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On a class of indecomposable modules with trivial source

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Abstract The concept of p-radical groups was introduced by Motose-Ninomiya [MN]. Later Tsushima [Ts] investigated p-radical blocks, a block-wise version of p-radical groups. Here we consider more general blocks and introduce module-theoretical viewpoint.

Introduction

Let p be a prime. Let k be an algebraically closed field of characteristic p. Tsushima [Ts] has defined p-radical blocks. In this paper we consider a more general concept and give a module theoretical consideration. We need to introduce some terminology. Let G be a group and P a p-subgroup of G. An indecomposable (right) kG-module S is said to be weakly P-radical if it is P-projective and the number of indecomposable summands (counting multiplicity) of S_P equals $\frac{\dim S|\mathrm{vx}(S)|}{|P|}$ (for notation, see below). A simple kG-module S is said to be P-radical if $(1_P)^G \simeq mS \oplus V$, where m is an integer and V is a kG-module not involving S, in which case m is positive, since $\mathrm{Hom}_{kG}(S, (1_P)^G) \neq 0$. We call a p-block B of G weakly P-radical if any simple kG-module in B is weakly P-radical. We call B P-radical if any simple kG-module in B is P-radical. Clearly B is P-radical if and only if $(1_P)^G e_B$ is semi-simple, where e_B is a block idempotent of kG corresponding to B. So when P is a Sylow p-subgroup of G, a P-radical block is a p-radical block in the sense of Tsushima [Ts].

In Section 1 we show for any p-subgroup P of G, B is P-radical if and only if B is weakly P-radical. In Section 2 we consider relationship between weakly P-radical simple modules and subgroups of G for a Sylow p-subgroup P of G. We obtain an alternative proof of a theorem of Laradji [La]. In Section 3 we consider D-radical blocks B for a defect group D of B and strengthen a theorem of Hida-Koshitani [HK].

For a k-module X dim X denotes the k-dimension of X. For an indecomposable kG-module S, let $\mathrm{vx}(S)$ be a vertex of S. For a group H and kH-modules X, Y, Hom (X, Y) denotes $\mathrm{Hom}_{kH}(X, Y)$ and let $P(X)$ be the projective cover of X. For subgroups H, K of G, $H\backslash G/K$ denotes a complete set of representatives of (H, K)-double cosets in G.

1. Weakly P-radical and P-radical modules

Let P be a p-subgroup of the group G. For an indecomposable kG-module S, let $n_{S,P}$ be the number of indecomposable summands of S_P (counting multiplicity), let $n'_{S,P}$ be the number of indecomposable summands of S_P (counting multiplicity) whose vertices are G-conjugate to $\mathrm{vx}(S)$. Note that $n'_{S,P}$ is positive, if S is P-projective, cf.[Fe,III 4.6]. Let $m_{S,P}$ be the multiplicity of S in $(1_P)^G$ as direct summands. If S is simple, let $k_{S,P}$ be the multiplicity of S in $(1_P)^G$ as irreducible constituents.

Lemma 1. *Let S be a P-projctive indecomposable kG-module. Then $n_{S,P} \le \frac{\dim S|\mathrm{vx}(S)|}{|P|}$ and the following are equivalent.*

(i) $n_{S,P} = \frac{\dim S|\mathrm{vx}(S)|}{|P|}$.

(ii) *S_P is a direct sum of modules of the form $(1_A)^P$ where A is a vertex of S contained in P.*

(iii) *$S_P \simeq \oplus_i (1_{Q_i})^P$, where Q_i are subgroups of P of the same order.*
If these conditions hold, then S has a trivial source.

Proof. We have $S_P = \oplus_{i=1}^{n_{S,P}} W_i{}^P$, where W_i are indecomposable kQ_i-modules for $Q_i \le P$ with $Q_i \le_G \mathrm{vx}(S)$. So $\dim S = \sum_i |P : Q_i| \dim W_i \ge (\sum_i \dim W_i)|P|/|\mathrm{vx}(S)| \ge n_{S,P}|P|/|\mathrm{vx}(S)|$. Thus $n_{S,P} \le \frac{\dim S|\mathrm{vx}(S)|}{|P|}$. The rest follows from [Fe, III 4.6]. □

As stated in Introduction, for a p-subgroup P of G, we say an indeccomposable kG-module S weakly P-radical if S is P-projective and $n_{S,P} = \frac{\dim S|\mathrm{vx}(S)|}{|P|}$. For this definition we have the following, which is straightforward to see.

