数理解析研究所講究録
第 1968 巻 2015 年 117-125



# On the Andreadakis conjecture of the automorphism groups of free groups

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**Abstract**

In this article, we consider a certain subgroup of the IA-automorphism group of a free group, which we call the upper-triangular IA-automorphism group of a free group, and denote it by $\mathrm{IA}_n^+$. We determine the images of the $k$-th Johnson homomorphism of $\mathrm{IA}_n^+$ for any $k \geq 1$ and $n \geq 2$. By using this result, we give an affirmative answer to the Andreadakis conjecture restricted to $\mathrm{IA}_n^+$. Namely, we show that the intersection of the Andreadakis-Johnson filtration and $\mathrm{IA}_n^+$ coincides with the lower central series of $\mathrm{IA}_n^+$.

In addition to this, we also consider the integral second (co)homology group of $\mathrm{IA}_n^+$. In particular, we construct non-trivial second homology classes of $\mathrm{IA}_n^+$ by observing its generators and relators, and show that the second cohomology group is not generated by cup products of the first cohomology groups.

In 1965, in his doctorial thesis, Andreadakis [1] introduced a descending filtration $\mathcal{A}_G(1) \supset \mathcal{A}_G(2) \supset \cdots$ of the automorphism group $\mathrm{Aut}\, G$ of a group $G$. We call it the Andreadakis-Johnson filtration of $\mathrm{Aut}\, G$. One of the remarkable properties of the filtration $\{\mathcal{A}_G(k)\}$ is central. More precisely, he [1] showed that the commutator subgroup of $\mathcal{A}_G(k)$ and $\mathcal{A}_G(l)$ is contained in $\mathcal{A}_G(k+l)$ for any $k, l \geq 1$. Hence the graded quotients $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ for each $k \geq 1$ is an abelian group. In particular, it is known that if $G$ is finitely generated, then so is $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ for any $k \geq 1$. In general, the graded quotients $\mathcal{A}_G(k)/\mathcal{A}_G(k+1)$ are considered to be a sequence of approximations of $\mathrm{Aut}\, G$, and are one of powerful tools to study the group structure of $\mathrm{Aut}\, G$.

Let $F_n$ be a free group of rank $n$ with basis $x_1, \ldots, x_n$. As is well known, one of the most basic and important groups is a free group in combinatorial group theory. Andreadakis [1] focused his interests on the

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study of the Andreadakis-Johnson filtration on $\mathrm{Aut}\, F_n$. For any group $G$, since the Andreadakis-Johnson filtration is central, the $k$-th subgroup $\mathcal{A}_G(k)$ contains that of the lower central series $\{\mathcal{A}'_G(k)\}$ of $\mathcal{A}_G(1)$ for each $k \geq 1$. Andreadakis [1] showed that $\mathcal{A}_{F_2}(k) = \mathcal{A}'_{F_2}(k)$ for any $k \geq 1$, and $\mathcal{A}_{F_3}(k) = \mathcal{A}'_{F_3}(k)$ for $k \leq 3$. In general, it is quite a difficult problem to determine whether $\mathcal{A}_G(k)$ coincides with $\mathcal{A}'_G(k)$ or not, even the case where $G = F_n$. It has been conjectured that $\mathcal{A}_{F_n}(k) = \mathcal{A}'_{F_n}(k)$ for any $n \geq 3$ and $k \geq 1$ by Andreadakis. Today, this conjecture is called the Andreadakis conjecture. For any $n \geq 2$, it is known that $\mathcal{A}_{F_n}(2) = \mathcal{A}'_{F_n}(2)$ due to Bachmuth [3], and that $\mathcal{A}'_{F_n}(3)$ has at most finite index in $\mathcal{A}_{F_n}(3)$ due to Pettet [29].

