ON THE $V$-TRANSVERSALITY CONSTRUCTION OF EQUIVARIANT FRAMED MAPS

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Abstract. Let $G$ be a finite group. For a pointed $G$-map $Y^+ \wedge V^* \rightarrow V^*$, we introduce the notion of $V$-transversality, where $Y$ is a compact (smooth) $G$-manifold, $V$ is a real $G$-module, $Y^+ = Y \coprod \{y_\infty\}$, and $V^* = V \cup \{v_\infty\}$. Every $V$-transversal $G$-map $Y^+ \wedge V^* \rightarrow V^*$ gives rise of a $G$-framed map with the target manifold $Y$. This yields a one-to-one correspondence from the set of pointed $G$-homotopy classes to the set of $G$-framed cobordisms. From this viewpoint, we discuss the Burnside ring of $G$ and the equivariant stable $G$-cohomotopy group $\omega^0_G(Y)$ which consists of equivalence classes of pointed $G$-maps $Y^+ \wedge V^* \rightarrow V^*$ for $V = \mathbb{C}[G]^m (m \gg 1)$.

1. INTRODUCTION

In this article, we report results obtained from discussions with Takashi Matsunaga and Yasuhiro Hara.

Let $G$ be a finite group throughout this paper. A $G$-framed map $f$ is a pair of a $G$-map $f : (X, \partial X) \rightarrow (Y, \partial Y)$ and a real $G$-vector bundle isomorphism $b : TX \oplus \varepsilon(\mathbb{R}^m) \rightarrow f^*TY \oplus \varepsilon(\mathbb{R}^m)$, where $X$ and $Y$ are compact (smooth) $G$-manifolds of same dimension. Let $\mathcal{N}(G, Y)$ be the set of $G$-framed cobordism classes of $G$-framed maps with the target manifold $Y$. Let $\omega^0_G(Y)$ be the equivariant stable cohomotopy group of $Y$, of which the definition will be given in Section 2. Let $V$ be a real $G$-module (of finite dimension). In this paper we introduce the notion that a

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$G$-map $Y^+ \wedge V^* \to V^*$ is $V$-transversal to $0$ in $V^*$, essentially due to T. Petrie [8], cf. [5, §5]. The next result indicates importance of the notion in equivariant surgery theory.

**Theorem 1.1.** The $V$-transversality construction gives a one-to-one correspondence

$$\Phi : \omega_G^0(Y) \to \mathfrak{M}(G, Y).$$

This is an equivariant version of [4, Theorems B and C].

Let $\omega_G^0$ denote the group $\omega_G^0(pt)$. Let $A(G)$ denote the Burnside ring, i.e. the Grothendieck group for the category of finite $G$-sets. By proofs in equivariant homotopy theory, e.g. [1], [6], we have seen that $A(G)$ is isomorphic to $\omega_G^0$. However the theorem above gives rise of a geometric proof.

**Theorem 1.2.** The $V$-transversality construction yields an isomorphism

$$\Psi : \omega_G^0 \to A(G).$$

Details of the theorem will be given in Section 4.

Let $S(G)$ denote the set of all subgroups of $G$. Let $\mathcal{F}$ be a set of subgroups of $G$. We call $\mathcal{F}$ lower closed if $H \in \mathcal{F}$ then $S(H) \subset \mathcal{F}$. We call $\mathcal{F}$ conjugation invariant if $gHg^{-1} \in \mathcal{F}$ holds for all $H \in \mathcal{F}$ and $g \in G$. Hereafter, we suppose that $\mathcal{F}$ is lower closed and conjugacy invariant. Let $\mathfrak{F} = \mathfrak{F}(G, \mathcal{F})$ be the category such that $\text{Obj}(\mathfrak{F})$ is same as $\mathcal{F}$ and $\text{Mor}(\mathfrak{F})$ consists of all $(H, g, K)$ with $H, K \in \mathcal{F}$, $g \in G$ and $gHg^{-1} \subset K$. Let $Y$ be a compact $G$-manifold. If $H \leq K \leq G$ then we have the restriction map $\text{res}^K_H : \omega_H^n(Y) \to \omega_K^n(Y)$. If $H \leq G$ and $g \in G$ then we have the conjugation map $c_g : H \to gHg^{-1}$ and hence the induced map $c_g^* : \omega^n_{gHg^{-1}}(Y) \to \omega_H^n(Y)$. For $(H, g, K) \in \text{Mor}(\mathfrak{F})$, the homomorphism $(H, g, K)^*$ is defined to be the composition

