HECKE ALGEBRAS, NEW VECTORS AND CHARACTERIZATION OF THE NEW SPACES

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This report is a summary of a recent joint work [2] with Prof. Moshe Baruch in which we characterize the space of new forms for $\Gamma_0(N)$ as a common eigenspace of certain Hecke operators which depend on primes dividing the level N.

Our approach is to study a certain p-adic Hecke algebra of functions on $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and find generators and relations for it. Casselman [3, 4] showed that there is a unique irreducible representation of K which contains a $K_0(p^n)$ fixed vector but does not contain a $K_0(p^k)$ fixed vector for $k < n$, such a vector is called a new vector and is unique up to scalar multiplication. Using p-adic Hecke algebra we explicitly find for any positive n the $n + 1$ irreducible representations of K which contain a $K_0(p^n)$ fixed vector including the unique representation that contains the "new vector" of level n.

This p-adic Hecke algebra sits inside the endomorphism algebra $\mathrm{End}_{\mathbb{C}}(A_{2k}(N))$, where $A_{2k}(N)$ is the space of adelic automorphic forms of weight $2k$ and level N and is well known [5] to be isomorphic to the classical space of cusp forms $S_{2k}(\Gamma_0(N))$. We use this isomorphism to translate the p-adic Hecke operators to their classical counterparts and obtain relations amongst them. This leads us to obtain the characterization results about the new and old spaces.

We view our work as a connection between the theory of new vectors described by Casselman and the theory of newforms by Atkin and Lehner [1]. We expect to have applications of these results to the Shimura correspondence and to the definition of new forms of half integral weight [7, 8, 9].

We present below one of our characterization results.

Theorem 1. *Let N be a square-free positive number. For any prime $p \mid N$, let $Q_p = p^{1-k} U_p W_p$ and $Q'_p = p^{1-k} W_p U_p$. Then the space of new forms $S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ is the intersection of the -1 eigenspaces of Q_p and Q'_p as p varies over the prime divisors of N.*

For a similar statement for general level N we will later introduce a certain "new" family of Hecke operators.

1. p-ADIC HECKE ALGEBRAS AND GENERATORS AND RELATIONS.

In this section we describe the Hecke algebra of functions on $K = \mathrm{GL}_2(\mathbb{Z}_p)$ which are bi-invariant with respect to $K_0(p^n)$ using generators and relations.

Let G denotes the group $\mathrm{GL}_2(\mathbb{Q}_p)$. Let $K_0(p^n)$ be the subgroup of K defined by

$$K_0(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \ : \ c \in p^n \mathbb{Z}_p \right\}.$$

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Note that the subgroup $K_0(p)$ denotes the usual Iwahori subgroup.

The Hecke algebra corresponding to $K_0(p^n)$ is defined as:

$$H(G//K_0(p^n)) = \{f \in C_c^\infty(G) : f(kgk') = f(g) \text{ for } g \in G, \ k, \ k' \in K_0(p^n)\},$$

it forms a $\mathbb{C}$-algebra under convolution which, for any $f_1, f_2 \in C_c^\infty(G)$, is defined by

$$f_1 * f_2(h) = \int_G f_1(g)f_2(g^{-1}h)dg = \int_G f_1(hg)f_2(g^{-1})dg,$$

where dg is the Haar measure on G such that the measure of $K_0(p^n)$ is one.

Let X_g be the characteristic function of the double coset $K_0(p^n)gK_0(p^n)$. Thus as a $\mathbb{C}$-vector space $H(G//K_0(p^n))$ is spanned by X_g as g varies over the double coset representatives of G modulo $K_0(p^n)$.

Let $\mu(g)$ denotes the number of disjoint left (right) $K_0(p^n)$ cosets in the double coset $K_0(p^n)gK_0(p^n)$. Then the following lemmas are well known [6, Corollary 1.1].

Lemma 1.1. *If* $\mu(g)\mu(h) = \mu(gh)$ *then* $X_g * X_h = X_{gh}$.

Lemma 1.2. *Let* $f_1, \ f_2 \in H(G//K_0(p^n))$ *such that* f_1 *is supported on* $K_0(p^n)xK_0(p^n) = \bigcup_{i=1}^m \alpha_i K_0(p^n)$ *and* f_2 *is supported on* $K_0(p^n)yK_0(p^n) = \bigcup_{j=1}^n \beta_j K_0(p^n)$. *Then*

$$f_1 * f_2(h) = \sum_{i=1}^m f_1(\alpha_i)f_2(\alpha_i^{-1}h)$$

where the nonzero summands are precisely for those i *for which there exist a* j *such that* $h \in \alpha_i\beta_j K_0(p^n)$.

