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A FREE BOUNDARY PROBLEM FOR A TWO SPECIES SUPERIOR-INFERIOR COMPETITION-DIFFUSION MODEL

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ABSTRACT. We discuss a free boundary problems for the two species competition-diffusion model in a one-dimensional habitat. We are concerned with the superior-inferior case with two free boundaries. Here, the two free boundaries which describe the spreading fronts of two competing species, respectively, may intersect each other. Our result shows, there exists a critical value such that the superior competitor always spreads successfully if its territory size is above this constant at some time. Otherwise, the superior competitor can be wiped out by the inferior competitor. Moreover, if the inferior competitor spreads not fast enough such that the superior competitor can catch up with it, the inferior competitor will be wiped out eventually and then a spreading-vanishing trichotomy is established. Some characterizations of the spreading-vanishing trichotomy via some parameters of the model are provided. On the other hand, when the superior competitor spreads successfully but with a sufficiently low speed, the inferior competitor can also spread successfully even the superior species is much stronger than the weaker one. It means that the inferior competitor can survive if the superior species cannot catch up with it.

Keywords: free boundary problem, superior-inferior, vanishing, spreading.

1. INTRODUCTION

In ecology, it is important to understand the spreading phenomenon of multiple competing species. For this, the classical works are to study the Cauchy problem, in particular, for traveling waves and two-front entire solutions. These are related to the so-called asymptotic spreading speed (cf. [13, 14, 17]). However, it is not realistic that a species lives in the entire space. Therefore, due to the boundedness of the habitat, a free boundary formulation was introduced recently by Du-Lin [5] for a single species by assuming the spreading front as a free boundary. It is assumed that the population density vanishes at the front and the mechanism of spreading is determined by the spatial population gradient at the front. A mathematical deduction can be found in [1]. For other related results, we refer to, for example, [2, 3, 4, 7, 8, 11, 12, 15, 18] and references cited therein.

In the case of superior-inferior competition, it is always the case that superior competitor wipe out the inferior competitor for the Cauchy problem. It is natural to ask what happen if two superior-inferior species live in a bounded habitat and spread only at the same direction but with different free boundaries. Does the superior competitor always wipe out the inferior one if it establishes persistent populations? If not, is it possible for weaker species to survive? Even more surprising, can inferior competitor wipe out the superior competitor? Note that these questions are under the assumption that two species are competing with a fully supplied resource from the environment.

In the joint work with Chang-Hong Wu ([10]), we study the following free boundary problem **(P)**:

$$u_t = d_1 u_{xx} + r_1 u(1-u-kv), \quad 0 < x < s_1(t),\ t > 0, \tag{1.1}$$

$$v_t = d_2 v_{xx} + r_2 v(1-v-hu), \quad 0 < x < s_2(t),\ t > 0, \tag{1.2}$$

$$u_x(0,t) = v_x(0,t) = 0, \quad t > 0, \tag{1.3}$$

$$u \equiv 0 \text{ for } x \geq s_1(t),\ t > 0; \quad v \equiv 0 \text{ for } x \geq s_2(t),\ t > 0, \tag{1.4}$$

$$s_1'(t) = -\mu_1 u_x(s_1(t),t),\ s_2'(t) = -\mu_2 v_x(s_2(t),t),\ t > 0, \tag{1.5}$$

$$s_1(0) = s_1^0,\ s_2(0) = s_2^0,$$

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x \in [0,\infty), \tag{1.6}$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of two competing species at the position x and time t; d_1, d_2 are diffusion rates of species u, v; r_1, r_2 are net birth rates of species u, v; h, k are competition coefficients of species u, v; the parameters μ_1 and μ_2 measure the intention to spread into new territories of u, v. All the parameters are positive and the initial data (u_0, v_0, s_1^0, s_2^0) satisfy

$$\begin{cases} s_1^0 > 0,\ s_2^0 > 0, \\ u_0 \in C^2[0, s_1^0],\ v_0 \in C^2[0, s_2^0],\ u_0'(0) = v_0'(0) = 0, \\ u_0(x) > 0 \text{ for } x \in [0, s_1^0),\ u_0(x) = 0 \text{ for } x \geq s_1^0, \\ v_0(x) > 0 \text{ for } x \in [0, s_2^0),\ v_0(x) = 0 \text{ for } x \geq s_2^0. \end{cases}$$

