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ON ISOMORPHISM FOR THE SPACE OF SOLENOIDAL VECTOR FIELDS AND ITS APPLICATION TO THE STOKES PROBLEM

Yasunori Maekawa
Mathematical Institute, Tohoku University

Hideyuki Miura
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology

1. Introduction

This article is a resume of the recent work [21, 22] by the authors.

We are interested in the algebraic and topological structure of the space of solenoidal vector fields in a domain $\Omega\subset\mathbb{R}^{n}$, $n\geq 2$. Under suitable conditions on Ω and its boundary $\partial\Omega$ the solenoidal vector field, which is a fundamental object in the analysis of incompressible flows, is characterized as a vector field $u=(u_{1},\cdots,u_{n})$ satisfying

$$\operatorname{div}u=0\text{ in }\Omega,\qquad u\cdot\mathbf{n}=0\text{ on }\partial\Omega, \tag{1.1}$$

where $\mathbf{n}$ is the unit exterior normal vector to $\partial\Omega$. Since (1.1) is considered as a boundary value problem of one partial differential equation, it is heuristically expected that the degree of freedom is $n-1$ for the space of solenoidal vector fields. The aim of the study here is to describe this natural observation with a mathematically rigorous setting. To be precise, it will be convenient to set up our problem in an abstract manner. Let Ω be a domain in $\mathbb{R}^{n}$ and let $X(\Omega)$ be a Banach space of functions in Ω satisfying $C_{0}^{\infty}(\Omega)\subset X(\Omega)\subset L_{loc}^{1}(\Omega)$. The space of solenoidal vector fields in $(X(\Omega))^{n}$, denoted by $X_{\sigma}(\Omega)$, is defined as

$$X_{\sigma}(\Omega)=\overline{\{u\in(C_{0}^{\infty}(\Omega))^{n}\mid\operatorname{div}u=0\text{ in }\Omega\}}^{\|u\|_{X(\Omega)}}. \tag{1.2}$$

Here we have written $\|u\|_{X(\Omega)}$ for $\|u\|_{(X(\Omega))^{n}}$ to simplify the notation. We call two Banach spaces X and Y *isomorphic* if there is a bounded and bijective linear operator $L:X\rightarrow Y$. We write $X\simeq Y$ when X and Y are isomorphic. Then our problem is to show that $X_{\sigma}(\Omega)\simeq(X(\Omega))^{n-1}$.

In the following paragraph we assume that $\Omega=\mathbb{R}^{n}$ or

$$\Omega=\{(x^{\prime},x_{n})\in\mathbb{R}^{n-1}\times\mathbb{R}\mid x_{n}>\eta(x^{\prime})\}, \tag{1.3}$$

where $\eta \in L^1_{loc}(\mathbb{R}^{n-1})$ is a given Lipschitz function, i.e., $\|\nabla'\eta\|_{L^\infty} < \infty$. When Ω is of the form (1.3) we introduce the anisotropic Lebesgue spaces $Y^{q,r}(\Omega)$ as in [20] by using the homeomorphism $\Phi : \Omega \ni x \mapsto y = \Phi(x) \in \mathbb{R}^n_+$:

$$\Phi_j(x) = \begin{cases} x_j, \ 1 \le j \le n-1, \\ x_n - \eta(x'), \ j = n. \end{cases} \tag{1.4}$$

That is, for $1 < q, r < \infty$ the Banach space $Y^{q,r}(\Omega)$ is defined as

(1.5)
$$Y^{q,r}(\Omega) = \left\{ f \in L^1_{loc}(\Omega) \mid \|f\|_{Y^{q,r}(\Omega)} = \|f \circ \Phi^{-1}\|_{L^q_{y_n}(\mathbb{R}_+; L^r_{y'}(\mathbb{R}^{n-1}))} < \infty \right\}$$

with the norm $\|\cdot\|_{Y^{q,r}(\Omega)}$. The space $Y^{q,r}(\Omega)$ can be naturally defined also for $\Omega = \mathbb{R}^n$ as

$$Y^{q,r}(\mathbb{R}^n) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) \mid \|f\|_{Y^{q,r}(\mathbb{R}^n)} = \|f\|_{L^q_{x_n}(\mathbb{R}; L^r_{x'}(\mathbb{R}^{n-1}))} < \infty \right\}. \tag{1.6}$$

We note that $Y^{q,q}(\Omega)$ coincides with $L^q(\Omega)$. Our result is stated as follows.

Theorem 1.1 ([21, Theorem 1.1]). *Let $1 < q, r < \infty$ and assume that Ω is of the form* (1.3) *with some $\eta \in L^1_{loc}(\mathbb{R}^{n-1})$ satisfying $\|\nabla'\eta\|_{L^\infty} < \infty$. Then $Y^{q,r}_\sigma(\Omega) \simeq (Y^{q,r}(\Omega))^{n-1}$.*