For $t \in \mathbb{Q}_p$ we consider the following elements:

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \ y(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \ w(t) = \begin{pmatrix} 0 & -1 \\ t & 0 \end{pmatrix},$$

$$d(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \ z(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

Let $N = \{x(t) : t \in \mathbb{Q}_p\}$, $\bar{N} = \{y(t) : t \in \mathbb{Q}_p\}$ and A be the group of diagonal matrices of G. Let $Z_G = \{z(t) : t \in \mathbb{Q}_p^*\}$ denote the center of G.

1.1. The Iwahori Hecke Algebra. We first look at the case when $n = 1$. We have following well-known lemma.

Lemma 1.3. *A complete set of representatives for the double cosets of* G *mod* $K_0(p)$ *are given by* $d(p^n)z(m)$, $w(p^n)z(m)$ *where* n, m *varies over integers.*

Using triangular decomposition of $K_0(p)$ we obtain following decomposition.

Lemma 1.4. *(1) For* $n \geq 0$ *we have*

$$K_0(p)d(p^n)K_0(p) = \bigsqcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} x(s)d(p^n)K_0(p).$$

(2) For $n \geq 1$ *we have*

$$K_0(p)d(p^{-n})K_0(p) = \bigsqcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} y(ps)d(p^{-n})K_0(p).$$

(3) For $n \geq 1$ we have

$$K_0(p)w(p^n)K_0(p) = \bigsqcup_{s \in \mathbb{Z}_p/p^{n-1}\mathbb{Z}_p} y(ps)w(p^n)K_0(p).$$

(4) For $n \geq 0$ we have

$$K_0(p)w(p^{-n})K_0(p) = \bigsqcup_{s \in \mathbb{Z}_p/p^{n+1}\mathbb{Z}_p} x(s)w(p^{-n})K_0(p).$$

Let $\mathcal{T}_n = X_{d(p^n)}$, $\mathcal{U}_n = X_{w(p^n)}$ and $\mathcal{Z} = X_{z(p)}$ be elements of the Hecke algebra $H(G//K_0(p))$. Note that $\mathcal{Z}$ commutes with every $f \in H(G//K_0(p))$ and that $\mathcal{Z}^n = X_{z(p^n)}$. We use Lemma 1.1, Lemma 1.2 and Lemma 1.4 to obtain the following relations in $H(G//K_0(p))$.

Lemma 1.5. *(1) If $n, m \geq 0$ or $n, m \leq 0$, then $\mathcal{T}_n * \mathcal{T}_m = \mathcal{T}_{n+m}$.*
(2) If $n \geq 0$ then $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ and $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{Z}^n * \mathcal{U}_{1-n}$.
(3) If $n \geq 0$ then $\mathcal{U}_1 * \mathcal{T}_{-n} = \mathcal{U}_{1-n}$ and $\mathcal{T}_{-n} * \mathcal{U}_1 = \mathcal{Z}^{-n} * \mathcal{U}_{1+n}$.
(4) If $n \geq 0$ then $\mathcal{U}_0 * \mathcal{T}_{-n} = \mathcal{U}_{-n}$ and $\mathcal{T}_n * \mathcal{U}_0 = \mathcal{Z}^n * \mathcal{U}_{-n}$.
(5) For $n \in \mathbb{Z}$, $\mathcal{U}_1 * \mathcal{U}_n = \mathcal{Z} * \mathcal{T}_{n-1}$ and $\mathcal{U}_n * \mathcal{U}_1 = \mathcal{Z}^n * \mathcal{T}_{1-n}$.
(6) For $n \geq 1$, $\mathcal{U}_0 * \mathcal{U}_n = \mathcal{T}_n$ and $\mathcal{U}_n * \mathcal{U}_0 = \mathcal{Z}^n * \mathcal{T}_{-n}$.
(7) $\mathcal{U}_0 * \mathcal{U}_0 = (p-1)\mathcal{U}_0 + p$

As a consequence we have the following well known theorem.

Theorem 2. *The Iwahori Hecke Algebra $H(G//K_0(p))$ is generated by $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{Z}$ with the relations:*
1) $\mathcal{U}_1^2 = \mathcal{Z}$
2) $(\mathcal{U}_0 - p)(\mathcal{U}_0 + 1) = 0$
3) $\mathcal{Z}$ commutes with $\mathcal{U}_0$ and $\mathcal{U}_1$

Remark 1. *The algebra $H(G//K_0(p))/\langle \mathcal{Z} \rangle$ is an algebra generated by $\mathcal{U}_0$ and $\mathcal{U}_1$ with the relations $\mathcal{U}_1^2 = 1$ and $(\mathcal{U}_0 - p)(\mathcal{U}_0 + 1) = 0$.*

1.2. A subalgebra. We now consider the case $n \geq 2$.

Let $H(K//K_0(p^n))$ denotes the subalgebra of $H(G//K_0(p^n))$ consisting of functions supported on K. We shall obtain generators and relations for $H(K//K_0(p^n))$.