Notice that the free boundaries $x = s_1(t)$ and $x = s_2(t)$ may intersect each other at some time. Also, the derivatives of u, v at the free boundary are considered as left derivatives. The case when $s_1 \equiv s_2$ was studied by a joint work with Wu ([9]) for weak competition case: $0 < h, k < 1$. See also the later improvement by Wang and Zhao [16]. In this work, we consider the case when u is a superior competitor and v is an inferior competitor, i.e., we assume

$$\textbf{(H)} \quad 0 < k < 1 < h.$$

The outline of this paper is as follows. We shall first describe the main results obtain in the work [10] in §2. Then some ideas of the proofs of these main theorems are given in §3. Finally, a brief discussion is presented in §4.

2. Main results

We first have the following global existence and uniqueness theorem.

Theorem 2.1. *The problem* **(P)** *admits a unique global in time solution* (u, v, s_1, s_2) *with* $s_1, s_2 \in C^{1+\alpha/2}([0,\infty))$ *and*

$$u \in C^{2,1}(\Omega_1) \cap C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_1), \quad v \in C^{2,1}(\Omega_2) \cap C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega}_2),$$

where $\alpha \in (0,1)$ *is arbitrary and* $\Omega_j := \{(x,t) : 0 \leq x \leq s_j(t),\ t > 0\}$, $j = 1, 2$.

Furthermore, we have the following a priori estimates

$$0 < u(x,t) \le K_1 := \max\{1, \|u_0\|_{L^\infty}\}, \quad x \in [0, s_1(t)), \ t \ge 0,$$
$$0 < v(x,t) \le K_2 := \max\{1, \|v_0\|_{L^\infty}\}, \quad x \in [0, s_2(t)), \ t \ge 0,$$
$$0 < s_1'(t) \le 2\mu_1 K_1 \max\left\{\sqrt{\frac{r_1}{2d_1}}, \ \frac{4}{3}, \ \frac{-4}{3}\left(\min_{x\in[0,s_1^0]} u_0'(x)\right)\right\}, \ t > 0,$$
$$0 < s_2'(t) \le 2\mu_2 K_2 \max\left\{\sqrt{\frac{r_2}{2d_2}}, \ \frac{4}{3}, \ \frac{-4}{3}\left(\min_{x\in[0,s_2^0]} v_0'(x)\right)\right\}, \ t > 0.$$

Also, the limits

$$s_{1,\infty} := \lim_{t\to\infty} s_1(t), \quad s_{2,\infty} := \lim_{t\to\infty} s_2(t)$$

are well-defined such that $s_{i,\infty} \le \infty$, $i = 1, 2$.

Remark 1. One of the classical method in dealing with the existence of solution for free boundary problem is to transform the free boundary into a fixed boundary. Due to the fact that two free boundaries may intersect each other at some time, these two free boundaries may not be straightened locally into two cylindrical domains together. Thus our proof here becomes more complicated than those of the above-mentioned related works.

We now define the notion of vanishing and spreading as follows.

Definition 1. The species u vanishes eventually if $s_{1,\infty} < +\infty$ and $\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,s_1(t)])} = 0$. The species u spreads successfully if $s_{1,\infty} = +\infty$ and the species u persists in the sense that there exist $\varepsilon > 0$ and two continuous curves $x = l_\pm(t)$ such that $l_+(t) - l_-(t) \ge \varepsilon$ for all large t and $u(x,t) \ge \varepsilon$ for all $x \in [l_-(t), l_+(t)]$ and for all large t. Same for v.