The isomorphism in Theorem 1.1 is constructed explicitly in terms of the Riesz transform $\nabla'(-\Delta')^{-1/2}$ and the Poisson semigroup $\{e^{-x_n(-\Delta')^{1/2}}\}_{x_n \ge 0}$ in $L^r(\mathbb{R}^{n-1})$, where the idea is motivated by Ukai [29] on the formula of the Stokes semigroup in the half space. Indeed, in the case $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ this isomorphism from $Y^{q,r}_\sigma(\Omega)$ to $(Y^{q,r}(\Omega))^{n-1}$ has a special structure; it is a restriction of a bounded linear operator $\mathbf{W} : (Y^{q,r}(\Omega))^n \to (Y^{q,r}(\Omega))^{n-1}$ which enjoys the property

$$\begin{aligned} &\{\nabla p \in (Y^{q,r}(\Omega))^n \mid p \in L^1_{loc}(\Omega)\,,\ \Delta p = 0 \text{ in } \Omega\} \\ &\qquad \subset \ \operatorname{Ker} \mathbf{W} = \{f \in (Y^{q,r}(\Omega))^n \mid \mathbf{W} f = 0\}\,. \end{aligned} \tag{1.7}$$

In fact, the kernel property such as (1.7) plays an important role in the analysis of the Stokes operator in [29]. Unfortunately, for general case $\Omega \ne \mathbb{R}^n, \mathbb{R}^n_+$, the isomorphism in Theorem 1.1 is not a restriction of the operator satisfying (1.7). Therefore, a natural question is whether we can construct an operator $\mathbf{W} : (Y^{q,r}(\Omega))^n \to (Y^{q,r}(\Omega))^{n-1}$ so that (1.7) holds and its restriction to $Y^{q,r}_\sigma(\Omega)$ defines an isomorphism from $Y^{q,r}_\sigma(\Omega)$ to $(Y^{q,r}(\Omega))^{n-1}$. Our next result is as follows.

Theorem 1.2 ([21, Theorem 1.2]). *Let $1 < q < \infty$ and assume that Ω is of the form* (1.3) *with some $\eta \in L^1_{loc}(\mathbb{R}^{n-1})$ satisfying $\|\nabla'\eta\|_{L^\infty} < \infty$. Then there is a bounded linear operator $\mathbf{W} : (Y^{q,2}(\Omega))^n \to (Y^{q,2}(\Omega))^{n-1}$ enjoying the following properties.*

(i) $\mathbf{W}$ *satisfies* (1.7) *for* $r=2$.
(ii) *The restriction* $\mathbf{W}|_{Y^{q,2}_\sigma(\Omega)} : Y^{q,2}_\sigma(\Omega) \to (Y^{q,2}(\Omega))^{n-1}$ *is an isomorphism.*

When $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ *the above assertion holds for* $Y^{q,r}(\Omega)$ *with* $1<q,r<\infty$.

In Theorem 1.2 so far we need a strong condition $r=2$ for the space $Y^{q,r}(\Omega)$ except for the case $\Omega=\mathbb{R}^n,\mathbb{R}^n_+$. This is because the regularity of η assumed here is rather mild, and moreover, η in Theorem 1.2 is allowed to behave wildly at infinity. For example, the boundary need not to be asymptotically flat (this means $|\nabla'\eta(x)| \to 0$ as $|x'|\to\infty$) and η may even grow linearly as $|x'|\to\infty$. It seems that the existence of $\mathbf{W}$ in Theorem 1.2 is closely related with the validity of the Helmholtz decomposition in $(Y^{q,r}(\Omega))^n$ for $r=2$ (cf. [20]), and therefore, the assertion as in Theorem 1.2 might fail for some $r\neq 2$ if one does not impose any other condition than $\|\nabla'\eta\|_{L^\infty}<\infty$; see, e.g., [4] for a counterexample of the Helmholtz decomposition in $L^p(\Omega)$ when Ω is of the form (1.3).

In these theorems we concretely construct an isomorphism in terms of the Poisson semigroup and the Dirichlet-Neumann map associated with the Laplace equations: $\Delta u=0$ in Ω, $u=g$ on $\partial\Omega$. This construction is particularly nontrivial when the boundary function η in (1.3) is not identically zero. The key tool here is a factorization of divergence form elliptic operators in [18, 19], which is considered as an operator theoretical description of the classical Rellich identity [27].

Thanks to (1.7), the isomorphism obtained in Theorem 1.2 is useful in the analysis of fluid equations. Indeed, it reduces the equations describing the motion of incompressible flows, which usually consists of $n+1$ equations due to the unknowns of the solenoidal velocity $u=(u_1,\cdots,u_n)$ and the scalar pressure p, into the equations of $n-1$ dependent variables. As a typical example, let us consider the Stokes equations

$$\text{(S)}\qquad \begin{cases} \partial_t u - \nu\Delta u + \nabla p = 0, & t>0,\ x\in\Omega,\\ \operatorname{div} u = 0, & t\geq 0,\ x\in\Omega,\\ u=0, & t>0,\ x\in\partial\Omega,\\ u|_{t=0}=a, & x\in\Omega. \end{cases}$$

Here $\nu>0$ is a viscosity coefficient and a is a given solenoidal vector field. By formally introducing the Helmholtz projection $\mathbf{P}$ and the Stokes operator $\mathbf{A}=-\mathbf{P}\Delta_D$ with the homogeneous Dirichlet boundary condition (these are well-defined at least in the L^2 functional framework), (S) is written in the abstract form

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \nu\mathbf{A}u = 0\,,\ t>0,\ u|_{t=0}=a\,. \tag{1.8}$$