The reason why we call $\{\mathcal{A}_G(k)\}$ the Andreadakis-Johnson filtration is that it should be mentioned not only Andreadakis's original works for $\mathrm{Aut}\, F_n$ but also Johnson's results for mapping class groups of surfaces. The mapping class group of a compact oriented surface with one boundary component can be embedded into the automorphism group of a free group by classical works of Dehn and Nielsen in the 1910s and in early 1920s. Hence we can consider a descending filtration of the mapping class group by restricting the Andreadakis-Johnson filtration to it. The first subgroup of this filtration is called the Torelli subgroup of the mapping class group. In the 1980s, Johnson studied the group structure of the Torelli subgroup in a series of works [15], [16], [17] and [18]. In particular, he gave a finite set of generators of the Torelli group, and he constructed a homomorphism $\tau$ to determine the abelianization of it. Today, his homomorphism $\tau$ is called the first Johnson homomorphism, and it is generalized to Johnson homomorphisms of higher degrees. Over the last two decades, good progress was made in the study of the Johnson homomorphisms of mapping class groups through the works of many authors including Morita [24], Hain [12] and others. The definition of the Johnson homomorphisms of the mapping class group can be easily generalized to those of $\mathrm{Aut}\, F_n$. To put it plainly, the Johnson homomorphisms are useful tools to study the graded quotients of the Andreadakis-Johnson filtration of $\mathrm{Aut}\, F_n$. (For details, see our survey papers [34] and [35].)

The first subgroup $\mathcal{A}_{F_n}(1)$ is called the IA-automorphism group of $F_n$, and usually denoted by $\mathrm{IA}_n$. Bachmuth [2] called $\mathrm{IA}_n$ the IA-automorphism group since that consists of automorphisms which induce identity automorphisms on the abelianized group $H$ of $F_n$. The letters I and A stands for "Identity" and "Automorphism" respectively. The subgroup $\mathrm{IA}_n$ reflects much richness and complexity of the structure of $\mathrm{Aut}\,F_n$, and plays important roles in various studies of $\mathrm{Aut}\,F_n$. In 1935, Magnus [21] showed that $\mathrm{IA}_n$ is finitely generated by automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j{}^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i$, $j \in \{1, 2, \ldots, n\}$ and

$$K_{ijl} : x_t \mapsto \begin{cases} x_i[x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i$, $j$, $l \in \{1, 2, \ldots, n\}$ such that $j > l$. The group structure of $\mathrm{IA}_n$ is, however, less well understood. For instance, no presentation for $\mathrm{IA}_n$ is known for $n \geq 3$. Krstić and McCool [20] showed that $\mathrm{IA}_3$ is not finitely presentable. For $n \geq 4$, it is not known whether $\mathrm{IA}_n$ is finitely presentable or not.

In this article, we consider a certain subgroup of $\mathrm{IA}_n$. Let $\mathrm{IA}_n^+$ be the subgroup of $\mathrm{IA}_n$ generated by $K_{ij}$ for $1 \leq j < i \leq n$ and $K_{ijl}$ for $1 \leq l < j < i \leq n$. The group $\mathrm{IA}_n^+$ is an IA-automorphism group analogue of the group of the upper triangular matrices. We call $\mathrm{IA}_n^+$ the upper-triangular IA-automorphism group of $F_n$. In our subsequent paper [36], we define the "upper-triangular" automorphism group $\mathrm{A}_n^+$, which is a subgroup of $\mathrm{Aut}\,F_n$, and show that $\mathrm{IA}_n^+$ coincides with the subgroup of $\mathrm{A}_n^+$ consisting of automorphisms which act on $H$ trivially. In the present paper, we give an affirmative answer to the Andreadakis conjecture restricted to $\mathrm{IA}_n^+$. Namely, set $\mathcal{A}_{F_n}(k)^+ := \mathcal{A}_{F_n}(k) \cap \mathrm{IA}_n^+$ for each $k \geq 1$, and let $\{\mathcal{A}'_{F_n}(k)^+\}$ be the lower central series of $\mathrm{IA}_n^+$. Then we show

**Theorem 1.** *For any $n \geq 2$ and $k \geq 1$, $\mathcal{A}_{F_n}(k)^+ = \mathcal{A}'_{F_n}(k)^+$.*

In order to prove this theorem, we use the Johnson homomorphisms

$$\tau_k'^{+} : \mathrm{gr}^k(\mathcal{A}_n'^{+}) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$$

of $\mathrm{IA}_n^+$ where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the $\mathbf{Z}$-linear dual group of $H$. Frankly, they are defined by restricting those of $\mathrm{Aut}\, F_n$ to the graded quotients of the lower central series $\{\mathcal{A}'_{F_n}(k)^+\}$. In particular, we completely determine their images as follows.