Lemma 2. *Let P be a p-subgroup of G and let S be an indecomposable kG-module. Let x be any element of G. Then $n_{S,P} = n_{S,P^x}, n'_{S,P} = n'_{S,P^x}, m_{S,P} = m_{S,P^x}$ and if S is simple $k_{S,P} = k_{S,P^x}$. In particular, if S is weakly P-radical, then S is weakly P^x-radical.*

Recall that a weight U for G is a projective simple $k[N_G(Q)/Q]$-module for a p-subgroup Q of G ([Al]). So as a $kN_G(Q)$-module U is indecomposable with trivial source and has Q as a vertex. The Green correspondent of U with respect to $(G, Q, N_G(Q))$ is said to be an Alperin (kG-)module.

The folowing strengthens Lemma 1 of [Al].

Theorem 3. *Let P be a p-subgroup of G. Let S be a P-projective indecomposable kG-module with trivial source. Then $m_{S,P} \le n'_{S,P}$ and the equality holds if and only if S is an Alperin module.*

Proof. We compute $n'_{S,P}$. Let Q be a vertex of S. Let U be the Green correspondent of S with respect to $(G, Q, N_G(Q))$. By Green's theorem, $U^G = S \oplus V$, where V is $\mathcal{X}$-projective for $\mathcal{X} = \{Q \cap Q^x; x \notin N_G(Q)\}$. Since V_P has no direct summands whose vertex is G-conjugate to Q, it suffices to consider $(U^G)_P$. By Mackey decomposition, $(U^G)_P \simeq \oplus_{x\in N_G(Q)\backslash G/P}((U^x)_{N_G(Q)^x\cap P})^P$. Assume that $((U^x)_{N_G(Q)^x\cap P})^P$ has an indecomposable summand with vertex G-conjugate to Q. Then for a vertex R of some indecomposable summand of $(U^x)_{N_G(Q)^x\cap P}$, we have $R^u \geq Q^g$ for some $u \in P$ and $g \in G$. Since U^x has a vertex Q^x, which is normal in $N_G(Q)^x$, we have $R \leq Q^x$. Therefore $R = Q^x$. Hence $P \geq Q^x$. Conversely assume $P \geq Q^x$. Then $N_G(Q)^x \cap P \geq Q^x$. Therefore $(U^x)_{N_G(Q)^x\cap P} = \overline{U^x}_{\overline{N_G(Q)^x\cap P}}$, where $\overline{N_G(Q)^x \cap P} = N_G(Q)^x \cap P/Q^x$ and $\overline{U^x}$ is the $N_G(Q^x)/Q^x$-module corresponding to U^x.

Since $\overline{U^x}$ is projective, we obtain $\overline{U^x}_{\overline{N_G(Q)^x\cap P}} \simeq \frac{\dim U}{|\overline{N_G(Q)^x\cap P}|}k[\overline{N_G(Q)^x \cap P}]$. Since $k[\overline{N_G(Q)^x \cap P}] = (1_{Q^x})^{N_G(Q)^x\cap P}$, we have $((U^x)_{N_G(Q)^x\cap P})^P \simeq \frac{\dim U}{|\overline{N_G(Q)^x\cap P}|}(1_{Q^x})^P$. Therefore $n'_{S,P} = \sum_{x\in N_G(Q)\backslash G/P, P\geq Q^x} \frac{\dim U}{|\overline{N_G(Q)^x\cap P}|}$.

Now we consider $m_{S,P}$. By the Burry-Carlson-Puig theorem, $m_{S,P}$ equals the multiplicity of U in $((1_P)^G)_{N_G(Q)}$ as direct summands. By Mackey decomposition, we have

$$((1_P)^G)_{N_G(Q)} \simeq \oplus_{x\in P\backslash G/N_G(Q)}(1_{P^x\cap N_G(Q)})^{N_G(Q)}.$$

Since U has vertex Q it suffices to consider those $x \in P\backslash G/N_G(Q)$ for which $Q \leq P^x$. Then, $(1_{P^x\cap N_G(Q)})^{N_G(Q)} = (1_{\overline{P^x\cap N_G(Q)}})^{\overline{N_G(Q)}}$, where $\overline{N_G(Q)} = N_G(Q)/Q$. Put $(1_{\overline{P^x\cap N_G(Q)}})^{\overline{N_G(Q)}} \simeq n_x\bar{U} \oplus V_x$, where $\bar{U}$ is the $\overline{N_G(Q)}$-module correspoding to U and

V_x has no summands isomorphic to $\bar{U}$. Then $m_{S,P} = \sum_{x\in P\backslash G/N_G(Q), Q\leq P^x} n_x$. Now

$$\dim \mathrm{Hom}((1_{\overline{P^x\cap N_G(Q)}})^{\overline{N_G(Q)}}, \bar{U}) = n_x \dim \mathrm{Hom}(\bar{U}, \bar{U}) + \dim \mathrm{Hom}(V_x, \bar{U}) \geq n_x.$$