Let $\mathbf{V}:(Y^{q,r}(\Omega))^{n-1}\to Y^{q,r}_\sigma(\Omega)$ be an isomorphism. Then by setting

$$w(t) = (w_1(t),\cdots,w_{n-1}(t))^\top = \mathbf{V}^{-1}u(t)\,,$$

we obtain the reduced equations

$$\text{(RS)} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} + \nu \mathbf{B} w = 0\,,\ t > 0\,,\ w|_{t=0} = \mathbf{V}^{-1} a\,,$$

where $\mathbf{B} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$. If $\mathbf{V}^{-1}$ is a restriction of $\mathbf{W}$ on $Y^{q,r}_\sigma(\Omega)$ satisfying (1.7) then we formally have

$$\mathbf{B} = -\mathbf{W}\mathbf{P}\Delta_D\mathbf{V} = -\mathbf{W}(\Delta_D - \mathbf{Q}\Delta_D)\mathbf{V} = -\mathbf{W}\Delta_D\mathbf{V}\,,$$

where $\mathbf{Q} = I - \mathbf{P}$ and Δ_D is the Laplace operator subject to the homogeneous Dirichlet boundary condition. As will be shown in [22], when Ω is of the form (1.3) with a smooth η our isomorphism provides

$$\text{(1.9)} \qquad \mathbf{B} = -\Delta_D + \text{lower order operators in } (Y^{q,2}(\Omega))^{n-1}.$$

Furthermore, we have $\mathbf{B} = -\Delta_D$ in $(Y^{q,r}(\Omega))^{n-1}$ when $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$; see [21]. In any cases, this reduction significantly simplifies the original equations. The idea to achieve such a reduction is inspired by the derivation of the solution formula for the Stokes problem in $\mathbb{R}^n_+$ in [29], although the characterization of the space of solenoidal vector fields as in Theorems 1.1 and 1.2 is not observed in [29]; see Remark 1.1 (i) below. When $\Omega = \mathbb{R}^n_+$ (and $\mathbb{R}^n$), due to the relation $\mathbf{V}^{-1}\mathbf{A}\mathbf{V} = -\Delta_D$, the Stokes semigroup $\{e^{-t\mathbf{A}}\}_{t\ge 0}$ associated with $-\mathbf{A}$ is expressed as $e^{-t\mathbf{A}} = \mathbf{V}e^{t\Delta_D}\mathbf{V}^{-1}$. Therefore, we have

Theorem 1.3 ([21, Theorem 1.3]). *Let $1 < q, r < \infty$ and let $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$. Then the Stokes semigroup in $Y^{q,r}_\sigma(\Omega)$ and the heat semigroup in $(Y^{q,r}(\Omega))^{n-1}$ are isomorphic. That is,*

$$\big(\{e^{-t\mathbf{A}}\}_{t\ge 0},\, Y^{q,r}_\sigma(\Omega)\big) \simeq \big(\{e^{t\Delta_D}\}_{t\ge 0},\, (Y^{q,r}(\Omega))^{n-1}\big).$$

For general Ω of the form (1.3) we have

Theorem 1.4 ([22]). *Let $1 < q < \infty$ and let Ω is of the form (1.3) with $\|\nabla^k \eta\|_{L^\infty} < \infty$ for $k = 1, 2, 3$. Then*

$$\Big(\{e^{-t\mathbf{A}}\}_{t\ge 0}\,, Y^{q,2}_\sigma(\Omega)\Big) \simeq \Big(\{e^{-t\mathbf{B}}\}_{t\ge 0}\,, Y^{q,2}(\Omega)^{n-1}\Big) \text{ for } 1 < q < \infty\,.$$

Here $\mathbf{B} = -\Delta_D + \mathbf{R}$ is the operator with the domain

$$D(\mathbf{B}) = D(-\Delta_D) = \{f \in Y^{q,2}(\Omega)^{n-1} \mid \nabla^k f \in Y^{q,2}(\Omega), k = 1, 2,\ f = 0 \text{ on } \partial\Omega\}\,,$$

where the linear operator $\mathbf{R}$ satisfies the estimate

$$\sup_{\lambda \gg 1} \lambda^\epsilon \|\mathbf{R}(\lambda - \Delta_D)^{-1}\|_{\mathcal{L}((Y^{q,2})^{n-1})} < \infty\,,\ 0 < \epsilon < \frac{1}{2}\,.$$

Remark 1.1. (i) In [29, Theorem 1.1] the solution formula for $u(t) = e^{-t\mathbf{A}}a$ in $\mathbb{R}^n_+$ is given as

$$u'(t) = e^{t\Delta_D}(a' + Sa_n) - SUe^{t\Delta_D}\big(-S\cdot a' + a_n\big), \tag{1.10}$$

$$u_n(t) = Ue^{t\Delta_D}\big(-S\cdot a' + a_n\big), \tag{1.11}$$

where

$$S = \nabla'(-\Delta')^{-1/2}, \tag{1.12}$$

$$(U\varphi)(x_n) = (-\Delta')^{1/2}\int_0^{x_n} e^{-(x_n-y_n)(-\Delta')^{1/2}}\varphi(\cdot, y_n)\,\mathrm{d}y_n\,. \tag{1.13}$$