**Theorem 2.** *For any $n \geq 2$ and $k \geq 1$, the image of $\tau_k'^+$ is a submodule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ generated by*

$$\{x_i^* \otimes [[\cdots [x_{j_1}, x_{j_2}], \ldots, x_{j_k}], x_i] \mid 1 \leq j_1, \ldots, j_k < i \leq n\}$$
$$\cup \{x_i^* \otimes [[\cdots [x_{j_1}, x_{j_2}], \ldots, x_{j_k}], x_{j_{k+1}}] \mid 1 \leq j_1, \ldots, j_{k+1} < i \leq n\}$$

*where $x_i^*$s are the dual basis of $H^*$. Furthermore, we have*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{Im}(\tau_k'^+)) = \sum_{i=2}^{n} r_{i-1}(k) + \sum_{i=2}^{n} r_{i-1}(k+1).$$

*Here $r_m(k)$ is the rank of the $k$-th graded quotient of the lower central series of $F_m$. More precisely, due to Witt [37], we have*

$$r_m(k) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}$$

*where $\mu$ is the Möbius function, and $d$ runs over all positive divisors of $k$.*

In [4], Bartholdi asserted that the "rational" Andreadakis conjecture is true by using the representation theory of the general linear group $\mathrm{GL}(n, \mathbf{Q})$. He also disprove the Andreadakis conjecture for $n = 3$ by giving brief descriptions of the procedure of a long computer calculation and its results. In general, to show $\mathcal{A}_{F_n}(k)/\mathcal{A}'_{F_n}(k) = 0$ is quite different thing to show $(\mathcal{A}_{F_n}(k)/\mathcal{A}'_{F_n}(k)) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$. Bartholdi's representation theoretical proof cannot be applied to a proof of the Andreadakis conjecture for general $n \geq 4$, and hence it is still open problem. To the best of our knowledge, to attack the Andreadakis conjecture directly is too difficult and complicated to solve. Perhaps it might be worth considering to use the decomposition theorem with $\mathrm{IA}_n^+$ on this problem. In general, "the upper-triangular type subgroup" has the much easier structure, and is useful to study the whole group. For example, the subgroup $\Lambda_n$ of the general linear group $\mathrm{GL}(n, \mathbf{Z})$ consisting of all upper-triangular matrices has a very simple presentation. By using the presentation of $\Lambda_n$, and by using a kind of decomposition theorem for $\mathrm{GL}(n, \mathbf{Z})$ with $\Lambda_n$, Magnus [21] obtained finitely many generators of $\mathrm{IA}_n$. If we consider to attack the Andreadakis conjecture for $\mathrm{IA}_n$ by constructing and using the decomposition theorem for $\mathrm{IA}_n$ with $\mathrm{IA}_n^+$, our results on the paper seem to play important roles on this problem as a foothold.

Next, as applications of our results mentioned above, we consider to detect non-trivial homology classes in the integral second homology groups of $\mathrm{IA}_n^+$. Let $F$ be the free group generated by $K_{ij}$ for $1 \leq j < i \leq n$ and $K_{ijl}$ for $1 \leq l < j < i \leq n$, and $\pi : F \to \mathrm{IA}_n^+$ the natural surjection. We denote by $R$ the kernel of $\pi$. Then by observing the homological five-term exact sequence of a group extension

$$1 \to R \to F \xrightarrow{\pi} \mathrm{IA}_n^+ \to 1,$$

we see $H_1(R, \mathbf{Z})_{\mathrm{IA}_n^+} \cong H_2(\mathrm{IA}_n^+, \mathbf{Z})$. For the lower central series $\Gamma_F(1) \supset \Gamma_F(2) \supset \cdots$ of $F$, set $R_k := R \cap \Gamma_F(k)$ and $\overline{R}_k := R/R_k$ for each $k \geq 1$. Then we have a surjective homomorphism

$$\psi_k : H_1(R, \mathbf{Z})_{\mathrm{IA}_n^+} \to H_1(R/R_{k+1}, \mathbf{Z})_{\mathrm{IA}_n^+}.$$