We first note the following lemma [4, Lemma 1].

Lemma 1.6. *A complete set of representatives for the double cosets of K mod $K_0(p^n)$ are given by 1, $w(1)$, $y(p)$, $y(p^2)$, $\ldots$ $y(p^{n-1})$.*

Let $\mathcal{U}_0 = X_{w(1)}$ and $\mathcal{V}_r = X_{y(p^r)}$ for $1 \leq r \leq n-1$ be the elements of $H(G//K_0(p^n))$. Then by the above lemma, $H(K//K_0(p^n))$ is spanned by $1, \mathcal{U}_0$ and $\mathcal{V}_r$ where $1 \leq r \leq n-1$.
We have the following lemma.

Lemma 1.7.

$$K_0(p^n)y(p^r)K_0(p^n) = \bigsqcup_{s \in \mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} d(s)y(p^r)K_0(p^n).$$

As a consequence of the above lemma and Lemma 1.2 we obtain the following relations.

Proposition 1.8. *We have the following relations in $H(K//K_0(p^n))$:*
(1) $\mathcal{V}_r^2 = p^{n-r-1}(p-1)(I + \sum_{j=r+1}^{n-1} \mathcal{V}_j) + p^{n-r-1}(p-2)\mathcal{V}_r$.
(2) $\mathcal{V}_r * \mathcal{V}_j = (p-1)p^{n-j-1}\mathcal{V}_r = \mathcal{V}_j * \mathcal{V}_r$ for $r+1 \leq j \leq n-1$.

(3) Let $\mathcal{Y}_{r+1} = I + \sum_{j=r+1}^{n-1} \mathcal{V}_j$. *Then*

$$\mathcal{V}_r * \mathcal{Y}_{r+1} = p^{n-r-1}\mathcal{V}_r = \mathcal{Y}_{r+1} * \mathcal{V}_r,$$

and so,

$$(\mathcal{V}_r - p^{n-r-1}(p-1))(\mathcal{V}_r + \mathcal{Y}_{r+1}) = 0.$$

For $1 \leq r \leq n - 1$, let $\mathcal{Y}_r = I + \sum_{j=r}^{n-1} \mathcal{V}_j$, take $\mathcal{Y}_n = I$. We have the following easy corollary.

Corollary 1.9. *(1)* $\mathcal{Y}_{n-r}^2 = p^r \mathcal{Y}_{n-r}$ *for all* $0 \leq r \leq n - 1$.
(2) $\mathcal{Y}_r * \mathcal{Y}_l = p^{n-r}\mathcal{Y}_l = \mathcal{Y}_l * \mathcal{Y}_r$ *for* $r \geq l$.

Next we obtain relations in $H(K//K_0(p^n))$ that involve $\mathcal{U}_0$.

Proposition 1.10. *(1)* $\mathcal{U}_0 * \mathcal{U}_0 = p^{n-1}(p-1)\mathcal{U}_0 + p^n \mathcal{Y}_1$.
(2) $\mathcal{U}_0 * \mathcal{Y}_r = p^{n-r}\mathcal{U}_0 = \mathcal{Y}_r * \mathcal{U}_0$ *for all* $1 \leq r \leq n$.
(3) $\mathcal{U}_0 * (\mathcal{U}_0 - p^n) * (\mathcal{U}_0 + p^{n-1}) = 0$.

As a consequence of the above relations have the following theorem.

Theorem 3. *The algebra* $H(K//K_0(p^n))$ *is an* $n + 1$ *dimensional commutative algebra with generators* $\{\mathcal{U}_0,\ \mathcal{Y}_1,\ \mathcal{Y}_2,\ \ldots,\ \mathcal{Y}_n\}$ *and relations given by Corollary 1.9 and Proposition 1.10.*

We should point out that we have not yet found an analogue of Theorem 2 for $H(G//K_0(p^n))$ for $n \geq 2$. However we would need the following relation later. Let $\mathcal{T}_m = X_{d(p^m)}$, $\mathcal{U}_m = X_{w(p^m)}$, $\mathcal{Z} = X_{z(p)}$ be the elements in $H(G//K_0(p^n))$. Then

Lemma 1.11. $(\mathcal{T}_1)^m * \mathcal{U}_m = \mathcal{T}_m * \mathcal{U}_m = \mathcal{Z}^m * \mathcal{U}_0$ *for all* $m \leq n$.

