AN ESTIMATE OF THE ISOVARIANT BORSUK-ULAM CONSTANT

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ABSTRACT. We shall discuss the isovariant Borsuk-Ulam constant determined from the weak isovariant Borsuk-Ulam theorem. We first illustrate some properties of the Borsuk-Ulam constant and next provide an estimate of the isovariant Borsuk-Ulam constant for the special unitary group $SU(n)$.

1. BACKGROUND

Borsuk-Ulam type results for $G$-maps between (linear) $G$-spheres were studied by many researchers and various generalizations were shown. In particular, the following generalization is well known; see [3] for example.

**Theorem 1.1.** Let $G$ be $(C_p)^k$ a product of cyclic groups of prime order $p$ or $T^k$ a ($k$-dimensional) torus. Suppose that $G$ acts smoothly and fixed-point-freely on spheres $S_1$ and $S_2$. If there exists a (continuous) $G$-map $f : S_1 \to S_2$, then the inequality

\[ \dim S_1 \leq \dim S_2 \]

holds.

On the other hand, T. Bartsch [1] proved that such a Borsuk-Ulam result does not hold for $G$ not being a $p$-toral group. A compact Lie group $G$ is called $p$-toral if there is an exact sequence $1 \to T \to G \to P \to 1$, where $T$ is a torus and $P$ is a finite $p$-group.

As a variation of the Borsuk-Ulam theorem, the isovariant Borsuk-Ulam theorem was first studied by A. G. Wasserman [9]. Let $G$ be a compact Lie group. A $G$-map $f : X \to Y$ is called $G$-isovariant if $f$ preserves the isotropy subgroups, i.e., $G_x = G_{f(x)}$ for any $x \in X$. In other words, it is a $G$-map such that $f_{|G(x)} : G(x) \to Y$ is injective on each orbit $G(x)$ of $x \in X$. From Wasserman's results, one sees the following.

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2010 Mathematics Subject Classification. Primary 55M20; Secondary 57S15, 57S25.

Key words and phrases. Borsuk-Ulam theorem; Borsuk-Ulam group; Borsuk-Ulam constant; isovariant map; representation theory.
Theorem 1.2 (Isovariant Borsuk-Ulam theorem). Let $G$ be a solvable compact Lie group. If there exists a $G$-isovariant map $f : SV \to SW$ between linear $G$-spheres, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

Wasserman conjectures that this theorem holds for all finite groups. This is unsolved at present; however, we showed a weak version of the isovariant Borsuk-Ulam theorem for an arbitrary compact Lie group.

Theorem 1.3 (Weak isovariant Borsuk-Ulam theorem ([5, 6])). There exists a positive constant $c > 0$ such that

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

for any pair of representations $V$ and $W$ with a $G$-isovariant map $f : SV \to SW$.

Definition. The isovariant Borsuk-Ulam constant $c_G$ of $G$ is defined to be the supremum of such a constant $c$. (If $G = 1$, then set $c_G = 1$ as convention.)

When $c_G = 1$, $G$ is called a Borsuk-Ulam group (BUG for short); namely, a Borsuk-Ulam group $G$ is a compact Lie group for which the isovariant Borsuk-Ulam theorem holds. In particular, a solvable compact Lie group is a Borsuk-Ulam group by Theorem 1.2, and several nonsolvable Borsuk-Ulam finite groups are also known; for the detail, see [7, 8, 9]. However, no one knows connected Borsuk-Ulam groups other than a torus. Therefore we would like to investigate $c_G$ and provide some estimates at least. We illustrate general properties of $c_G$ in section 2 and we provide an estimate $c_G$ for $G = U(n)$ in section 3; in fact, we notice

$$c_{U(n)} \geq \frac{n}{n + 1}$$

whose complete proof will be written elsewhere.

2. Properties of $c_G$

The following result is a generalization of Wasserman’s result and is proved by a similar argument as in [9].

Proposition 2.1. If $1 \to K \to G \to Q \to 1$ is an exact sequence of compact Lie groups, then

$$\min \{c_K, c_Q\} \leq c_G \leq c_Q.$$  

In particular, if $K$ is a Borsuk-Ulam group, then $c_G = c_Q$.  

Using this inductively, we have

**Corollary 2.2.** If \( 1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G \), then

\[
\min_{1 \leq i \leq r} \{ c_{H_i/H_{i-1}} \} \leq c_G.
\]

As an example, one sees the following.

**Example 2.3.** It follows that \( c_{U(n)} = c_{SU(n)} = c_{PSU(n)} \). In particular, \( c_{SU(2)} = c_{SO(3)} \) since \( PSU(2) \cong SO(3) \).