On the other hand,

$$\begin{aligned} \dim \mathrm{Hom}((1_{\overline{P^x\cap N_G(Q)}})^{\overline{N_G(Q)}}, \bar{U}) &= \dim \mathrm{Hom}((1_{\overline{P^x\cap N_G(Q)}}, \bar{U}_{\overline{P^x\cap N_G(Q)}}) \\ &= \frac{\dim U}{|\overline{P^x \cap N_G(Q)}|}, \end{aligned}$$

since $\bar{U}$ is projective. Therefore

$$\sum_{x\in P\backslash G/N_G(Q), Q\leq P^x} \dim \mathrm{Hom}(1_{\overline{P^x\cap N_G(Q)}})^{\overline{N_G(Q)}}, \bar{U}) = \sum_{x\in P\backslash G/N_G(Q), Q\leq P^x} \frac{\dim U}{|\overline{P^x \cap N_G(Q)}|} = n'_{S,P}.$$

Here the last equality follows by considering the correspondence $x \mapsto x^{-1}$. Hence $n'_{S,P} \geq \sum_x n_x = m_{S,P}$. If the equality holds then $m_{S,P} = n'_{S,P} \neq 0$, since S is P-projective,cf, [Fe,III 4.6]. So $n_x \neq 0$ for some x. Thus $\dim \mathrm{Hom}\ (\bar{U}, \bar{U}) = 1$. Since $\bar{U}$ is projective we see $\bar{U}$ is simple and S is Alperin. Conversely assume U is simple. Then equality holds throughout. Hence $m_{S,P} = n'_{S,P}$. The proof is complete. □

Corollary 4. *Let S be a P-projective simple module with trivial source. Then*

$$m_{S,P} \le n'_{S,P} \le n_{S,P} \le k_{S,P} = \frac{\dim P(S)}{|P|}.$$

Proof. The first inequality follows from Theorem 3. The second is trivial. To prove the third, put $S_P \simeq \oplus_{i=1}^{n_{S,P}} (1_{Q_i})^P$ for suitable Q_i. Then $n_{S.P}$ =dim Hom$(S_P, 1_P)$ =dim Hom $(S, (1_P)^G) \le k_{S,P}$. Further $k_{S,P}$ = dim Hom $(P(S), (1_P)^G)$ = dim Hom $(P(S)_P, 1_P) = \frac{\dim P(S)}{|P|}$, since $P(S)_P$ is projective. □

Proposition 5. *Let S be a simple module. If S is P-radical, then S is weakly P-radical.*

Proof. Since S is P-projective and has trivial source, by Corollary 4 we have $m_{S,P} \le n'_{S,P} \le n_{S,P} \le k_{S,P}$. By assumption $m_{S,P} = k_{S,P}$, so that $n'_{S,P} = n_{S,P}$. Thus S is weakly P-radical. □

When P is a Sylow p-subgroup of G, a weakly P-radical module is said to be just a weakly radical module. The same is true for other terminology. Also $n_{S,P}$ is denoted by n_S etc. Such convention is justified by Lemma 2. Note that then radical blocks are p-radical blocks as defined by Tsushima [Ts]. .

Corollary 6. ([Ok2, Lemma 1]) *A radical (simple) module is weakly radical.*

The following is fundamental.

Proposition 7. ([Ok1, Lemma 2.2]). *A simple kG-module with trivial source is an Alperin module.*

Lemma 8. *If S is an Alperin module, then* $(\dim S)_p = |P : \mathrm{vx}(S)|$, *where P is a Sylow p-subgroup of G with* $\mathrm{vx}(S) \le P$.

Proof. See the proof of Lemma 2.2 of [Ok1]. □

Proposition 9. *Let S be a simple kG-module. Let P be a Sylow p-subgroup of G.*

(i) *If S is weakly radical,* $\dim P(S) \ge |\mathrm{vx}(S)|\dim S \ge |P|(\dim S)_{p'}$.

Furthermore the following conditions are equivalent.

(ii) *S is radical.*

(iii) *S is weakly radical and* $\dim P(S) = |P|(\dim S)_{p'}$.

(iv) *S is weakly radical and* $\dim P(S) = |\mathrm{vx}S|\dim S$.

(v) $n_S = \frac{\dim P(S)}{|P|}$.

Proof. We may assume $P \ge \mathrm{vx}(S)$.

(i) By Corollary 4 we have $\frac{\dim P(S)}{|P|} = k_S \ge n_S = \frac{\dim S}{|P:\mathrm{vx}(S)|}$, from which the first inequality follows. The second inequality follows from Green's theorem.