1.3. Representations of K having a $K_0(p^n)$ fixed vector. We are interested in irreducible representations of K having a $K_0(p^n)$ fixed vector. Let

$$I(n) := \mathrm{Ind}_{K_0(p^n)}^K 1 = \{\phi : K \to \mathbb{C} : \phi(k_0 k) = \phi(k) \text{ for } k_0 \in K_0(p^n),\ k \in K\}.$$

Then $I(n)$ is a right representation of K, via right translation, denoted by π_R, where $\pi_R(k)(\phi)(k') = \phi(k'k)$, and the dimension of this representation is $[K : K_0(p^n)] = p^{n-1}(p+1)$. It follows from Frobenius Reciprocity that every (smooth) irreducible representation of K which has a nonzero $K_0(p^n)$ fixed vector is isomorphic to a subrepresentation of $I(n)$. We shall therefore decompose $I(n)$ into sum of irreducible representations. We note the following easy lemma.

Lemma 1.12. *We have* $I(n)^{K_0(p^n)} = H(K//K_0(p^n))$ *and consequently the dimension of* $I(n)^{K_0(p^n)}$ *is* $n + 1$.

Further, using induction argument and Frobenius reciprocity we can check that the representation $I(n)$ is a sum of $n + 1$ distinct irreducible representations.

We shall now explicitly describe the irreducible subrepresentations of $I(n)$. Consider the following action π_L of $H(K//K_0(p^n))$ on $I(n)$: for $f \in H(K//K_0(p^n))$ and $\phi \in I(n)$ set

$$\pi_L(f)(\phi)(g) = \int_K f(k)\phi(k^{-1}g)dk \quad \text{for all } g \in K.$$

In particular, if $\phi \in I(n)^{K_0(p^n)}$ which by Lemma 1.12 is same as the algebra $H(K//K_0(p^n))$ then we have $\pi_L(f)(\phi) = f * \phi$. It is easy to check that the action π_L commutes with the action π_R. It now follows by Schur's Lemma that for each $f \in H(K//K_0(p^n))$ the

operator $\pi_L(f)$ acts as a scalar operator on an irreducible subrepresentation of $I(n)$. We use this to distinguish the irreducible components of $I(n)$.

Note that if σ is any irreducible subrepresentation of $I(n)$ then σ contains a $K_0(p^n)$ fixed vector, that is there exists a non-zero vector $v_\sigma \in \sigma \cap I(n)^{K_0(p^n)}$. Thus v_σ is a linear combination of $\mathcal{U}_0$ and $\mathcal{Y}_r$ for $1 \le r \le n$. Since $\pi_L(f)$ acts as a scalar for every $f \in H(K//K_0(p^n))$ the vector v_σ will be an eigenvector under the action of $\pi_L(\mathcal{U}_0)$ and $\pi_L(\mathcal{Y}_r)$ for all $1 \le r \le n$. For each σ we can compute these eigenvectors v_σ and their corresponding eigenvalues using the relations in Corollary 1.9 and Proposition 1.10. Thus we obtain the following proposition.

Proposition 1.13. *A basis of eigenvectors for $H(K//K_0(p^n))$ under the above action is given by:*
$v_1 = \mathcal{U}_0 + \mathcal{Y}_1$
$v_2 = \mathcal{U}_0 - p\mathcal{Y}_1$
$w_k = \mathcal{Y}_k - p\mathcal{Y}_{k+1}$ *for* $1 \le k \le n-1$,
with eigenvalues given as follows:
$\mathcal{U}_0 * v_1 = p^n v_1$, $\mathcal{Y}_i * v_1 = p^{n-i} v_1$ *for all* $1 \le i \le n$.
$\mathcal{U}_0 * v_2 = -p^{n-1} v_2$, $\mathcal{Y}_i * v_2 = p^{n-i} v_2$ *for all* $1 \le i \le n$.
$\mathcal{U}_0 * w_k = 0$, $\mathcal{Y}_i * w_k = 0$ *for all* $1 \le i \le k$, $\mathcal{Y}_i * w_k = p^{n-i} w_k$ *for all* $k < i \le n$.