In fact, the map $\mathbf{W} : (Y^{q,r}(\mathbb{R}^n_+))^n \to (Y^{q,r}(\mathbb{R}^n_+))^{n-1}$ is given as $\mathbf{W} = E' + SE_n$, where $E'u := u'$ and $E_n u := u_n$ when $\Omega = \mathbb{R}^n_+$. In the argument of [29] the relation $(E' + SE_n)e^{-t\mathbf{A}} = e^{t\Delta_D}(E' + SE_n)$ is already found and it is a key to derive (1.10) - (1.11) in [29]. On the other hand, our argument in Theorem 1.2 reveals that $E' + SE_n$ actually defines an isomorphism from $Y^{q,r}_\sigma(\mathbb{R}^n_+)$ onto $(Y^{q,r}(\mathbb{R}^n_+))^{n-1}$, and it is also shown that such a description is generic for a wide class of domains like (1.3) including the whole space. Moreover, our approach leads to the isomorphic formulation between the Stokes semigroup in $Y^{q,r}_\sigma(\Omega)$ and the heat (or perturbed heat) semigroup in $(Y^{q,r}(\Omega))^{n-1}$ as in Theorem 1.3 (or in [22]).
(ii) When $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$, it is not difficult to see that the Laplace operator generates an analytic semigroup in $(Y^{q,r}(\Omega))^{n-1}$. Therefore from Theorem 1.3 we see that the Stokes operator also generates an analytic semigroup in $Y^{q,r}_\sigma(\Omega)$. For the case $r = q$, this fact has already proved in many literature, e.g., [28, 23, 12, 29, 8, 5]. In fact, by the same reason, it is proved that the Stokes operator admits a bounded H^∞-calculus in $Y^{q,r}_\sigma(\Omega)$. See [26, 5] for the fact that the Laplace (or Stokes) operator in $L^p(\Omega)$ (or $L^p_\sigma(\Omega)$ respectively) admits a bounded H^∞-calculus.
(iii) For the case of general domains with graph boundary, unless $\|\nabla'\eta\|_{L^\infty}$ is small, it is not known whether the Stokes operator generates an analytic semigroup or admits a bounded H^∞-calculus even in the space $L^p_\sigma(\Omega)$ $(p \neq 2)$ [8, 24]. Theorem 1.4 addressed this question in the functional setting $Y^{q,2}_\sigma(\Omega)$ by justifying the formula (1.9), since one can show that the Laplace operator (with the Dirichlet boundary condition) admits a bounded H^∞-calculus in the space $Y^{q,2}_\sigma(\Omega)$; see [22] for details, where the result of [19] is essentially used to obtain the estimate of the operator R. We also refer to [1, 7, 2, 10, 11] for the recent progress of the Stokes problem in some domains with noncompact boundary. In particular, [1] recently established the analyticity of the Stokes semigroup in the L^∞ space in a special two-dimensional unbounded domain such that the Helmholtz decomposition does not hold.

(iv) As a byproduct of our construction of the isomorphism, we obtain a projection onto the space of solenoidal vector fields in the domain of the form (1.3), which was found in [29, Remark 1.5] for $\Omega = \mathbb{R}^n_+$ and is different from the standard Helmholtz projection; see Remark 3.1.

This article is organized as follows. In Section 2 we recall the result of [18] on the factorization of some class of elliptic operators. In Section 3 we will show how to construct the isomorphism in Theorem 1.2 without the details of the proofs.

2. Preliminaries - Elliptic operator of divergence form -

In this section we recall the result of [18] for some class of second order elliptic operators of divergence form. By the coordinate transform (1.4) the Laplace operator Δ is transformed to an elliptic operator of divergence form whose coefficients are independent of one variable. Taking this into mind, we consider the second order elliptic operator in $\mathbb{R}^n = \{(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}\}$,

$$\mathcal{A} = -\nabla \cdot A\nabla, \; A = A(x') = \big(a_{i,j}(x')\big)_{1\le i,j\le n} \, . \tag{2.1}$$

Here $n \in \mathbb{N}$, $\nabla = (\nabla', \partial_n)^\top$ with $\nabla' = (\partial_1, \cdots, \partial_{n-1})^\top$, and each $a_{i,j}$ is always assumed to be *t-independent*. We further assume that A is a *real symmetric* matrix and each component $a_{i,j}$ is a measurable function satisfying the uniformly elliptic condition

$$\langle A(x')\eta, \eta\rangle \ge \nu_1|\eta|^2, \; |\langle A(x')\eta, \zeta\rangle| \le \nu_2|\eta||\zeta| \tag{2.2}$$

for all $\eta, \zeta \in \mathbb{R}^n$ and for some constants ν_1, ν_2 with $0 < \nu_1 \le \nu_2 < \infty$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathbb{R}^n$, i.e., $\langle \eta, \zeta\rangle = \sum_{j=1}^n \eta_j \zeta_j$ for $\eta, \zeta \in \mathbb{R}^n$. For later use we set $b = a_{n,n}$, which satisfies $\nu_1 \le b \le \nu_2$ due to (2.2). We also denote by $\mathbf{a}$ the vector $\mathbf{a}(x') = (a_{1,n}(x'), \cdots, a_{n-1,n}(x'))^\top$.

We write $D_H(T)$ for the domain of a linear operator T in a Banach space H. Under the condition (2.2) the standard theory of sesquilinear forms gives a realization of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$, denoted again by $\mathcal{A}$, such as

$$\begin{aligned} D_{L^2}(\mathcal{A}) = \{&w \in H^1(\mathbb{R}^n) \mid \text{there is } F \in L^2(\mathbb{R}^n) \text{ such that} \\ &\langle A\nabla w, \nabla v\rangle_{L^2(\mathbb{R}^n)} = \langle F, v\rangle_{L^2(\mathbb{R}^n)} \text{ for all } v \in H^1(\mathbb{R}^n)\}, \end{aligned} \tag{2.3}$$

and $\mathcal{A}w = F$ for $w \in D_{L^2}(\mathcal{A})$. Here $H^1(\mathbb{R}^n)$ is the usual Sobolev space and $\langle w, v\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} w(x',t)v(x',t)\,\mathrm{d}x'\,\mathrm{d}t$.