(ii)⇒(iii): By Corollary 6, S is weakly radical. Also by Corollary 4 $n_S = k_S$. We have $n_S = \frac{\dim S}{|P:\mathrm{vx}(S)|} = (\dim S)_{p'}$ by Proposition 7 and Lemma 8. And $k_S = \frac{\dim P(S)}{|P|}$. Thus the equality holds.

(iii)⇒(iv): This follows from (i).

(iv)⇒(v): Since S is weakly radical, $n_S = \frac{\dim S}{|P:\mathrm{vx}(S)|}$. The result follows.

(v)⇒(ii): Write $S_P = \oplus_{i=1}^{n_S}(W_i)^P$, where each W_i is an indecomposable kQ_i-module for some $Q_i \leq P$. Then dim Hom $(1_P, S_P) = \sum_i$ dim Hom $(1_P, (W_i)^P) = \sum_i$dim Hom $(1_{Q_i}, W_i) \geq n_S$. So we have

$$n_S \leq \dim \mathrm{Hom}(1_P, S_P) = \dim \mathrm{Hom}((1_P)^G, S) \leq k_S = \frac{\dim P(S)}{|P|}.$$

Hence equality holds throughout. Likewise we have dim Hom$(S, (1_P)^G) = k_S$. Hence there exist submodules U and V of $(1_P)^G$ with the following properties: $U \simeq k_S S$, $(1_P)^G/V \simeq k_S S$ and V does not involve S. Then $U \cap V = 0$ and hence $(1_P)^G = U \oplus V$. Thus S is radical. The proof is complete. □

Corollary 10. *Let S be a simple kG-module for a p-solvable group G. Then S is radical if and only if S is weakly radical.*

Proof. "only if" part: This follows from Corollary 6.
"if" part: Let P be a Sylow p-subgroup of G. Since G is p-solvable dim $P(S) = |P|(\dim S)_{p'}$ by Fong's theorem [Na, Corollary10.14]. Thus Proposition 9 yields the result. □

Remark. There does exist a simple kG-module which is weakly radical but not radical. Indeed, clearly 1_G is always weakly radical. Let G be the alternating group of degree 5 and $p = 3$. Then dim $P(1_G) = 6$([HB, p.222]). So by Proposition 9, 1_G is not radical.

Corollary 11. If B is radical, then $(1_P)^G e_B \simeq \oplus_S (\dim S)_{p'} S$, where S runs through simple modules in B up to isomorphism.

Theorem 12. *Let P be a p-subgroup of G. Then B is P-radical if and only if B is weakly P-radical.*

Proof. "if" part: Let $(1_P)^G e_B \simeq \oplus_S m_{S,P} S \oplus X$, where S runs through simple modules in B up to isomorphism. Assume $X \neq 0$ and let T be a simple submodule of X. Then $\dim \mathrm{Hom}(T, (1_P)^G) > m_{T,P}$. But $\dim \mathrm{Hom}(T, (1_P)^G) = \dim \mathrm{Hom}(T_P, 1_P) = n_{T,P} = n'_{T,P} = m_{T,P}$ by Proposition 7 and Theorem 3, a contradiction. Hence $X = 0$ and B is P-radical.
"only if" part: This follows from Proposition 5. □

The group G is said to be p-radical, if $(1_P)^G$ is semi-simple for a Sylow p-subgroup P of G ([Ts,p.80]),

Corollary 13. G is p-radical if and only if any simple kG-module is weakly radical.

Lemma 14. *If an Alperin module S is weakly radical, then S is simple.*

Proof. By Theorem 3 $m_S = n_S$. From $(1_P)^G = m_S S \oplus V$, we have

$$n_S = \dim \mathrm{Hom}(S_P, 1_P) = \dim \mathrm{Hom}(S, (1_P)^G) = m_S \dim \mathrm{Hom}(S, S) + \dim \mathrm{Hom}(S, V)$$

Thus Hom $(S, S) = k$ and Hom $(S, V) = 0$. Let T be a simple module in the head of S. Since Hom $(T, (1_P)^G)$= Hom $(T_P, 1_P) \neq 0$, T is a submodule of V or S. The former is impossible, since Hom $(S, V) = 0$. Thus the latter holds. Then there is a non-zero homomorphism $\varphi : S \to \mathrm{Soc}(S)$. Of course $\varphi(J(S)) = 0$. Since Hom $(S, S) = k$, φ must be a monomorphism. Therefore $J(S) = 0$. Thus S is simple.

Proposition 15. *Let B be a block of G. Assume that Alperin's weight conjecture [Al] is true for B. Then the following are equivalent.*
(i) *B is radical.*
(ii) *$(1_P)^G e_B$ is a direct sum of weakly radical indecomposable modules.*
(iii) *All Alperin modules in B are weakly radical.*

Proof. (i)$\Rightarrow$(ii): Any simple module S in B is radical. Hence S is weakly radical by Corollary 6.