Corollary 1.14. *The representation $I(n)$ is a sum of $n+1$ irreducible subspaces given by:* $S_1 = \mathrm{Span}(\pi_R(K)v_1)$, $S_2 = \mathrm{Span}(\pi_R(K)v_2)$ *and* $T_k = \mathrm{Span}(\pi_R(K)w_k)$ *where* $1 \le k \le n-1$ *such that* $\dim(S_1) = 1$, $\dim(S_2) = p$, $\dim(T_k) = p^{k-1}(p^2 - 1)$. *Consequently, T_{n-1} is the unique irreducible representation of K that has a $K_0(p^n)$ fixed vector w_{n-1} but does not have $K_0(p^k)$ fixed vector for $k < n$ that is, w_{n-1} is the "new" vector of level n.*

2. TRANSLATION FROM THE ADELIC TO THE CLASSICAL SETTING.

Let $G_\infty = \mathrm{GL}_2(\mathbb{R})^+$. Then G_∞ acts on the upper half plane $\mathbb{H}$ using Möbius transformation. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$, $z \in \mathbb{H}$ and functions f on $\mathbb{H}$ recall the automorphy factor and the slash operator $|_{2k} g$,

$$j(g, z) = det(g)^{-1/2}(cz + d), \quad f|_{2k} g = j(g, z)^{-2k} f\left(\frac{az + b}{cz + d}\right).$$

Let N be a positive integer and $K_p = K_0(p^\alpha)$ for a prime p such that $p^\alpha \| N$. Let K_f be the subgroup of $\mathrm{GL}_2(\mathbb{A})$ defined by

$$K_f(N) = \prod_{q < \infty} K_q.$$

By the strong approximation theorem we have

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q}) G_\infty K_f(N)$$

Let $A_{2k}(N)$ be the space of functions $\Phi \in L^2(Z_\mathbb{A} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}))$ satisfying the following properties:

(1) $\Phi(gk) = \Phi(g)$ for all $g \in \mathrm{GL}_2(\mathbb{A})$, $k \in K_f(N)$.

(2) $\Phi(gr(\theta)) = e^{-i2k\theta}\Phi(g)$ where $r(\theta) = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \in \mathrm{SO}(2)$.

(3) Φ is smooth as a function of G_∞ and satisfies the differential equation $\Delta\Phi = -k(k-1)\Phi$ where Δ is the Casimir operator.

(4) Φ is cuspidal, that is $\int_{\mathbb{Q}\backslash\mathbb{A}} \Phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} g\right) da = 0$ for all $g \in \mathrm{GL}_2(\mathbb{A})$.

By Gelbart [5, Proposition 3.1] there exists an isomorphism

$$A_{2k}(N) \to S_{2k}(\Gamma_0(N))$$

given by $\Phi \mapsto f_\Phi$ where for $z \in \mathbb{H}$,

$$f_\Phi(z) = \Phi(g_\infty)j(g_\infty, i)^{2k}$$

where $g_\infty \in G_\infty$ is such that $g_\infty(i) = z$. The inverse map is given by $f \mapsto \Phi_f$ where for $g \in \mathrm{GL}_2(\mathbb{A})$ if $g = \gamma g_\infty k$ (using strong approximation),

$$\Phi_f(g) = f(g_\infty(i))j(g_\infty, i)^{-2k}.$$

This isomorphism induces a ring isomorphism of spaces of linear operators by

$$q : \mathrm{End}_{\mathbb{C}}(A_{2k}(N)) \to \mathrm{End}_{\mathbb{C}}(S_{2k}(\Gamma_0(N)))$$

given by

$$q(\mathcal{T})(f) = f_{\mathcal{T}(\Phi_f)}.$$

Let $N = p^n M$ where p is a prime coprime to M and $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We note that the $H(G//K_0(p^n))$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A_{2k}(N))$ via the following action:

for $\mathcal{T} \in H(G//K_0(p^n))$ and $\Phi \in A_{2k}(N)$, $\quad \mathcal{T}(\Phi)(g) = \int_G \mathcal{T}(x)\Phi(gx)dx.$

Then we have following proposition.

Proposition 2.1. *Let $N = p^n M$ where $n \geq 1$ and $p \nmid M$. Let $f \in S_{2k}(\Gamma_0(N))$. Consider operators $\mathcal{T}_1$, $\mathcal{U}_m \in H(G//K_0(p^n))$ where $m \leq n$. If $n \geq 2$, further consider $\mathcal{V}_r \in H(G//K_0(p^n))$ where $1 \leq r \leq n-1$. Then,*

(1) $q(\mathcal{T}_1)(f)(z) = p^{-k}\sum_{s=0}^{p-1} f((z+s)/p) = \tilde{U}_p(f)(z)$.

(2) If $f \in S_{2k}(\Gamma_0(p^r M))$ where $r \leq n$ then $q(\mathcal{U}_r)(f)(z) = p^{n-r}f|_{2k}W_{p^r}(z)$ where $W_{p^r} = \begin{pmatrix} p^r\beta & 1 \\ p^r M\gamma & p^r \end{pmatrix}$ is an integer matrix of determinant p^r. In particular, $q(\mathcal{U}_n)(f)(z) = f|_{2k}W_{p^n}(z)$.