Definition 2.1. (i) *For a given $h \in \mathcal{S}'(\mathbb{R}^{n-1})$ we denote by $M_h : \mathcal{S}(\mathbb{R}^{n-1}) \to \mathcal{S}'(\mathbb{R}^{n-1})$ the multiplication $M_h u = hu$.*
(ii) *We denote by $E_\mathcal{A} : H^{1/2}(\mathbb{R}^{n-1}) \to \dot{H}^1(\mathbb{R}^n_+)$ the $\mathcal{A}$-extension operator, i.e., $w = E_\mathcal{A}\varphi$ is the solution to the Dirichlet problem*

$$\begin{cases} \mathcal{A}w = 0 \text{ in } \mathbb{R}^n_+, \\ \quad w = \varphi \text{ on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}. \end{cases} \tag{2.4}$$

The one parameter family of linear operators $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$, *defined by* $E_{\mathcal{A}}(t)\varphi = (E_{\mathcal{A}}\varphi)(\cdot,t)$ *for* $\varphi \in H^{1/2}(\mathbb{R}^{n-1})$, *is called the Poisson semigroup associated with* $\mathcal{A}$.
(iii) *We denote by* $\Lambda_{\mathcal{A}} : H^{1/2}(\mathbb{R}^{n-1}) \to \dot{H}^{-1/2}(\mathbb{R}^{n-1}) = \big(\dot{H}^{1/2}(\mathbb{R}^{n-1})\big)^*$ *the Dirichlet-Neumann map associated with* $\mathcal{A}$, *which is defined through the sesquilinear form*

$$(2.5)\qquad \langle \Lambda_{\mathcal{A}}\varphi, g\rangle_{\dot{H}^{-\frac{1}{2}},\dot{H}^{\frac{1}{2}}} = \langle A\nabla E_{\mathcal{A}}\varphi, \nabla E_{\mathcal{A}} g\rangle_{L^2(\mathbb{R}^n_+)},\ \ \varphi, g \in H^{1/2}(\mathbb{R}^{n-1}).$$

Here $\langle\cdot,\cdot\rangle_{\dot{H}^{-1/2},\dot{H}^{1/2}}$ *denotes the duality coupling of* $\dot{H}^{-1/2}(\mathbb{R}^{n-1})$ *and* $\dot{H}^{1/2}(\mathbb{R}^{n-1})$.

Remark 2.1. (i) Eq. (2.4) is considered in a weak sense; cf. [18, Section 2.1]. The proof of the existence of the extension operator $E_{\mathcal{A}}$ is well known. As is shown in [18, Proposition 2.4], $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ is a strongly continuous and analytic semigroup in $H^{1/2}(\mathbb{R}^{n-1})$. We denote its generator by $-\mathcal{P}_{\mathcal{A}}$, and $\mathcal{P}_{\mathcal{A}}$ is called a *Poisson operator* associated with $\mathcal{A}$. (ii) Since A is Hermite and satisfies the uniformly elliptic condition (2.2), the theory of the sesquilinear forms [16, Chapter VI. §2] shows that $\Lambda_{\mathcal{A}}$ is extended as a self-adjoint operator in $L^2(\mathbb{R}^{n-1})$.

The next result plays a fundamental role in our argument.

Theorem 2.1. *Let* $\mathcal{A}$ *be the elliptic operator defined in* (2.1) *with a real symmetric matrix* A *satisfying* (2.2). *Then* $D_{L^2}(\Lambda_{\mathcal{A}}) = H^1(\mathbb{R}^{n-1})$ *holds with equivalent norms, and the Poisson semigroup* $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ *in* $H^{1/2}(\mathbb{R}^{n-1})$ *is extended as a strongly continuous and analytic semigroup in* $L^2(\mathbb{R}^{n-1})$, *where its generator* $-\mathcal{P}_{\mathcal{A}}$ *satisfies*

(2.6)
$$D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^{n-1}),\ \ -\mathcal{P}_{\mathcal{A}}\varphi = -M_{1/b}\Lambda_{\mathcal{A}}\varphi - M_{\mathbf{a}/b}\cdot\nabla'\varphi,\ \ \varphi \in H^1(\mathbb{R}^{n-1}).$$

Furthermore, the realization $\mathcal{A}'$ *in* $L^2(\mathbb{R}^{n-1})$ *and the realization* $\mathcal{A}$ *in* $L^2(\mathbb{R}^n)$ *are respectively factorized as*

$$(2.7)\qquad \mathcal{A}' = M_b\mathcal{Q}_{\mathcal{A}}\mathcal{P}_{\mathcal{A}},\ \ \mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_b\mathcal{P}_{\mathcal{A}})^*,$$

$$(2.8)\qquad \mathcal{A} = -M_b(\partial_t - \mathcal{Q}_{\mathcal{A}})(\partial_t + \mathcal{P}_{\mathcal{A}}).$$

Here $(M_b\mathcal{P}_{\mathcal{A}})^*$ *is the adjoint of* $M_b\mathcal{P}_{\mathcal{A}}$ *in* $L^2(\mathbb{R}^{n-1})$.