(ii)$\Rightarrow$(iii):Let S be an Alperin module in B. Then $m_S = n'_S > 0$ by Theorem 3 and [Fe,III 4.6]. Hence S is weakly radical.

(iii)$\Rightarrow$(i): Let S be an Alperin module in B. Then S is weakly radical. Hence S is simple by Lemma 14. Thus, by Alperin's weight conjecture, any simple module T in B is an Alperin module. Hence T is weakly radical. So B is weakly radical and radical by Theorem 12.

Proposition 16. *Let S be an indecomposable kG-module. If* dim *S is prime to p, then S is weakly radical if and only if G/*Ker*S is a p'-group.*

Proof. (i) "only if" part: Let P be a Sylow p-subgroup of G. By Lemma 1, we have $S_P \simeq \oplus_i (1_{Q_i})^P$, where Q_i are vertices of S. Thus $Q_i = P$ for all i and $P \leq$Ker$S$.

"if" part: Since $P \leq$ Ker$S$, the result follows by Lemma 1.

2. Weakly radical simple modules and subgroups

In this section we consider relationship between weakly radical simple modules and subgroups.

Proposition 17. *Let S be a simple kG-module with trivial source. Let H be a subgroup of G and let U be a simple kH-module such that $S \simeq U^G$.*

(i) *If S is weakly radical, then U is weakly radical.*
(ii) *Let P be a Sylow p-subgroup of G. The following are equivalent.*
(iia) *S is radical and $P(S) \simeq P(U)^G$.*
(iib) $\dim \mathrm{Inv}_{P^x \cap H}(U) = \frac{\dim P(U)}{|P^x \cap H|}$ *for any $x \in G$.*
(iic) *U is radical and S is weakly radical.*

Proof. (i) Choose a Sylow p-subgroup P of G such that $Q = P \cap H$ is a Sylow p-subgroup of H. We have $(U_Q)^P | S_P$ by Mackey decomposition. Since S has a trivial source, so does U. So we can put $U_Q \simeq \oplus_i (1_{R_i})^Q$ for some subgroups R_i of Q. Then $(U_Q)^P \simeq \oplus_i (1_{R_i})^P$. Since S is weakly radical, all R_i have the same order. Thus U is weakly radical by Lemma 1.

(iia)⇒(iib): We have $n_S = \dim \mathrm{Hom}(1_P, S_P) = \dim \mathrm{Hom}((1_P)^G, S) = \dim \mathrm{Hom}(((1_P)^G)_H, U) = \sum_{x \in P \backslash G/H} \dim \mathrm{Hom}((1_{P^x \cap H})^H, U)$. Here

$$\dim \mathrm{Hom}((1_{P^x \cap H})^H, U) = \dim \mathrm{Hom}(1_{P^x \cap H}, U_{P^x \cap H}) = \dim \mathrm{Inv}_{P^x \cap H}(U).$$

And

$$\begin{aligned} \dim \mathrm{Hom}((1_{P^x \cap H})^H, U) &\leq \dim \mathrm{Hom}(P(U), (1_{P^x \cap H})^H) \\ &= \dim \mathrm{Hom}(P(U)_{P^x \cap H}, 1_{P^x \cap H}) = \frac{\dim P(U)}{|P^x \cap H|}. \end{aligned}$$

Further, $\sum_x \frac{|H|_p}{|P^x \cap H|} = |G : H|_{p'}$. Therefore $n_S \leq \frac{\dim P(U) |G:H|_{p'}}{|H|_p} = \frac{\dim P(S)}{|G|_p} = k_S$. Since S is radical, equality holds throughout by Proposition 9, and the result follows.

(iib)⇒(iia): From the above proof we obtain $n_S = \frac{\dim P(U)|G:H|}{|G|_p}$. Since $P(S) | P(U)^G$, $\frac{\dim P(U)|G:H|}{|G|_p} \geq \frac{\dim P(S)}{|G|_p} = k_S$. Therefore $n_S = k_S$ by Corollary 4, and S is radical by Proposition 9. Further, $P(S) \simeq P(U)^G$.

(iia)⇒(iic): Since S is weakly radical by Corollary 6, U is weakly radical by (i). So by Proposition 9 it suffices to show dim $P(U) = |\mathrm{vx}(U)| \dim U$. We have $\dim P(S) = |G : H| \dim P(U)$. Since S is radical, by Proposition 9 dim $P(S) = |\mathrm{vx}(S)| \dim S = |\mathrm{vx}(S)||G : H| \dim U$. Since $\mathrm{vx}(S) =_G \mathrm{vx}(U)$, the result follows.