(3) $q(\mathcal{V}_r)(f)(z) = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} f|_{2k}A_s$ where $A_s \in \mathrm{SL}_2(\mathbb{Z})$ is any matrix of the form $\begin{pmatrix} a_s & b_s \\ p^r M & p^{n-r} - sM \end{pmatrix}$.

(4) If $f \in S_{2k}(\Gamma_0(p^r M))$ then $q(\mathcal{V}_r)(f) = p^{n-r-1}(p-1)f$, consequently, $q(\mathcal{Y}_r)(f) = p^{n-r}f$.

We give below a proof of statement (3).
Let $n \geq 2$. Using Lemma 1.7 we have

$$\mathcal{V}_r(\Phi)(g) = \int_G X_{y(p^r)}\Phi(gh)dh = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \Phi(gd(s)y(p^r)).$$

Let $z \in \mathbb{H}$ be such that $z = g_\infty i$ for some $g_\infty \in G_\infty$. Then,

$$q(\mathcal{V}_r)(f)(z) = \sum_{s\in\mathbb{Z}_p^*/1+p^{n-r}\mathbb{Z}_p} \Phi_f(g_\infty d(s)y(p^r))j(g_\infty, i)^{2k}.$$

By strong approximation, $g_\infty d(s)y(p^r) = A_s^{-1} h_\infty k_f$ for some $A_s \in GL_2(\mathbb{Q})$, $h_\infty \in G_\infty$ and $k_f \in K_f(N)$. For any $s \in \mathbb{Z}_p^*$, we have $\gcd(p^r M, p^{n-r} - sM) = 1$, so there exists integers a_s, b_s such that $a_s(p^{n-r} - sM) - b_s p^r M = 1$. Take

$$A_s = \begin{pmatrix} a_s & b_s \\ p^r M & p^{n-r} - sM \end{pmatrix} \in SL_2(\mathbb{Z}),$$

then A_s belongs to K_q for $q \neq p$ and

$$A_s d(s)y(p^r) = \begin{pmatrix} a_s + b_s p^r & b_s \\ p^n & p^{n-r} - sM \end{pmatrix} \in K_0.$$

Consequently,

$$q(\mathcal{V}_r)(f)(z) = \sum_{s \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p} f(A_s z)j(A_s, z)^{-2k} = \sum_{s \in \mathbb{Z}_p^*/1 + p^{n-r}\mathbb{Z}_p} f|_{A_s}(z).$$

Remark 2. *The operator $q(\mathcal{U}_n)$ is the usual Atkin-Lehner operator W_{p^n} while the operator $q(\mathcal{T}_1)$ is the operator $\tilde{U}_p = p^{1-k}U_p$ where U_p is the usual Hecke operator. It is obvious that $q(\mathcal{Z})$ is the identity operator.*

Let $N = pM$ where $p \nmid M$. Let $Q_p = q(\mathcal{U}_0)$ where $\mathcal{U}_0 \in H(G//K_0(p))$. Then using Lemma 1.5 we have

Corollary 2.2. $Q_p = p^{1-k}U_p W_p$ and $(Q_p - p)(Q_p + 1) = 0$.

Now consider $N = p^n M$ where $n \geq 2$. Let $Q_{p^m} = (\tilde{U}_p)^m W_{p^m}$ for $m \leq n$ where W_{p^m} is the Atkin-Lehner operator on $S_{2k}(\Gamma_0(p^m M))$. Using Lemma 1.11 and Propositions 2.1 and 1.10 we have

Corollary 2.3. *For $\mathcal{U}_0 \in H(G//K_0(p^n))$, we have $Q_{p^n} = q(\mathcal{U}_0)$ and hence $Q_{p^n}(Q_{p^n} - p^n)(Q_{p^n} + p^{n-1}) = 0$. Further for $m \leq n$ we have $Q_{p^n} = (\tilde{U}_p)^m q(\mathcal{U}_m)$, hence if $f \in S_{2k}(\Gamma_0(p^m M)) \subseteq S_{2k}(\Gamma_0(N))$ then $Q_{p^n}(f) = p^{n-m}Q_{p^m}(f)$.*

Let $S_{p^n, r} = q(\mathcal{Y}_r)$ where $\mathcal{Y}_r \in H(G//K_0(p^n))$, $1 \leq r \leq n$. Using relations in Corollary 1.9, we have

Corollary 2.4. $S_{p^n, r}(S_{p^n, r} - p^{n-r}) = 0$ for $1 \leq r \leq n$.