$$\omega_K^n(Y) \xrightarrow{\text{res}^K_{gHg^{-1}}} \omega_{gHg^{-1}}(Y) \xrightarrow{c_g^*} \omega_H^n(Y).$$

Let $\omega_G^n(Y)$ denote the inverse limit

$$\text{inv-lim}_{\mathfrak{F}} \omega^n_H(Y) \quad (H \in \mathcal{F}).$$

We set $\omega_G^n = \omega_G^n(pt)$. It is interesting to study the image and the kernel of the canonical map

$$\text{res}_{\mathcal{F}} : \omega_G^0(Y) \to \omega_G^0(Y).$$
Theorem 1.3 (Y. Hara-M. M.). Let $G$ be a nontrivial nilpotent group and $\mathcal{F} = S(G) \setminus \{G\}$. The canonical map $\text{res}_\mathcal{F} : \omega^0_G \to \omega^0_{G,\mathcal{F}}$ is surjective if and only if $G$ is a cyclic group of order a prime or a product of distinct primes.

This implies

**Corollary 1.4.** Let $G$ and $\mathcal{F}$ be as above and let $Y$ be a compact $G$-manifold such that $Y^G \neq \emptyset$. If the canonical map $\text{res}_\mathcal{F} : \omega^0_G(Y) \to \omega^0_{G,\mathcal{F}}(Y)$ is surjective then $G$ is a cyclic group of order a prime or a product of distinct primes.

Let $k_G$ be the product of the primes $p$ such that $G$ contains a normal subgroup $N$ with index $p$. Here $k_G$ is understood to be 1 if $G$ is a perfect group. The next lemma is essentially due to R. Oliver [7, Lemma 8].

**Lemma 1.5.** There exists an element $\gamma_G = [X_1] - [X_2]$ of $A(G)$ such that $|X_1^G| - |X_2^G| = k_G$ and $\text{res}^G_H \gamma_G = 0$ for all $H < G$, where $X_1$ and $X_2$ are finite $G$-sets.

**Proposition 1.6.** Let $G$ be a nontrivial group and $\mathcal{F} = S(G) \setminus \{G\}$. Then the kernel of $\text{res}_\mathcal{F} : \omega^0_G \to \omega^0_{G,\mathcal{F}}$ is generated by $\gamma_G$.

**Corollary 1.7.** Let $G$ and $\mathcal{F}$ be as above. Then $\gamma_G \omega^0_G(Y)$ is contained in the kernel of $\omega^0_G(Y) \to \omega^0_{G,\mathcal{F}}(Y)$.

2. The Equivariant Stable Cohomotopy Group $\omega^n_G(Y)$

Let $Y$ be a compact $G$-manifold. We denote by $Y^+$ the disjoint union $Y \amalg \{y_\infty\}$ and the point $y_\infty$ is regarded as the base point of $Y^+$. Let $V$ be a real $G$-module. We denote by $V^*$ the one-point compactification $V \cup \{v_\infty\}$ and $v_\infty$ is regarded as the base point of $V^*$. The smash product

$$Y^+ \wedge V^* = \frac{Y^+ \times V^*}{(Y^+ \times v_\infty) \cup (y_\infty \times V^*)}$$

can be regarded as the Thom space of real $G$-vector bundle $\pi : Y \times V \to Y$ with fiber $V$.