Then we can detect non-trivial second homology classes of $\mathrm{IA}_n^+$ through $\psi^k$ by studying the structure of each $H_1(R/R_{k+1}, \mathbf{Z})_{\mathrm{IA}_n^+}$. In the paper, we especially consider the case where $k = 2$ and 3. For $k = 2$, we easily see that $H_1(R/R_3, \mathbf{Z})_{\mathrm{IA}_n^+} = R/R_3$, and that $R/R_3$ is a free abelian group of rank $n(n^2-1)(n-2)^2(n^2+5n+9)/72$. Furthermore, by studying a group structure of $H_1(R/R_4, \mathbf{Z})_{\mathrm{IA}_n^+}$, we obtain $H_2(\mathrm{IA}_n^+, \mathbf{Z}) \ncong R/R_3$, and show

**Theorem 3.** *For $n \geq 3$, $H_2(\mathrm{IA}_n^+, \mathbf{Z})$ contains a free abelian group of rank*

$$\frac{1}{72}n(n^2-1)(n-2)^2(n^2+5n+9) + \frac{1}{2}(n-1)(n-2).$$

By considering the dual version of the above argument, we also see that $H^2(\mathrm{IA}_n^+, \mathbf{Z})$ contains a free abelian group of rank $n(n^2-1)(n-2)^2(n^2+5n+9)/72 + (n-1)(n-2)/2$. In addition to this, we see that the cup product

$$\cup : \Lambda^2 H^1(\mathrm{IA}_n^+, \mathbf{Z}) \to H^2(\mathrm{IA}_n^+, \mathbf{Z})$$

is not surjective. Namely,

**Theorem 4.** *For any* $n \geq 3$, $H^2(\mathrm{IA}_n^+, \mathbf{Z}) \neq \mathrm{Im}(\cup)$.

Finally, we give some remarks related to our results. We show that the natural homomorphisms

$$\mathcal{A}'_{F_n}(k)^+/\mathcal{A}'_{F_n}(k+1)^+ \to \mathcal{A}'_{F_n}(k)/\mathcal{A}'_{F_n}(k+1)$$

induced from the inclusion map $\mathrm{IA}_n^+ \to \mathrm{IA}_n$ are injective for any $k \geq 1$. The group $\mathrm{P}\Sigma_n^+$ of $\mathrm{IA}_n^+$ generated by $K_{ij}$ for any $1 \leq j < i \leq n$ is called the upper-triangular McCool group. Let $\mathrm{P}\Sigma_n(1)^+ \supset \mathrm{P}\Sigma_n(2)^+ \supset \cdots$ be the lower central series of $\mathrm{P}\Sigma_n^+$. Cohen, Pakianathan, Vershinin and Wu [9] completely determined the structure of the graded quotients $\mathrm{P}\Sigma_n(k)^+/\mathrm{P}\Sigma_n(k+1)^+$ for each $k \geq 1$. By using the Johnson homomorphisms, we show that the natural homomorphisms

$$\mathrm{P}\Sigma_n(k)^+/\mathrm{P}\Sigma_n(k+1)^+ \to \mathcal{A}'_{F_n}(k)/\mathcal{A}'_{F_n}(k+1)$$

induced from the inclusion map $\mathrm{P}\Sigma_n^+ \to \mathrm{IA}_n$ are injective for any $k \geq 1$. We remark that Part (2) of this proposition is the answer to a problem listed in [9]. (See Section 10 of [9].)

**Acknowledgments**

This work was supported by JSPS KAKENHI Grant Number 24740051.

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