Let $Q'_{p^n} = W_{p^n} Q_{p^n} W_{p^n}^{-1}$ and $S'_{p^n, r} = W_{p^n} S_{p^n, r} W_{p^n}^{-1}$. Then Q'_{p^n} and $S'_{p^n, r}$ also satisfy the above cubic and quadratic relations.

3. CHARACTERIZATION RESULTS.

The following is a restatement (slightly general) of Theorem 1.

Theorem 1. Let $N = M_1^2 M$ where M_1 and M are square free and coprime. Then $f \in S_{2k}^{new}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p dividing M and $Q_{p^2}(f) = 0 = Q'_{p^2}(f)$ for all primes p dividing M_1.

For a general N we need to use the family of operators $S_{p^n, r}$ to obtain a similar characterization result.

Theorem 4. *Let N be a positive integer. Then the space of new forms $S_{2k}^{new}(\Gamma_0(N))$ is the intersection of the -1 eigenspaces of Q_p and Q'_p where p varies over the primes such that $p\|N$ and the 0 eigenspaces of $S_{p^\gamma, \gamma-1}$ and $S'_{p^\gamma, \gamma-1}$ for primes p such that $p^\gamma\|N$ with $\gamma \geq 2$. That is, $f \in S_{2k}^{new}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q'_p(f)$ for all primes p such that $p\|N$ and $S_{p^\gamma, \gamma-1}(f) = 0 = S'_{p^\gamma, \gamma-1}(f)$ for all primes p such that $p^\gamma\|N$ for $\gamma \geq 2$.*

Let $q = e^{2\pi i z}$ and $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}(\Gamma_0(m))$. Let p be an odd prime. Define

$$R_p(f)(z) = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) a_n q^n, \qquad R_\chi(f)(z) = \sum_{n=1}^{\infty} \left(\frac{-1}{n}\right) a_n q^n.$$

By [1, Lemma 33], R_p and R_χ are operators on $S_{2k}(\Gamma_0(m))$ provided that $p^2 \mid m$ and $16 \mid m$ respectively.

Theorem 5. *Let $N = 2^\beta M_1 M_2$ where $M_1 M_2$ is odd such that M_1 is square free and any prime divisor of M_2 divides it with a power at least 2. Let $\beta \geq 4$. Then $f \in S_{2k}^{\mathrm{new}}(\Gamma_0(N))$ if and only if $Q_p(f) = -f = Q_p'(f)$ for all primes p dividing M_1, $(R_\chi)^2(f) = f$ and $(R_p)^2(f) = f$ for all primes p dividing M_2, and $S_{p^\gamma, \gamma-1}(f) = 0$ for all primes p such that $p^\gamma \| 2^\beta M_2$.*

Remark 3. *We can similarly characterize the subspaces of old forms $V(d)S_{2k}^{\mathrm{new}}(\Gamma_0(M))$ that appear in the direct sum decomposition of the old space $S_{2k}^{\mathrm{old}}(\Gamma_0(N))$. In particular for N square-free and M, $M' > 1$ such that $MM' \mid N$ we have $f \in V(M')S_{2k}^{\mathrm{new}}(\Gamma_0(M))$ if and only if $Q_p(f) = -f = Q_p'f$ for all $p \mid M$, $Q_q'(f) = qf$ for all $q \mid M'$ and $Q_q(f) = qf$ for all $q \mid (N/MM')$.*

3.1. Sketch of proof.

We now sketch a proof of the Theorem 4 for a particular case when any prime divisor of N divides it with a power at least 2. Let $N = p^n M$ where $n \geq 2$ and $(p, M) = 1$. Recall the family of operators that we defined: for $1 \leq r \leq n$,

$$S_{p^n, r}(f) = f + \sum_{j=r}^{n-1} \sum_{s \in \mathbb{Z}_p^* / 1 + p^{n-j} \mathbb{Z}_p} f|_{2k} A_{s,j},$$

where $A_{s,j} = \begin{pmatrix} a_{s,j} & b_{s,j} \\ p^j M & p^{n-j} - sM \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and the quadratic relation they satisfy

$$S_{p^n, r}(S_{p^n, r} - p^{n-r}) = 0.$$

We have the following lemma.

Lemma 3.1. *For $1 \leq r \leq n$, a set of right coset representatives for $\Gamma_0(N)$ in $\Gamma_0(p^r M)$ consists of the identity element and elements of the form*

$$A_{s,j} = \begin{pmatrix} a_{s,j} & b_{s,j} \\ p^j M & p^{n-j} - sM \end{pmatrix} \quad \text{where } r \leq j \leq n-1 \text{ and } s \in \mathbb{Z}_p^* / 1 + p^{n-j} \mathbb{Z}_p \,.$$

Consequently, the operator $S_{p^n, r}$ takes the space $S_{2k}(\Gamma_0(N))$ to $S_{2k}(\Gamma_0(p^r M))$.