We denote by $\mathbb{R}$ the 1-dimensional real vector space with trivial $G$-action. For a finite $G$-CW complex $X$ and an integer $n$, we define $\omega^n_G(X)$ by

$$\omega^n_G(X) = \lim_{m \to \infty} [A, B]^G_m$$
where

\[
A = \begin{cases}
X^+ \wedge M_m^* & (n \geq 0) \\
X^+ \wedge (\mathbb{R}^{[n]} \oplus M_m)^* & (n < 0),
\end{cases}
\]

\[
B = \begin{cases}
(\mathbb{R}^n \oplus M_m)^* & (n \geq 0) \\
M_m^* & (n < 0),
\end{cases}
\]

\[M_m = \mathbb{C}[G]^{\oplus m},\]

and \([-, -]_G^n\) stands for the set of all homotopy classes of maps in the category of pointed \(G\)-spaces. We set \(\omega^n_G = \omega^n_G(pt)\). Define the map \(\deg_H : \omega^0_G \rightarrow \mathbb{Z}\) by

\[\deg_H([f : M^*_m \rightarrow M^*_m]) = \deg(f^H : M^H_m \rightarrow M^H_m).\]

It is known that the map

\[\prod_{H \in S(G)} \deg_H : \omega^n_G \rightarrow \prod_{H \in S(G)} \mathbb{Z}\]

is injective and

\[[V^*, V^*_0]_0 \cong \omega^0_G\]

via the canonical map whenever \(V \supset \mathbb{C}[G]\).

3. \(V\)-TRANSVERSALITY OF \(G\)-MAPS \(Y^+ \wedge V^* \rightarrow V^*\)

In this section we introduce the notion of \(V\)-transversality due to T. Petrie [8]. Let \(V\) be a real \(G\)-module with a \(G\)-invariant inner product. For \(H \leq G\), \(V\) is decomposed into \(V^H \oplus V_H\) as real \(N_G(H)\)-modules, where \(V^H\) is the \(H\)-fixed point set of \(V\). A base point preserving \(G\)-map \(\alpha : Y^+ \wedge V^* \rightarrow V^*\) is called \(V\)-transversal to 0 in the target space \(V^*\) if the following conditions are fulfilled.

(1) \(\alpha\) is smooth on a neighborhood of \(X\).

(2) \(\alpha\) is transversal to 0 in \(V\), i.e. \(d_x\alpha : T_x(Y \times V) \rightarrow T_0 V\) is surjective at every \(x \in X\).

(3) the normal derivative \(\nu_x(\alpha) : V_H \rightarrow V_H\) at \(x\), where \(H = G_x\), coincides with the identity map \(V_H \rightarrow V_H\) for every \(x \in X\),

where \(X = \alpha^{-1}(0)\) and \(\nu_x(\alpha)\) is defined to be the composition

\[V_H \xrightarrow{\text{incl}} V = T_{xV(x)}V \xrightarrow{\text{incl}} T_x(Y \times V) \xrightarrow{d_x\alpha} T_0(V) \xrightarrow{\text{proj}} V_H.\]
Lemma 3.1. Let $Y$ be a compact $G$-manifold with $G$-invariant Riemannian metric, $A$ a closed $G$-subset of $Y$, $V$ a real $G$-module, and $\varepsilon : Y^+ \wedge V^* \to \mathbb{R}$ a $G$-invariant positive function. Let $\alpha : Y^+ \wedge V^* \to V^*$ be a base-point preserving $G$-map such that $\alpha : A^+ \wedge V^* \to V^*$ is $V$-transversal to 0 in $V^*$. Then there exists a base-point preserving $G$-homotopy $H : I \times (Y^+ \wedge V^*) \to V^*$, where $I = [0, 1]$, from $\alpha$ to $\beta : Y^+ \wedge V^* \to V^*$ satisfying the following conditions.

1. $H(t, (x, v)) = \alpha(x, v)$ for all $t \in I$, $x \in A$, and $v \in V$.
2. $d(\alpha(y, v), H(t, (y, v))) < \varepsilon(y, v)$ for all $(y, v) \in Y \times V$.
3. $\beta$ is $V$-transversal to 0 in $V^*$.

A $G$-framed map $f = (f, b)$ consists of a $G$-map $f : X \to Y$, where $X$ and $Y$ are compact (smooth) $G$-manifolds, and a real $G$-vector bundle isomorphism $b : T(X) \oplus \mathbb{R}^m \to f^*T(Y) \oplus \mathbb{R}^m$ for some integer $m \geq 0$.

