ldots, n\}$

$N = \text{the group of permutation matrices of order } n$

$N \times N = \text{the permutation group on } \mathcal{L} \text{ defined by } L(P, Q) = P^T L Q$

Lemma. Let $C$ be a circulant matrix of order $n$ whose first row is $(a_1, a_2, \ldots, a_n)$ with $\{a_1, a_2, \ldots, a_n\} = \Omega$. Let $T \in N$ be a circulant permutation matrix whose first row is $(0, 1, 0, \ldots, 0)$. If $Q, R \in N$ and $QC = CR$ then $Q = R$ and $Q \in \langle T \rangle$.

Lemma 7. Assume $C \in \mathcal{L}$ and $C$ is circulant. Then,

(i) The number of Latin squares in $\mathcal{L}$ equivalent to $C$ is $(n!)^2/n$;

(ii) If $n \geq 4$, then there exists a Latin square of $\mathcal{L}$ not equivalent to circulant one.

\[ \vdots \] By Theorem III.1.19 of [1], $|\mathcal{L}_n| > (n!)^{2n}/n^{n^2}$. As $(n!)^{2n}/n^{n^2} > (n-1)!(n!)^2/n$, $(n \geq 4)$, the lemma holds.

Non class regular STDs

Theorem. Let $p > 3$ be a prime and $A_L$ the SCT($p^{2e-1}, p^{2e}, p, p^{2e}$) matrix defined in Theorem 6. Then the STD($p^{2e-1}$,$p^{2e}$,$p$) obtained from $A_L$ is not class regular.

Proof. By Lemma 7, there exists a Latin square $L$ not equivalent to a circulant one. Let $(\mathbb{P}, \mathbb{B})$ be the STD($p^{2e-1}$,$p^{2e}$,$p$) obtained from $A_L$ and let $G$ be the SCT($p^{2e-1}$,$p^{2e}$,$p$, $p^{2e}$) automorphism group of order $p^{2e}$. Suppose false and let $U$ be a class regular automorphism group of $(\mathbb{P}, \mathbb{B})$. Then, as $G$ normalizes $U$ and $|U| = p$, $G$ centralizes $U$. The direct product $G := G \times U$ contains a $(p^{2e}, p, p^{2e}, p^{2e-1})$-RDS corresponding to $(\mathbb{P}, \mathbb{B})$. By Proposition 5, $L$ must be equivalent to a circulant Latin square, a contradiction.
§5 RDS and $\lambda$-planar functions

In this section we define a $\lambda$-planar function as a generalization of planar functions.

**Theorem.** Let $G = GU$ be a group of order $mu$ and $G, U$ its subgroups with $|G| = m, |U| = u$ and $G \triangleright U$. Let $D$ be a $(m, u, k, \lambda)$-RDS in $G$ relative to $U$. Then there exists a $k$-subset $C$ of $G$ and a function $f : C \to U$ satisfying the following.

(i) $D = \{xf(x) | x \in C\}$

(ii) $\#\{x \in C | ax \in C, f(ax)^\varphi(a)f(x)^{-1} = b\} = \lambda$

for any $a \in G \setminus \{1\}$ and $b \in U$.

**Proposition.** Let $G, U$ be groups of order $m, u$, respectively. Let $\varphi$ be a homomorphism from $G$ to $\text{Aut}(U)$ and $f$ a function from $C$ to $U$ for a $k$-subset $C$ of $G$. Assume that for any $a \in G \setminus \{1\}$ and $b \in U$

\[
(*) \quad \#\{x \in C | ax \in C, f(ax)^\varphi(a)f(x)^{-1} = b\} = \lambda.
\]

Then $D = \{xf(x) | x \in C\}$ is a $(m, u, k, \lambda)$-RDS in a semi-direct product $G = GU$ of $G$ by $U$ with respect to $\varphi$.

**Definition.** Let $G$ and $U$ be groups. Let $C$ be a subset of $G$ and $\varphi \in \text{Hom}(G, \text{Aut}(U))$. We call a function $f : C \to U$ a $\lambda$-planar function relative to $(C, U, \varphi)$ if $f$ satisfies $(\star)$. If $\varphi$ is a trivial homomorphism, we say $f$ is a $\lambda$-planar function relative to $(C, U)$. We note that a 1-planar function relative to $(G, U)$ is just a planar function in the usual sense (see Pott [5]).

**Example.** Let $q = p^e$ be a power of a prime $p$ and set $G = F = (GF(q^2), +) \supset U = K = (GF(q), +)$. Then a function

\[
f(x) = x^{q+1}
\]

from $G$ to $U$ is a $q$-planar function relative to $(G, U)$.

\[
\therefore \text{Let } 0 \neq a \in G \text{ and } b \in U. \text{ Then, } f(a + x) - f(x) = b \iff (a^q + x^q)(a + x) - x^{q+1} = b
\]

\[
\iff ax^q + a^q x = b - a^{q+1} \quad (\star \star).
\]

As $ax^q + a^q x = ax^q + (ax^q)^q = \text{Tr}_{F/K}(ax^q)$, $(\star \star)$ has exactly $q$ solutions in $G$. Thus $f$ is a $q$-planar function relative to $(G, U)$.
\(\lambda\)-planar functions, SCTs, and RDSs

**Theorem 8.** Let \(G\) be a group of order \(m\) and \(U\) a group of order \(u\). Let \(D_y\) be subsets of \(G\) for each \(y \in U\). If a \(u \times u\) matrix \(D = [D_{yz^{-1}}]_{y,z \in U}\) over \(\mathbb{Z}[G]\) is an SCT\((m, u, k, \lambda)\) matrix, then the following holds.

(i) Set \(C = \bigcup_{y \in U} D_y (\subset G)\). Then \(|C| = k, G = \langle C \rangle\) and a function \(f : C \rightarrow U\) defined by \(f(D_y) = y\) (\(y \in U\)) is a \(\lambda\)-planar function relative to \((C, U)\).

(ii) Set \(D = \{(x, f(x) | x \in C\}\}. Then \(D\) is an \((m, u, k, \lambda)\)-RDS in \(G \times U\) relative to \(1 \times U\).

**Remark.** A \((u\lambda, u, u\lambda, \lambda)\)-RDS is called semiregular. It is conjectured that any forbidden subgroup of a semiregular RDS is a \(p\)-group for a prime \(p\). Concerning this we can show the following as an application of Theorems 6 and 8.

**Theorem.** Any \(p\)-group can be a forbidden subgroup of a semiregular RDS.

As a corollary we have the following, which gives another proof of de Launey's result on generalized Hadamard matrices (cf. [1], Theorem 5.9).

**Corollary** There exists a \(\text{GH}(p^m, p^{2e-m})\) matrix over any group of order \(p^m\) whenever \(e \geq m\).

**References**