Let

$$s_* := \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}, \ s^* := \frac{\pi}{2}\sqrt{\frac{d_1}{r_1}}\frac{1}{\sqrt{1-k}}, \ s^{**} := \frac{\pi}{2}\sqrt{\frac{d_2}{r_2}}.$$

Note that $s_* < s^*$. Then the following theorem gives some simple criteria of vanishing-spreading via the asymptotical habitat sizes.

Theorem 2.2. *Assume* **(H)**. *Then the followings hold:*

(i) *If $s_{1,\infty} \le s_*$, then the species u vanishes eventually.*
In this case, the species v vanishes eventually, if $s_{2,\infty} \le s^{**}$; if $s_{2,\infty} > s^{**}$, v spreads successfully and

(2.1) $$\lim_{t\to\infty} v(\cdot,t) = 1 \quad \textit{uniformly } \forall \textit{ bdd subset of } [0,\infty).$$

(ii) *If $s_{1,\infty} \in (s_*, s^*]$, then u vanishes eventually and v spreads successfully with behavior* (2.1).

(iii) *If $s_{1,\infty} > s^*$, then u spreads successfully. Furthermore,*

$$\liminf_{t\to\infty} u(\cdot,t) \ge 1 - k\rho_2 \quad \textit{uniformly } \forall \textit{ bdd subset of } [0,\infty),$$

where $\rho_2 := \limsup_{t\to\infty} \|v(\cdot,t)\|_{C[0,s_2(t)]} \in [0,1]$.

Remark 2. Theorem 2.2 shows that the inferior competitor may win the competition if the the territory of the superior species does not cross over $[0, s^*]$. However, u is always unbeatable if its territory exceeds $[0, s^*]$. A natural question arises: does the weaker species v always die out eventually if u spreads successfully? Intuitively, if the superior competitor spreads faster than the inferior competitor, the inferior competitor would have no chance to survive eventually even its initial population and initial habitat size are large.

We recall a result from [1] that given $a, b, d, \mu > 0$ there exists a unique $c_0 = c_0(a, b, d, \mu) \in (0, 2\sqrt{ad})$ such that the problem

$$c_0 U' = dU'' + U(aU - b) \text{ in } (0, \infty),\ U(0) = 0,\ U(\infty) = \frac{b}{a} \tag{2.2}$$

has a unique positive solution $U = U_{c_0}$ with $\mu U'_{c_0}(0) = c_0$. Moreover, c_0 is strictly increasing in a and μ, respectively, and is strictly decreasing in b such that

$$\lim_{\frac{a\mu}{bd}\to 0} \frac{c_0}{\sqrt{ad}} \frac{bd}{a\mu} = \frac{1}{\sqrt{3}},\quad \lim_{\frac{a\mu}{bd}\to\infty} \frac{c_0}{\sqrt{ad}} = 2. \tag{2.3}$$

For every $d_i > 0$, $r_i > 0$ $(i = 1, 2)$, $0 < k < 1 < h$, define

$$\mathcal{A} := \{(\mu_1, \mu_2) \in (0, \infty) \times (0, \infty) :\ c_0(r_1(1-k), r_1, d_1, \mu_1) > c_0(r_2, r_2, d_2, \mu_2)\}.$$

Then $\mathcal{A}$ is non-empty. Indeed, by (2.3),

$$\lim_{\mu_1\to\infty} c_0(r_1(1-k), r_1, d_1, \mu_1) = 2\sqrt{d_1 r_1(1-k)},$$
$$\lim_{\mu_2\to 0} c_0(r_2, r_2, d_2, \mu_2) = 0,$$

there exist $\mu_1^* > 0$ and $\mu_2^* > 0$ such that $[\mu_1^*, \infty) \times (0, \mu_2^*] \subset \mathcal{A}$.

The next theorem provides a spreading and vanishing trichotomy.