Thus we have the following corollary.

Corollary 3.2. *For $1 \leq r \leq n$, the p^{n-r} eigenspace of $S_{p^n, r}$ is precisely the subspace $S_{2k}(\Gamma_0(p^r M))$.*

Proposition 3.3. *Let $1 \leq r \leq n$. Then for each $r < \alpha \leq n$, the space $S_{2k}^{\mathrm{new}}(\Gamma_0(p^\alpha M))$ is contained in the 0 eigenspace of $S_{p^n, r}$.*

Proof. For a prime q with $(q, N) = 1$, the Hecke operator T_q on $S_{2k}(\Gamma_0(N))$ corresponds to the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_q) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_q)$ which belongs to the q-adic Hecke algebra. Since $\mathcal{Y}_r$ belongs to the p-adic Hecke algebra $H(K_0(p^n))$, it follows that the operators $S_{p^n, r}$ and T_q on $S_{2k}(\Gamma_0(N))$ commute.

Let $r < \alpha \leq n$ and $f \in S_{2k}^{\text{new}}(\Gamma_0(p^\alpha M))$ be a primitive form. Thus f is an eigenform with respect to T_q for any q coprime to N. Now since $S_{p^n,r}$ and T_q commute we get that $S_{p^n,r}(f)$ is also an eigenfunction with respect to all such T_q having the same eigenvalue as f.

By the above lemma, $S_{p^n,r}(f) \in S_{2k}(\Gamma_0(p^r M))$ and as $r < \alpha$, it is an old form in the space $S_{2k}(\Gamma_0(p^\alpha M))$. It now follows from Atkin and Lehner that $S_{p^n,r}(f) = 0$. $\square$

We have the following lemma.

Lemma 3.4. *For $1 \leq r \leq n$, the operator W_{p^n} maps $S_{2k}(\Gamma_0(p^r M))$ onto the space $V(p^{n-r})S_{2k}(\Gamma_0(p^r M))$.*

Consequently, the p^{n-r} eigenspace of $S'_{p^n,r}$ is precisely the space $V(p^{n-r})S_{2k}(\Gamma_0(p^r M))$.

Applying above results to the case $r = n - 1$ we have the following corollary.

Corollary 3.5. *The space $S_{2k}(\Gamma_0(p^{n-1}M))$ is the p eigenspace of $S_{p^n,n-1}$ and the space $V(p)S_{2k}(\Gamma_0(p^{n-1}M))$ is the p eigenspace of $S'_{p^n,n-1}$. Moreover, the space $S_{2k}^{\text{new}}(\Gamma_0(N))$ is contained in the intersection of the 0 eigenspaces of $S_{p^n,n-1}$ and $S'_{p^n,n-1}$.*

Finally we need the following proposition.

Proposition 3.6. *The operators $S_{p^n,n-1}$ and $S'_{p^n,n-1}$ are self-adjoint with respect to Petersson inner product.*

Proof of Theorem 4. Let $N = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$ where q_j are distinct primes and $\alpha_j \geq 2$ for all $1 \leq j \leq s$. We have already seen one side implication. Conversely suppose $f \in S_{2k}(\Gamma_0(N))$ is such that $S_{q_j^{\alpha_j},\alpha_j-1}(f) = 0 = S'_{q_j^{\alpha_j},\alpha_j-1}(f)$ for all $1 \leq j \leq s$. It follows from from Corollary 3.5 that for each $1 \leq j \leq s$, the subspace $S_{2k}(\Gamma_0(N/q_j))$ is contained in the q_j eigenspace of $S_{q_j^{\alpha_j},\alpha_j-1}$ and $V(q_j)S_{2k}(\Gamma_0(N/q_j))$ is contained in the q_j eigenspace of $S'_{q_j^{\alpha_j},\alpha_j-1}$.

Since $S_{q_j^{\alpha_j},\alpha_j-1}$, $S'_{q_j^{\alpha_j},\alpha_j-1}$ are self-adjoint operators we get that f is orthogonal to $S_{2k}(\Gamma_0(N/q_j))$ and $V(q_j)S_{2k}(\Gamma_0(N/q_j))$ for each prime divisor q_j of N. Thus f is orthogonal to the old space, that is, $f \in S_{2k}^{\text{new}}(\Gamma_0(N))$. $\square$

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