(iic)⇒(iia):Since U is radical, dim $P(U)^G = |G : H||\mathrm{vx}(U)| \dim U$. Since S is weakly radical, by Proposition 9 dim $P(S) \geq |\mathrm{vx}(S)| \dim(S) = |\mathrm{vx}(S)||G : H| \dim U$. Hence dim $P(S) \geq \dim P(U)^G$. But $P(S) | P(U)^G$. So the equality holds throughout. Therefore $P(S) \simeq P(U)^G$ and S is radical by Proposition 9.
□

Theorem 18([La,Theorem]) *Let P be a Sylow p-subgroup of G. The following are equivalent.*

(i) *G is p-radical.*

(ii) *For any simple kG-module S, there are a subgroup H of G and a simple kH-module U with the following properties: $S = U^G, \mathrm{vx}(U) \leq \mathrm{Ker} U, P^x \cap H$ is a Sylow p-subgroup of H for any $x \in G$.*

(iii) *For any simple kG-module S, there are a subgroup H of G and a simple kH-module U with the following properties: $S = U^G, \mathrm{vx}(S) \leq \mathrm{Ker} U, P^x \cap H$ is a Sylow p-subgroup of H for any $x \in G$.*

Proof. (i) $\Rightarrow$ (ii) G is p-solvable by [Ok2]. So there are H and U as above such that $S = U^G$ and that dim U is a p'-number by [Na,Theorem 10.11]. Since G is p-solvable, $P(S) \simeq P(U)^G$ by Fong's theorem [Na,Corollary 10.14]. Hence U is radical by Proposition 17. Therefore $\mathrm{vx}(U) \leq \mathrm{Ker}\, U$ by Corollary 6 and Proposition 16. Further, for any $x \in G$, $\dim U = \dim \mathrm{Inv}_{P^x \cap H}(U) = \frac{\dim P(U)}{|P^x \cap H|} = \frac{|H|_p \dim U}{|P^x \cap H|}$ by Proposition 16, Proposition 17 (iib) and Fong's theorem [Na,Corollary 10.14]. So $P^x \cap H$ is a Sylow p-subgroup of H for any $x \in G$.

(ii)$\Rightarrow$ (i) By Corollary 13, it suffices to show S is weakly radical. From the condition that $\mathrm{vx}(U) \leq \mathrm{Ker}(U)$, we see $U|(1_{\mathrm{Ker}(U)})^H$. This implies U is weakly radical. We have $S_P \simeq \sum_{x\in H\backslash G/P}(U^x_{H^x\cap P})^P$. Since U^x is a weakly radical kH^x-module and $H^x \cap P$ is a Sylow p-subgroup of H^x, we have $U^x_{H^x\cap P} \simeq \oplus_i(1_{Q_{x,i}})^{H^x\cap P}$ and $|Q_{x,i}| = |\mathrm{vx}U|$. Therefore $S_P \simeq \oplus_{x,i}(1_{Q_{x,i}})^P$. So S is weakly radical by Lemma 1.

(ii)$\Rightarrow$(iii). Since $\mathrm{vx}(U)$ is a vertex of S, the result follows.

(iii)$\Rightarrow$(ii). Since $\mathrm{vx}(S) \leq \mathrm{Ker}U$, $\mathrm{vx}(S) \leq \mathrm{vx}(U)$ for a vertex of U([NT,Theorem 4.7.8 (i)]). But $\mathrm{vx}(S) =_G \mathrm{vx}(U)$. So $\mathrm{vx}(U) = \mathrm{vx}(S) \leq \mathrm{Ker}U$. The proof is complete. □

In case of normal subgroups we have the following

Proposition 19. *Let N be a normal subgroup of G. Let S (resp. X) be a simple kG-(resp. kN-)module.*
(i) If $S|X^G$ and X is weakly radical, then S is weakly radical.
(ii) If $X|S_N$ and S is weakly radical, then X is weakly radical.

Proof. Let P be a Sylow p-subgroup of G.

(i) We have $S_P|(X^G)_P$. By Mackey decomposition,

$$(X^G)_P \simeq \oplus_{x_i\in N\backslash G/P}((X^{x_i})_{P\cap N})^P.$$

It is straightforward to check that for each x_i, X^{x_i} is also weakly radical. So by Lemma 1, for each i, $(X^{x_i})_{P\cap N} \simeq \oplus_j(1_{Q_{ij}})^{P\cap N}$, where Q_{ij} are subgroups of $P \cap N$ such that $|Q_{ij}| = |\mathrm{vx}(X)|$. Hence S is weakly radical by Lemma 1.