Suppose $\alpha : Y^+ \wedge V^* \to V^*$ is $V$-transversal to 0 in $V^*$. Then set $X = \alpha^{-1}(0)$ and let $f : X \to Y$ be the composition of the inclusion $j_X : X \to Y \times V$ and the projection $\pi_Y : Y \times V \to Y$. There is a canonical isomorphism

$$T(Y \times V)|_X = (\pi_Y TY \oplus \pi_V TV)|_X = f^*TY \oplus \mathbb{R}^m.$$

We also have an isomorphism

$$T(Y \times V)|_X = TX \oplus \nu(X, Y \times V) \cong TX \oplus (\alpha|_X)^*\nu(0, V) = TX \oplus \mathbb{R}^m.$$

Thus we get a $G$-vector bundle isomorphism

$$\beta : TX \oplus \mathbb{R}^m \to f^*TY \oplus \mathbb{R}^m$$

such that $(\beta|_x)(x, v) = (x, v) \in \mathbb{R}^m$ for all $x \in X$ and $v \in V_H$, where $H = G_x$. By Lück-Madsen [3, Appendix, Proposition (A2)] and [5, §6], in the case $m > \dim Y$, we obtain a $G$-vector bundle isomorphism

$$b : TX \oplus \mathbb{R}^m \to f^*TY \oplus \mathbb{R}^m$$

such that $\beta$ and $b$ are stably regularly $G$-homotopic. This procedure obtaining $f = (f, b)$ from $\alpha$ is called the $V$-transversality construction of $G$-framed maps. The construction may also be called Pontryagin-Petrie construction of $G$-framed maps, cf. [4, §7].
4. THE ISOMORPHISM $\Psi : \omega_G^0 \to A(G)$

Let $\alpha : V^* \to V^*$ be a base-point preserving $G$-map such that $V \supset \mathbb{C}[G]$. Then we have $\dim V^G \geq 2$. Suppose $\alpha$ is $V$-transversal to 0 in $V^*$. The $V$-transversality construction yields $X = \alpha^{-1}(0)$, $f = \alpha|_X : X \to \{pt\}$, $\beta : \epsilon(V) \to \epsilon(V)$ and $b : \epsilon(V^G) \to \epsilon(V^G)$ such that $\beta|_x : V \to V$ coincides with $d\alpha_x : V \to V$ for $x \in X$, and $\beta$ is regularly $G$-homotopic to $b \oplus id_{\epsilon(V^G)}$. Decompose the $G$-set $X = \alpha^{-1}(0)$ into the disjoint union of $G$-orbits $X_i = Gx_i$, where $i = 1, \ldots, k$; i.e. $X = \coprod_{i=1}^{k} X_i$. We define

$$\epsilon(d\alpha_{x_i}) = \begin{cases} 1 & \text{(if } d\alpha_{x_i} \text{ is orientation preserving),} \\ -1 & \text{(if } d\alpha_{x_i} \text{ is orientation reversing).} \end{cases}$$

Set

$$X_+ = \coprod_{i} Gx_i : \epsilon(d\alpha_{x_i}) = 1,$$
$$X_- = \coprod_{i} Gx_i : \epsilon(d\alpha_{x_i}) = -1.$$  

Then we obtain an element $[X_+] - [X_-]$ of $A(G)$. The correspondence $[f] \mapsto [X_+] - [X_-]$ gives the map $\Psi : \omega_G^0 \to A(G)$. We have a canonical one-to-one correspondence $\Xi : \mathfrak{N}(G, pt) \to A(G)$. Moreover the diagram

$$\begin{array}{ccc}
\omega_G^0 & \xrightarrow{\phi} & \mathfrak{N}(G, pt); \\
\Psi & \cong & \Xi \\
A(G) & \xrightarrow{\phi} & [f, b]
\end{array} \begin{array}{ccc}
[f] & \xrightarrow{\Psi} & [X_+] - [X_-]
\end{array}$$

commutes. Once it was admitted that $\Phi$ is a one-to-one correspondence, the map $\Psi$ is bijective and hence an isomorphism.

REFERENCES


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