Theorem 2.3. *Assume* **(H)** *and* $d_i > 0$, $r_i > 0$ *are given,* $i = 1, 2$. *If* $(\mu_1, \mu_2) \in \mathcal{A}$, *then the dynamics of* **(P)** *satisfies the following trichotomy:*

(i) *both two species vanish eventually:* $s_{1,\infty} \le s_*$ *and* $s_{2,\infty} \le s^{**}$,

(ii) u *vanishes eventually and* v *spreads successfully:* $s_{1,\infty} \le s^*$,

(iii) u *spreads successfully and* v *vanishes eventually.*

For the characterization of the set $\mathcal{A}$, we have

Theorem 2.4. *Assume* **(H)** *and* $d_i > 0$, $r_i > 0$ *are given,* $i = 1, 2$. *Then there exist a strictly increasing* C^1 *function* $\Lambda(\cdot)$ *with* $\Lambda(0^+) = 0$ *such that the following hold:*

- *If* $\sqrt{r_1 d_1(1-k)} \ge \sqrt{r_2 d_2}$, *then*
$$(\mu_1, \mu_2) \in \mathcal{A} \iff \mu_1 > \Lambda(\mu_2),\quad \mu_2 \in (0, \infty);$$
- *If* $\sqrt{r_1 d_1(1-k)} < \sqrt{r_2 d_2}$, *then*
$$(\mu_1, \mu_2) \in \mathcal{A} \iff \mu_1 > \Lambda(\mu_2),\quad \mu_2 \in (0, \nu_2)$$
for some $\nu_2 \in (0, \infty)$ *such that* $\Lambda(\nu_2^-) = \infty$.

The followings are some sufficient conditions on initial data for vanishing-spreading.
Let (u, v, s_1, s_2) be a solution of **(P)**. Then

- If $s_1^0 < s_*$ and $\|u_0\|_{L^\infty}$ is small enough, then u vanishes eventually.

- When u vanishes eventually, the following hold:
  (a-1) if $s_2^0 < s^{**}$, then v also vanishes eventually as long as $\|v_0\|_{L^\infty}$ is small enough;
  (a-2) if $s_2^0 < s^{**}$, then v spreads successfully as long as $\|v_0\|_{L^\infty}$ is large enough;
  (a-3) if $s_2^0 \geq s^{**}$, then v always spreads successfully regardless of its initial population.
- Suppose that $s_1^0 > s^*$. Then u spreads successfully and v vanishes eventually as long as

$$\mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0,\infty), \quad \text{if } \sqrt{r_1 d_1(1-k)} \geq \sqrt{r_2 d_2}.$$
$$\mu_1 > \Lambda(\mu_2), \quad \mu_2 \in (0,\nu_2), \quad \text{if } \sqrt{r_1 d_1(1-k)} > \sqrt{r_2 d_2},$$

regardless of their initial population size, where ν_2 and Λ are defined in Theorem ??.

The following result is the case of both species spread successfully.

Theorem 2.5. *Assume* **(H)**. *Given d_1, μ_2, r_i, $i = 1,2$, u_0 and v_0 with $s_1^0 < s_2^0$ and $(v_0)'(x) \leq 0$ for all $x \in [s_1^0, s_2^0]$. Suppose that $s_{1,\infty} > s^*$ (e.g., $s_1^0 > s^*$). Then there exists $\bar{d} > 0$ depending on $d_1, \mu_2, r_1, r_2, u_0$ and v_0 such that if $d_2 > \bar{d}$, then both two species spread successfully as long as*

$$\mu_1 \leq \bar{\mu}, \quad s_2^0 - s_1^0 > 2\pi \left[\sqrt{\frac{r_2}{d_2}\left(1 - \frac{\bar{d}}{d_2}\right)} \right]^{-1},$$

for some positive constant $\bar{\mu}$ depending only on d_2 and $\bar{d}$.

Remark 3. Theorem 2.5 shows that if the superior competitor spreads too slow to catch up with the inferior competitor, it may leave enough space for the inferior competitor to survive. Notice that u is assumed to be superior to v.