For the proof of Theorem 2.1, see, e.g. [18, Theorem 1.3, Theorem 4.2]. The identities (2.7) and (2.8) are considered as an operator-theoretical description of the classical Rellich identity [27], but when the matrix A is not real symmetric and possesses a limited smoothness the verification of this identity becomes a delicate problem. The Rellich type identity is verified and used by [14] when A is real symmetric and by [3] when $\mathbf{r_2} = 0$ without any extra regularity condition on A. See also [25, 15, 13] for the study of the elliptic boundary value problem in relation to the Rellich identity.

3. Isomorphism in Theorem 1.2

In this article we only state how to construct the isomorphism in Theorem 1.2 without proofs. For details and the application to the Stokes problem, see [21, 22].

When Ω is of the form (1.3), through the standard transformation

$$u = \tilde{u} \circ \Phi^{-1}\,, \ g = \tilde{g} \circ \Phi^{-1}\,, \ \Phi \text{ is as in (1.4)}\,,$$

the Laplace equations, $-\Delta \tilde{u} = 0$ in Ω and $\tilde{u} = \tilde{g}$ on $\partial\Omega$, are transformed to the elliptic equations in the half space

$$\mathcal{A}u = 0 \text{ in } \mathbb{R}^n_+\,, \ u = g \text{ on } \partial\mathbb{R}^n_+\,. \tag{3.1}$$

Here $\mathcal{A} = -\nabla \cdot A\nabla$ and A is a real symmetric and positive definite matrix with $\mathbf{a} = -\nabla'\eta$, $b = 1 + |\nabla'\eta|^2$, and $A' = (a_{i,j})_{1\le i,j\le n-1} = I'$ (the identity matrix). Note that each coefficient of A is independent of the y_n variable, and hence we can apply the result of Section 2. It is straightforward to see that the matrix A is written as $A = B^\top B$, where $B = \big(b_{i,j}\big)_{1\le i,j\le n}$ with $b_{i,j} = \delta_{ij}$ for $1 \le i,j \le n-1$, $b_{i,n} = -\partial_i\eta$ for $1 \le i \le n-1$, $b_{n,j} = 0$ for $1 \le j \le n-1$, and $b_{n,n} = 1$. The matrix $B^\top$ is the transpose of B. The key point here is that the solenoidal property in the original variables

$$\operatorname{div} \tilde{u} = 0 \text{ in } \Omega\,, \ \tilde{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

is equivalent with

$$\operatorname{div} B^\top u = 0 \text{ in } \mathbb{R}^n_+\,, \ \gamma(B^\top u)_n = 0 \text{ on } \partial\mathbb{R}^n_+ \tag{3.2}$$

in the new variables, where γ is the trace to the boundary $\partial\mathbb{R}^n_+$. Thus it is natural to introduce the space $Y^{q,r}_{\tilde{\sigma}}(\mathbb{R}^n_+)$ as

$$\begin{aligned} Y^{q,r}_{\tilde{\sigma}}(\mathbb{R}^n_+) &= \big\{\tilde{u} \circ \Phi^{-1} \in (Y^{q,r}(\mathbb{R}^n_+))^n \mid \tilde{u} \in Y^{q,r}_\sigma(\Omega)\big\} \\ &= \big\{u \in (Y^{q,r}(\mathbb{R}^n_+))^n \mid \operatorname{div} B^\top u = 0 \text{ in } \mathbb{R}^n_+,\ \gamma(B^\top u)_n = 0 \text{ on } \partial\mathbb{R}^n_+\big\}\,, \end{aligned} \tag{3.3}$$

where the characterization (3.3) is due to [21, Lemma 2.1].

For a vector $v = (v', v_n)^\top \in \mathbb{R}^{n-1} \times \mathbb{R}$ we define the $(n-1) \times n$ matrix E' and the $1 \times n$ matrix E_n by the relation

$$E'v = (B^\top v)' = v'\,, \ E_n v = (B^\top v)_n = v_n - M_{\nabla'\eta} \cdot v'\,. \tag{3.4}$$

Set

$$S = \big(\nabla' + M_{\nabla'\eta}\mathcal{P}_{\mathcal{A}}\big)\Lambda_{\mathcal{A}}^{-1}\,, \tag{3.5}$$

and also set

$$(U\varphi)(\cdot, y_n) = \Lambda_{\mathcal{A}} \int_0^{y_n} e^{-(y_n - z_n)\mathcal{P}_{\mathcal{A}}} \varphi(z_n)\, \mathrm{d}z_n\,, \tag{3.6}$$

for a function $\varphi(\cdot, z_n) = \varphi(z', z_n)$. Next we define the operator $Z = (Z', Z_n)^\top : (C_0^\infty(\mathbb{R}^n_+))^{n-1} \to (\mathcal{D}'(\mathbb{R}^n_+))^n$ as

$$Z'[h] = h + SU\Lambda_{\mathcal{A}}^{-1}\nabla' \cdot h\,, \tag{3.7}$$

$$Z_n[h] = -U\Lambda_{\mathcal{A}}^{-1}\nabla' \cdot h + M_{\nabla'\eta} \cdot Z'[h]\,. \tag{3.8}$$

The operator Z is shown to be the isomorphism between $Y^{q,2}_{\tilde{\sigma}}(\mathbb{R}^n_+)$ and $Y^{q,2}(\mathbb{R}^n_+)^{n-1}$ as follows.