(ii) We have $X_{P\cap N}|S_{P\cap N}$. Put $S_P \simeq \oplus_i(1_{Q_i})^P$ for suitable $Q_i \leq P$. Then for each i,

$$((1_{Q_i})^P)_{P\cap N} \simeq \oplus_{u\in Q_i\backslash P/P\cap N}(1_{N\cap(Q_i)^u})^{P\cap N},$$

Since Q_i are G-conjugate, $|N \cap (Q_i)^u|$ are the same for all i and u. Thus X is weakly radical by Lemma 1. The proof is complete. □

3. D-radical blocks

Let B be a block of G with defect group D. D-radical blocks have been investigated in [Hida-Koshitani]

Lemma 20. *Let P and Q be p-subgroups of G.*

(i) *If S is a weakly P-radical module and $P \leq Q$, then S is weakly Q-radical.*

(ii) *If S is a P-radical module and $P \leq Q$, then S is Q-radical. In particular, if B is D-radical, then B is radical.*

(iii) *If B is P-radical, P contains a defect group of B.*

Proof (i) Let X be an indecomposable summand of S_Q. Then, since S is weakly P-radical, $(1_{Q_i})^P | X_P$ for some $Q_i \leq P$ with $Q_i =_G \mathrm{vx}(S)$. Then there is a vertex $\mathrm{vx}(X)$ of X with $\mathrm{vx}(X) \geq Q_i$. But $\mathrm{vx}(X) \leq_G \mathrm{vx}(S)$, so $\mathrm{vx}(X) = Q_i$. Since X has trivial source, we obtain $X = (1_{Q_i})^Q$. Thus S is weakly Q-radical.

(ii) Since there is an epi $(1_P)^Q \to 1_Q$, there is an epi $\varphi : (1_P)^G \to (1_Q)^G$. We have $(1_P)^G = U \oplus V$, where $U \simeq mS$ for some integer m and V does not involve S. Then $(1_Q)^G = \varphi(U) + \varphi(V)$. Here $\varphi(U) \simeq m'S$ for some integer m' and $\varphi(V)$ does not involve S. Hence $(1_Q)^G = \varphi(U) \oplus \varphi(V)$, and S is Q-radical.

(iii) Let S be a simple module in B with vertex D. Then S is P-radical, and S is weakly P-radical. Thus P contains a vertex of S, and the result follows. The proof is complete. □

Lemma 21. *Let S be an indecomposable kG-module. Let $\mathrm{vx}(S) = Q \leq P$ for a p-subgroup P of G. The following are equivalent.*

(i) *S is weakly P-radical and Q is strongly closed in P with respect to G.*

(ii) *S is weakly P-radical and Q is weakly closed in P with respect to G.*

(iii) *$S_P \simeq n(1_Q)^P$ for some integer n and $Q \lhd P$.*

(iv) *$Q \leq \mathrm{Ker}S$.*

Proof. (i)$\Rightarrow$ (ii): This is trivial.

(ii)$\Rightarrow$(iii): We have $S_P \simeq \oplus_i (1_{Q_i})^P$, where $Q_i =_G Q$ for each i. Since $Q, Q_i \leq P$, we obtain $Q_i = Q$. Therefore $S_P \simeq n(1_Q)^P$ for some integer n. Clearly $Q \lhd P$.

(iii)$\Rightarrow$(iv): Clearly $S_Q \simeq m1_Q$ for some integer m.

(iv)$\Rightarrow$(i): We have $S_Q \simeq m1_Q$ for some integer m, so that S is weakly Q-radical. Thus S is weakly P-radical by Lemma 20. Put $N = \mathrm{Ker}S$. Then $S_N \simeq m1_N$ and S is N-projective. Hence S and 1_N have a common vertex. Thus Q is a Sylow p-subgroup of N. Since $Q \leq P \cap N \leq N$, we obtain $Q = N \cap P$. Then for any $g \in G, Q^g \cap P \leq N \cap P = Q$. Thus Q is strongly closed in P with respect to G. The proof is complete. □

Let $B_0(G)$ be the principal block of G.

Theorem 22 (Okuyama). *If $B_0(G)$ is radical, G is p-solvable.*

Proof. See the proof of Theorem 1 of [Ok2]. □

Let $R_p(G)$ be the maximal normal p-solvable subgroup of G.

The following strengthens Theorem 1.1 of [HK].

Theorem 23. *Let P be a Sylow p-subgroup of G with $P \geq D$. The following are equivalent.*

(i) *B is D-radical.*

(ii) *B is weakly D-radical.*

(iii) *There is a p-solvable normal subgroup N of G such that: B covers $B_0(N)$, D is a Sylow p-subgroup of N, and $B_0(N)$ is radical.*

(iv) *For a block b of $R_p(G)$ covered by B, it holds that: D is a defect group of b, b is D-radical, and $G = N_G(D)R_p(G)$.*

(v) *B is radical and D is strongly closed in P with respect to G.*

(vi) *B is radical and D is weakly closed in P with respect to G.*

(vii) *B is radical and there is a simple kG-module S in B with* $\mathrm{Ker}S \geq D$.