**Proof.** There is an exact sequence

\[
1 \to C_n \to \mathbb{S}^1 \times SU(n) \to U(n) \to 1.
\]

Since \( C_n \) is a Borsuk-Ulam group, it follows from Proposition 2.1 that \( c_{U(n)} = c_{\mathbb{S}^1 \times SU(n)} \). Next, there is an exact sequence

\[
1 \to \mathbb{S}^1 \to \mathbb{S}^1 \times SU(n) \to SU(n) \to 1.
\]

Since \( \mathbb{S}^1 \) is a Borsuk-Ulam group, it follows that \( c_{\mathbb{S}^1 \times SU(n)} = c_{SU(n)} \). Thus \( c_{U(n)} = c_{SU(n)} \). Since the center of \( SU(n) \) is isomorphic to \( C_n \), it follows that \( c_{PSU(n)} = c_{SU(n)} \). \qed

### 3. Estimation of \( c_{U(n)} \)

Let \( T \) denote the maximal torus \( T \) of \( U(n) \) given by diagonal matrices:

\[
T = \left\{ \begin{pmatrix} t_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_n \end{pmatrix} \middle| t_i \in \mathbb{S}^1 (\subset \mathbb{C}) \right\}.
\]

We set

\[
d_{U(n)} = \sup \left\{ \frac{\dim U^T}{\dim U} \middle| U : \text{nontrivial irreducible } U(n)-\text{representation} \right\}.
\]

In order to estimate \( c_{U(n)} \), we use the fact \( c_{U(n)} \geq 1 - d_{U(n)} \) deduced from a result of [6].

**Theorem 3.1.** \( d_{U(n)} = \frac{1}{n+1} \), and hence \( c_{U(n)} \geq \frac{n}{n+1} \).

This is proved by representation theory. The irreducible complex representations of \( U(n) \) are parametrized by \( \lambda \) in

\[
\Lambda = \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \cdots \geq \lambda_n \}.
\]

Let \( V_\lambda \) denote the irreducible \( U(n) \)-representation corresponding to \( \lambda \in \Lambda \). (Then \( \lambda \) is the highest weight of \( V_\lambda \).) Since \( \text{Res}_T : R(U(n)) \to R(T)^{W_n} \) is isomorphic, where \( W_n \cong S_n \) is

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the Weyl group of $U(n)$, the character $\chi_{\lambda}$ of $\text{Res}_{T}V_{\lambda}$ is a homogenous symmetric Laurent polynomial in $\mathbb{Z}[t_{1}^{\pm 1}, \cdots, t_{n}^{\pm 1}]$ with a form

$$\chi_{\lambda}(t) = \sum_{\mu \in \mathbb{Z}^{n}} m_{\lambda}(\mu)t^{\mu} = \sum_{\mu \in \mathbb{Z}^{n}} m_{\lambda}(\mu)t_{1}^{\mu_{1}} \cdots t_{n}^{\mu_{n}} \quad (t = \text{diag}(t_{1}, \cdots, t_{n}) \in T).$$

The coefficient $m_{\lambda}(\mu)$ is the multiplicity of a weight $\mu$, i.e., the dimension of the weight space corresponding to $\mu$:

$$m_{\lambda}(\mu) = \dim \{v \in V_{\lambda} | t \cdot v = t^{\mu}v \text{ for all } t \in T\} \geq 0.$$

Let $M_{\lambda} := \{\mu \in \mathbb{Z}^{n} | |\mu| = |\lambda| \text{ and } \mu \preceq \lambda\}$, which is a finite set. Here $|\mu| = \sum_{i=1}^{n} \mu_{i}$, and $\preceq$ is the dominant order on $\mathbb{Z}^{n}$ defined by

$$\mu \preceq \lambda \iff \sum_{i=1}^{k} \mu_{i} \leq \sum_{i=1}^{k} \lambda_{i} \quad (1 \leq \forall k \leq n).$$

The following results can be found in [2, 4].

**Proposition 3.2.** Let $\lambda \in \Lambda$ and $\mu \in \mathbb{Z}^{n}$.

1. $m_{\lambda}(\mu) \neq 0 \iff \mu \in M_{\lambda}$.
2. $m_{\lambda}(\lambda) = 1$ for $\lambda \in \Lambda$.
3. $m_{\lambda}(w \cdot \mu) = m_{\lambda}(\mu)$ for any $w \in W_{n}$, where $w \cdot \mu = (\mu_{w^{-1}(1)}, \cdots, \mu_{w^{-1}(n)})$.
4. $W_{n}$ acts on $M_{\lambda}$ by permutation as in (3) and for any $\mu \in M_{\lambda}$, $W_{n}(\mu) \cap \Lambda$ consists of one element. Therefore $M_{\lambda} \cap \Lambda$ is a complete system of representatives of $M_{\lambda}/W_{n}$.

Thus the character has a form

$$\chi_{\lambda}(t) = \sum_{\mu \in M_{\lambda}} m_{\lambda}(\mu)t^{\mu} = \sum_{\mu \in M_{\lambda} \cap \Lambda} m_{\lambda}(\mu)P_{\mu}(t),$$

where $P_{\mu}(t) = \sum_{\nu \in W_{n}(\mu)} t^{\nu}$.