Proposition 3.1 ([21, Corollary 3.1]). *Let $1 < q < \infty$. Then the operator Z defined by (3.7) - (3.8) is extended as a bounded and bijective operator from $(Y^{q,2}(\mathbb{R}^n_+))^{n-1}$ onto $Y^{q,2}_{\tilde{\sigma}}(\mathbb{R}^n_+)$. Moreover, we have $Z^{-1} = E' + SE_n$.*

For the proof of Proposition 3.1 see [21]. We emphasize here that, in order to justify the condition $\operatorname{div} B^\top Z[h] = 0$, we need to use the identity (2.7). Indeed, in the proof of Proposition 3.1 the identity

$$\nabla' \cdot S = -\Lambda_{\mathcal{A}}\mathcal{P}_{\mathcal{A}}\Lambda_{\mathcal{A}}^{-1} \tag{3.9}$$

was essentially used, which is formally derived from the computation

$$\begin{aligned}
\nabla' \cdot S &= (\Delta' + \nabla' \cdot M_{\nabla'\eta}P_{\mathcal{A}})\Lambda_{\mathcal{A}}^{-1} \\
&= \big(-M_b\mathcal{Q}_{\mathcal{A}}P_{\mathcal{A}} + \nabla' \cdot M_{\nabla'\eta}P_{\mathcal{A}}\big)\Lambda_{\mathcal{A}}^{-1} \\
&= \big(-(\Lambda_{\mathcal{A}} + \nabla' \cdot M_{\nabla'\eta})P_{\mathcal{A}} + \nabla' \cdot M_{\nabla'\eta}P_{\mathcal{A}}\big)\Lambda_{\mathcal{A}}^{-1} \\
&= -\Lambda_{\mathcal{A}}P_{\mathcal{A}}\Lambda_{\mathcal{A}}^{-1}\,.
\end{aligned}$$

Here we firstly used the factorization of $-\Delta'$ stated in (2.7) and then used (2.6). Once (3.9) is derived, we observe that

$$\nabla' \cdot Z'[h] = \nabla' \cdot h - \Lambda_{\mathcal{A}}\mathcal{P}_{\mathcal{A}}\Lambda_{\mathcal{A}}^{-1}U\Lambda_{\mathcal{A}}^{-1}\nabla' \cdot h\,, \tag{3.10}$$

while we have from the definition of E_n and U,

$$\partial_n E_n Z[h] = -\partial_n U\Lambda_{\mathcal{A}}^{-1}\nabla' \cdot h = -\nabla' \cdot h + \Lambda_{\mathcal{A}}\mathcal{P}_{\mathcal{A}}\Lambda_{\mathcal{A}}^{-1}U\Lambda_{\mathcal{A}}^{-1}\nabla' \cdot h\,. \tag{3.11}$$

These identities imply $\operatorname{div} B^\top Z[h] = 0$. In order to establish the estimate in $Y^{q,2}(\mathbb{R}^n_+)$ the following estimates between the Poisson operator and the Dirichlet-Neumann map are essential:

$$\|\nabla' f\|_{L^2(\mathbb{R}^{n-1})} \le C\|\Lambda_{\mathcal{A}}f\|_{L^2(\mathbb{R}^{n-1})} \le C'\|\mathcal{P}_{\mathcal{A}}f\|_{L^2(\mathbb{R}^{n-1})} \le C''\|\nabla f\|_{L^2(\mathbb{R}^{n-1})} \tag{3.12}$$

for all $f \in H^1(\mathbb{R}^{n-1})$. These inequalities are known as variants of the classical Rellich identity [27, 25, 14, 17]; see also [20, Proposition 2] for a short proof in relation with the Helmholtz decomposition.

The operator $E' + SE_n$ has a special property about its kernel, which plays an important role in the relation between the Stokes operator and the Laplace operator.

Proposition 3.2 ([21, Lemma 3.4]). *Let $1 < q < \infty$. Assume that $p \in L^1_{loc}(\mathbb{R}+; L^2(\mathbb{R}^{n-1}))$ satisfies $\nabla p \in (Y^{q,2}(\mathbb{R}^n_+))^n$ and $\mathcal{A}p = 0$ in $\mathbb{R}^n_+$ in the sense of distributions. Then $(E' + SE_n)B\nabla p = 0$.*

From Propositions 3.1, 3.2 the isomorphism in Theorem 1.2 is constructed as follows. Let us define the bounded linear operators $\mathbf{V} : (Y^{q,2}(\Omega))^{n-1} \to (Y^{q,2}(\Omega))^n$ and $\mathbf{W} : (Y^{q,2}(\Omega))^n \to (Y^{q,2}(\Omega))^{n-1}$ as

$$(\mathbf{V}w)(x) = \big(Z[w \circ \Phi^{-1}]\big)(\Phi(x)) . \tag{3.13}$$

$$(\mathbf{W}u)(x) = \big((E' + SE_n)[u \circ \Phi^{-1}]\big)(\Phi(x)) . \tag{3.14}$$

Here Z is defined as (3.7) - (3.8), while E', E_n, and S are defined as (3.4) - (3.5). Then by Proposition 3.1 it follows that $\operatorname{Ran}(\mathbf{V}) = Y^{q,2}_\sigma(\Omega)$ and $\mathbf{V}$ is also invertible. In particular, $\mathbf{V}^{-1} = \mathbf{W}$ on $Y^{q,2}_\sigma(\Omega)$. Finally, $\mathbf{W}$ satisfies (i) of Theorem 1.2 by Proposition 3.2.