3. Some ideas

For the proofs of the theorems stated in §2, we utilize the results of [5, 8] on the spreading-vanishing dichotomy for one species competition case. Also, we borrow some ideas of [6] on the 2 species competition system with one free boundary. The key idea is to construct some suitable super/sub-solutions for a related single equation and use some a priori estimates and the regularity theory of parabolic problem.

More precisely, we recall from results of [5, 8] as follows. Let (w, h) be a solution of

$$\begin{cases} w_t = dw_{xx} + w(a - bw), \ 0 < x < h(t), \ t > 0, \\ w_x(0,t) = 0, \ w(h(t),t) = 0, \ t > 0, \\ h'(t) = -\mu w_x(h(t),t), \ t > 0, \\ h(0) = h_0, \ w(x,0) = w_0(x), \ 0 < x < h_0, \end{cases}$$

where $h_0 > 0$, $w_0 \in C^2[0,h_0]$ and $w_0(x) > 0 = w_0'(0) = 0 = w_0(h_0)$ for $x \in [0,h_0)$. Then the following hold:

(i) (Spreading-vanishing dichotomy) Either

$$\lim_{t\to\infty} h(t) = \infty, \quad \lim_{t\to\infty} w(x,t) = \frac{a}{b}$$

uniformly in any bounded subset of $[0,\infty)$ or

$$\lim_{t\to\infty} h(t) \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}, \quad \lim_{t\to\infty} \|w(\cdot,t)\|_{C[0,h(t)]} = 0.$$

(ii) When $\lim_{t\to\infty} h(t) = \infty$, $h(t)/t \to c_0(a,b,d,\mu)$ as $t \to \infty$ and

$$\lim_{t\to\infty} \sup_{x\in[0,h(t)]} |w(x,t) - U_{c_0}(h(t)-x)| = 0,$$

where c_0 and U_{c_0} are defined in (2.2).

Moreover, for the proof of Theorem 2.5, we need the following lemma.

Lemma 3.1. *Suppose that* $s_1(t) < s_2(t)$ *for* $t \in [0,\tau_1]$ *and* $\eta(t) := [s_1(t)+s_2(t)]/2$. *Then* $v_x(x,t) < 0$ *for all* $x \in [\eta(t), s_2(t)]$ *and for all* $t \in (0,\tau_1]$ *as long as* $(v_0)'(x) \le 0$ *for all* $x \in [s_1^0, s_2^0]$.

This lemma, the monotonicity of the profile $v(\cdot,t)$ near its free boundary, can be proved by applying the reflection argument. From this, we also construct a suitable sub-solution for v to finish the proof of Theorem 2.5.

4. Discussion

In this work, we consider a free boundary problem for a two species competition system in which a superior and a inferior species have different spreading fronts. Our main goal is to investigate its dynamics. However, there are difficulties due to the possible intersections of two free boundaries. Surprisingly, unlike the case of Cauchy problem, the superior species is not always the winner.

If the territory size of the superior cannot cross some critical value, it can lose the competition, while if its territory is above this critical value, then spreading occurs. Interestingly, when spreading of the superior competitor occurs, our model shows the weaker species does not necessarily die out eventually. In fact, if the superior competitor spreads too slow to catch up with the inferior competitor, it may leave enough space for the inferior competitor to establish persistent population.

Finally, we mention two open questions as follows. If one species vanishes eventually, then it can be reduced to the one species case and the spreading speed can be understood as in the work of [5]. However, if both species spread successfully, it would be interesting to determine the spreading speed. Another open question is the case of higher dimension habitat. Note that our method works well for general non-symmetric case in 1d habitat and radial symmetric case for higher space dimension. However, for the general higher dimension case, the Stefan condition (1.5) for species u can be replaced by

$$\Phi_t = \mu \nabla_x u \cdot \nabla_x \Phi$$

if the free boundary is represented by

$$\Gamma(t) = \{x \in \mathbf{R}^N : \Phi(x,t) = 0\}$$

for some suitable function Φ. This case is still open for 2 species case.

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