(viii) *B is radical and there is a normal subgroup N of G such that D is a Sylow p-subgroup of N.*

Proof. (i)$\Leftrightarrow$(ii) This follows from Theorem 12.

(ii)$\Rightarrow$(iii): Let S_1 be a simple kG-module in B with vertex D. Put $N = \mathrm{Ker}S_1$. Since S_1 is weakly D-radical, $(S_1)_D \simeq n1_D$ for some integer n. So $D \leq N$. Since B covers $B_0(N)$, D is a defect group of $B_0(N)$. Thus D is a Sylow p-subgroup of N. For any simple kN-module X in $B_0(N)$, choose a simple kG-module S in B lying over X. Then, since S is weakly D-radical, we see X is weakly radical by Proposition 19 and Lemma 20. So $B_0(N)$ is radical by Theorem 12 and N is p-solvable by Theorem 22.

(iii)$\Rightarrow$(iv): Let b be a block of $R_p(G)$ covered by B. Since $N \leq R_p(G)$ and b covers $B_0(N)$, we may assume D is a defect group of b. By the Frattini argument $G = N_G(D)N = N_G(D)R_p(G)$. Let S be a simple module in b. For any irreducible constituent X of S_N, X lies in $B_0(N)$ and X is weakly D-radical. Thus S is weakly D-radical. So b is weakly D-radical and hence D-radical by Theorem 12.

(iv)$\Rightarrow$(ii): For any simple kG-module S in B, let X be an irreducible constituent in b of $S_{R_p(G)}$. Then, since b is D-radical and hence weakly D-radical, $X_D \simeq \oplus_i (1_{Q_i})^D$, where $Q_i =_{R_p(G)} \mathrm{vx}(X)$. S_D is a direct sum of of the modules of the form $(X^g)_D, g \in G$. Now there is $n \in N_G(D)$ such that $X^g \simeq X^n$. Then

$$(X^g)_D \simeq (X^n)_D \simeq (X_D)^n \simeq \oplus_i (1_{Q_i{}^n})^D.$$

Since $|Q_i{}^n| = |\mathrm{vx}(X)|$, S is weakly D-radical. Hence (ii) follows.

(v) $\Rightarrow$(vi): This is trivial.

(vi)$\Rightarrow$(v): Let S be a simple module in B with vertex D. Since S is weakly radical and D is weakly closed in P with respect to G, D is strongly closed in P with respect to G by Lemma 21.

(v)$\Rightarrow$ (ii): Let S be a simple module in B. We have $S_P \simeq \oplus_i (1_{Q_i})^P$, where Q is a vertex of S and $Q_i = Q^{x_i}$, $x_i \in G$. We may assume $Q \leq D$. $((1_{Q_i})^P)_D \simeq \oplus_{u \in Q_i \backslash P/D} (1_{Q_i^u \cap D})^D$. We see $Q_i^u = Q^{x_i u} \leq D^{x_i u} \cap P \leq D$ by (v). Therefore $((1_{Q_i})^P)_D \simeq \oplus_u (1_{Q_i^u})^D$. Hence S is weakly D-radical.

(i) and (iii)$\Rightarrow$ (vii): By Lemma 20, B is radical. Let S be a simple module in B lying over 1_N. Then $D \leq N \leq \mathrm{Ker}S$.

(vii)$\Rightarrow$ (viii): Let $N = \mathrm{Ker}S$. Then B covers $B_0(N)$. Therefore $D = D \cap N$ is a defect group of $B_0(N)$.

(viii)$\Rightarrow$ (v): This follows from the fact that $D = P \cap N$. The proof is complete. □

Remark. The implication (i)$\Rightarrow$ (ii) has been proved in Lemma 7 of [Ko] in a different way.

Corollary 24 ([HK], Corollary 1.3). *If* $\mathrm{vx}(S) \leq \mathrm{Ker}S$ *for any simple module* S *in* B, *then* B *is* D*-radical.*

Proof. Let S be a simple module in B. By Lemma 21 S is weakly D-radical. Hence B is weakly D-radical, and B is D-radical by Theorem 23. □

The following extends Theorem 22.

Corollary 25. *Let* B *be a radical block of* G *with defect group* D. *If* D *is a Sylow* p*-subgroup of* G, *then* G *is* p*-solvable.*

Proof. We see B is D-radical. If N is as in (iii) of Theorem 23, then N is p-solvable and G/N is a p'-group. Hence G is p-solvable. □

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