Remark 3.1 (Ukai's projection). Let $\mathbf{V}$ and $\mathbf{W}$ be the operators given as (3.13) and (3.14). Theorem 1.2 implies that the operator

$$\mathbf{P_0} = \mathbf{V}\mathbf{W} : (Y^{q,2}(\Omega))^n \to Y^{q,2}_\sigma(\Omega) \tag{3.15}$$

is a continuous projection from $(Y^{q,2}(\Omega))^n$ onto $Y^{q,2}_\sigma(\Omega)$. In the case $\eta = 0$ (i.e., $\Omega = \mathbb{R}^n_+$), through a short calculation, this projection coincides with the one found by [29, Remark 1.5]:

$$(\mathbf{P_0}u)' = u' + Su_n - SU(-S \cdot u' + u_n) , \ (\mathbf{P_0}u)_n = U(-S \cdot u' + u_n) ,$$

where S and U are defined as in Remark 1.1 (ii). In the case of general η we have used a factorization of the elliptic operators. The projection $\mathbf{P_0}$ is different from the well-known Helmholtz projection, which is orthogonal in $(L^2(\Omega))^n$ while $\mathbf{P_0}$ is not, as is observed in [29] for the case $\eta = 0$.

When $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$ we can simply take $\Phi(x) = x$, and hence, the isomorphism $\mathbf{V}$ coincides with Z (which is defined as (3.20) - (3.21) below) in both cases. Moreover, the matrices E', E_n, and the operators S, K are respectively defined as

$$E'v = v', \ E_n v = v_n, \ v = (v', v_n)^\top \in \mathbb{R}^{n-1} \times \mathbb{R}, \tag{3.16}$$

$$S = \nabla'(-\Delta')^{-\frac{1}{2}}, \ K = -(-\Delta')^{-\frac{1}{2}}\nabla' \cdot E' + E_n. \tag{3.17}$$

When $\Omega = \mathbb{R}^n$ the operator U is defined as

$$(U\varphi)(\cdot, y_n) = (-\Delta')^{\frac{1}{2}} \int_{-\infty}^{y_n} e^{-(y_n - z_n)(-\Delta')^{\frac{1}{2}}} \varphi(z_n)\, \mathrm{d}z_n, \tag{3.18}$$

while when $\Omega = \mathbb{R}^n_+$ we set

$$(U\varphi)(\cdot, y_n) = (-\Delta')^{\frac{1}{2}} \int_0^{y_n} e^{-(y_n - z_n)(-\Delta')^{\frac{1}{2}}} \varphi(z_n)\, \mathrm{d}z_n. \tag{3.19}$$

With these operators $\mathbf{V} = (V', V_n)$ is given as in (3.7) - (3.8), that is,

$$V'[w] = w + SU(-\Delta')^{-\frac{1}{2}}\nabla' \cdot w, \tag{3.20}$$

$$V_n[w] = -U(-\Delta')^{-\frac{1}{2}}\nabla' \cdot w. \tag{3.21}$$

On the other hand, the operator $\mathbf{W}$ is given as

$$\mathbf{W} = E' + SE_n\,. \tag{3.22}$$

Note that, when $\Omega = \mathbb{R}^n_+$, the Dirichlet-Neumann map and the Poisson operator coincide with the fractional Laplacian $(-\Delta')^{1/2}$. It is classical that $\nabla'(-\Delta')^{-1/2}$ and $(-\Delta')^{-1/2}\nabla'$ define the singular integral operators. Hence, for any $1 < r < \infty$, the operators S and K are respectively bounded from $L^r(\mathbb{R}^{n-1})$ to $(L^r(\mathbb{R}^{n-1}))^{n-1}$ and from $(L^r(\mathbb{R}^{n-1}))^{n-1}$ to $L^r(\mathbb{R}^{n-1})$. Moreover, it is also well known that the Poisson semigroup $\{e^{-t(-\Delta')^{1/2}}\}_{t\geq 0}$ admits the maximal regularity estimates in $L^q(\mathbb{R}_+; L^r(\mathbb{R}^{n-1}))$, $1 < q, r < \infty$. Hence we have

$$\|U\varphi\|_{Y^{q,r}(\Omega)} \leq C\|\varphi\|_{Y^{q,r}(\Omega)}, \quad 1 < q, r < \infty, \quad \Omega = \mathbb{R}^n \text{ or } \mathbb{R}^n_+. \tag{3.23}$$

Then it is easy to see that the counterparts of Propositions 3.1 and 3.2 hold with $Y^{q,r}(\Omega)$ when $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$ as follows.

Proposition 3.3. *Let $1 < q, r < \infty$ and let $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$. Then the operator $\mathbf{V}$ defined as (3.20) - (3.21) is bounded and bijective from $(Y^{q,r}(\Omega))^{n-1}$ onto $Y^{q,r}_\sigma(\Omega)$. Moreover, we have $\mathbf{V}^{-1} = \mathbf{W}$, where $\mathbf{W}$ is defined as (3.22).*

Proposition 3.4. *Let $1 < q, r < \infty$. Assume that $p \in L^1_{loc}(\mathbb{R}+; L^r(\mathbb{R}^{n-1}))$ satisfies $\nabla p \in (Y^{q,r}(\mathbb{R}^n_+))^n$ and $\Delta p = 0$ in $\mathbb{R}^n_+$ in the sense of distributions. Then $(E' + SE_n)\nabla p = 0$.*

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