**Proposition 3.3.** Let $G = U(n)$ and $\lambda \in \Lambda$.

1. $\dim V_{\lambda} = \chi(1) = \sum_{\mu \in M_{\lambda}} m_{\lambda}(\mu)$.
2. $\dim V_{\lambda}^{T} = m_{\lambda}(0)$, the constant term of $\chi_{\lambda}(t)$.
3. $\dim V_{\lambda}^{T} > 0 \iff 0 \in M_{\lambda} \iff \lambda \in \Lambda_{0} := \{\lambda = (\lambda_{1}, \cdots, \lambda_{n}) \in \mathbb{Z}^{n} | \lambda_{1} \geq \cdots \geq \lambda_{n}, \sum_{i} \lambda_{i} = 0\}$.

Furthermore, the dimension of $V_{\lambda}$ is described in terms of the highest weight $\lambda \in \Lambda$.

**Proposition 3.4** (dimension formula for $U(n)$ ([2, 4])).

$$\dim V_{\lambda} = \frac{\prod_{i<j}(\lambda_{i} - \lambda_{j} + j - i)}{\prod_{i<j}(j - i)}.$$
On the other hand, computation of the multiplicity is not so easy (if \( \lambda \) is large); however several multiplicity formulas are known; for example, Freudenthal formula, Kostant formula, and combinatorially \( m_{\lambda}(\mu) \) can be given as a Kostka number (= the number of certain semi-standard Young tableaux). We use Freudenthal's multiplicity formula; see [4] for example.

3.1. Outline of proof of Theorem 3.1. We may assume \( \lambda \in \Lambda_{0} \) and \( \lambda \neq 0 \), since \( \dim V_{\lambda}^{T} = 0 \) if \( \lambda \notin \Lambda_{0} \). Let \((-,-)\) denote the (standard) inner product on \( \mathbb{R}^{n} \). Let \( \alpha_{ij} = e_{i} - e_{j} \) for \( i \neq j \), where \( e_{i} \) is the \( i \)-th fundamental unit vector. All \( \alpha_{ij} \) form the root system of type \( A_{n-1} \). Let \( R_{+} = \{ \alpha_{ij} \mid 1 \leq i < j \leq n \} \) the set of positive roots and set

\[
\rho := \frac{1}{2} \sum_{\alpha \in R_{+}} \alpha = \left( \frac{n-1}{2}, \frac{n-3}{2}, \cdots, \frac{n-3}{2}, -\frac{n-1}{2} \right).
\]

Applying Freudenthal's multiplicity formula to \( \mu = 0 \), we have an inequality

\[
(*) : m_{\lambda}(0)K_{\lambda} \leq 2n(n-1)d \sum_{k=1}^{d} m_{\lambda}(\mu_{k}),
\]

where \( K_{\lambda} := \| \lambda \|^{2} + 2(\lambda, \rho) \) and \( \mu_{k} := k\alpha_{1n} = (k, 0, \cdots, 0, -k) \in \Lambda_{0} \). Since \( \mu_{k} \in M_{\lambda} \) \((1 \leq k \leq d)\), \( \chi_{\lambda}(t) \) has a form

\[\chi_{\lambda}(t) = m_{\lambda}(0) + \sum_{k=1}^{d} m_{\lambda}(\mu_{k})P_{\mu_{k}}(t) + \text{other terms},\]

where \( P_{\mu_{k}}(t) = \sum_{i \neq j} t^{k\alpha_{ij}} = \sum_{i \neq j} t_{i}^{k} t_{j}^{-k} \), which has \( n(n-1) \) terms. This shows

\[\dim V_{\lambda} = \chi_{\lambda}(1) \geq m_{\lambda}(0) + \sum_{k=1}^{d} m_{\lambda}(\mu_{k})n(n-1).\]

Using the inequality (*), we obtain

\[\dim V_{\lambda} \geq \left( 1 + \frac{K_{\lambda}}{2d} \right) m_{\lambda}(0).\]

Since \( K_{\lambda} = \| \lambda \|^{2} + 2(\lambda, \rho) \geq \lambda_{1}^{2} + \lambda_{n}^{2} + (n-1)(\lambda_{1} - \lambda_{n}) \), it follows that

\[\frac{\dim V_{\lambda}^{T}}{\dim V_{\lambda}} \leq \frac{1}{n+1}.\]

On the other hand, applying the multiplicity formula to \( \lambda = \mu_{1} \), one sees

\[\dim V_{\mu_{1}}^{T} = n - 1,\]

and by the dimension formula, \( \dim V_{\mu_{1}} = (n + 1)(n - 1) \). Hence it follows that

\[\frac{\dim V_{\mu_{1}}^{T}}{\dim V_{\mu_{1}}} = \frac{1}{n+1}.\]
Thus we have \( d_{U(n)} = \frac{1}{n+1} \).

Remark. In case of \( n = 2 \), the theorem provide an estimate \( c_{U(2)} \geq 2/3 \); however, this may be improved by a further argument; in fact, we show that \( c_{U(2)} \geq 4/5 \) in [6].

REFERENCES


