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Norm computation and analytic continuation of vector-valued holomorphic discrete series representations

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Abstract

The holomorphic discrete series representations is realized on the space of vector-valued holomorphic functions on the complex bounded symmetric domains. When the parameter is sufficiently large, then its norm is given by the converging integral, but when the parameter becomes small, then the integral does not converge. However, if once we compute the norm explicitly, then we can consider its analytic continuation, and can discuss its properties, such as unitarizability. In this article we treat the results on explicit norm computation.

1 Introduction: Holomorphic discrete series of $SU(1,1)$

Let $D:=\{w\in\mathbb{C}:|w|<1\}$, $G:=SU(1,1)$, and $\lambda\in\mathbb{C}$. Then the universal covering group $\tilde{G}$ acts on $\mathcal{O}(D)$ by

$$\tau_\lambda\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1}\right)f(w):=(cw+d)^{-\lambda}f\left(\frac{aw+b}{cw+d}\right).$$

This action preserves the sesquilinear form

$$\langle f,h\rangle_\lambda:=\frac{\lambda-1}{\pi}\int_D f(w)\overline{h(w)}(1-|w|^2)^{\lambda-2}dw.$$

If $\operatorname{Re}\lambda>1$, then for any polynomial f,h, we have $|\langle f,h\rangle_\lambda|<\infty$. Thus τ_λ is a unitary representation of $\tilde{G}$ if $\lambda>1$. This is called the holomorphic discrete series representation. On the other hand, if $\operatorname{Re}\lambda\le 1$, then $\langle f,h\rangle_\lambda$ does not converge if $f,h\not\equiv 0$. However, when $\operatorname{Re}\lambda>1$ and $f=\sum_{m=0}^\infty a_mw^m$, we can compute the norm explicitly as

$$\|f\|_\lambda^2=\sum_{m=0}^\infty\frac{m!}{(\lambda)_m}|a_m|^2 \qquad \text{where} \qquad (\lambda)_m:=\lambda(\lambda+1)\cdots(\lambda+m-1).$$

This expression is available even when $\operatorname{Re}\lambda\le 1$, and is positive definite for $\lambda>0$. That is, τ_λ defines a unitary representation of $\tilde{G}$ when $\lambda>0$. This example shows that if once the norm is explicitly computed, we can treat the analytic continuation of the holomorphic discrete series representation.

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2 Holomorphic discrete series of general Hermitian Lie group

From now on, we let G be a general simple Lie group, and $K \subset G$ be its maximal compact subgroup. We denote the Cartan involution of G corresponding to K by ϑ, and extend anti-holomorphically on $G^{\mathbb{C}}$. We assume that K has a non-discrete center. In this case, (G, K) is called of Hermitian type. Also we assume that G has a complexification $G^{\mathbb{C}}$. We denote the corresponding Lie algebras of G, K, $G^{\mathbb{C}}$ by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{g}^{\mathbb{C}}$. Then we can take an element $z \in \mathfrak{z}(\mathfrak{k})$ (the center of $\mathfrak{k}$) such that the eigenvalues of $ad(z)$ are $+\sqrt{-1}$, 0, $-\sqrt{-1}$. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-$ be the corresponding eigenspace decomposition. Then there exists a domain $D \subset \mathfrak{p}^+$ which is diffeomorphic to G/K via the following diagram.

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, dashed, "\wr"'] & G^{\mathbb{C}}/K^{\mathbb{C}}P^- \\ D \arrow[r, hook] & \mathfrak{p}^+ \arrow[u, "\exp"'] \end{tikzcd}$$

Let (τ, V) be a holomorphic representation of $K^{\mathbb{C}}$, and χ be a suitable character of $\tilde{K}^{\mathbb{C}}$, the universal covering group of $K^{\mathbb{C}}$. Then the space of holomorphic sections of the vector bundle on G/K with fiber $V \otimes \chi^{-\lambda}$ is isomorphic to the space of V-valued holomorphic functions on D.

$$\Gamma_{\mathcal{O}}(G/K, \tilde{G} \times_{\tilde{K}} (V \otimes \chi^{-\lambda})) \simeq \mathcal{O}(D, V).$$

Via this identification, the universal covering group $\tilde{G}$ acts on $\mathcal{O}(D, V)$ by the form

$$\tau_\lambda(g)f(w) = \chi(\kappa(g^{-1}, w))^\lambda \tau(\mu(g^{-1}, w))^{-1} f(g^{-1}w)$$

($g \in G$, $w \in D$), using some smooth map $\kappa : \tilde{G} \times D \to \tilde{K}^{\mathbb{C}}$. This action preserves the sesquilinear form

$$\langle f, g\rangle_{\lambda,\tau} := \frac{c_\lambda}{\pi^n} \int_D \left(\tau(B(w)^{-1})f(w), g(w)\right)_\tau \chi(B(w))^{\lambda-p} dw$$

($f, g \in \mathcal{O}(D, V)$), where $n = \dim \mathfrak{p}^+$, p is an integer determined from $\mathfrak{g}$ which we will define later, and $B : \mathfrak{p}^+ \supset D \to \tilde{K}^{\mathbb{C}}$ is some smooth map. Also we determine the constant c_λ so that $\|v\|_{\lambda,\tau} = |v|_\tau$ holds for any constant function v. Then this norm converges for any nonzero polynomial if $\operatorname{Re}\lambda$ is sufficiently large.

Example 2.1. *Let*

$$G = \left\{ g \in GL(2r, \mathbb{C}) : g \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \ g \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \bar{g} \right\},$$

which is isomorphic to $Sp(r, \mathbb{R})$. Then G/K is diffeomorphic to

$$D = \{w \in \mathrm{Sym}(r, \mathbb{C}) : I - ww^* \text{ is positive definite.}\}.$$

Let (τ, V) be a representation of $K^{\mathbb{C}} = GL(r, \mathbb{C})$. Then $\tilde{G}$ acts on $\mathcal{O}(D, V)$ by

$$\tau_\lambda\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}\right) f(w) := \det(Cw + D)^{-\lambda} \tau\left({}^t(Cw + D)\right) f\left((Aw + B)(Cw + D)^{-1}\right).$$

This preserves the sesquilinear form

$$\langle f, g\rangle_{\lambda,\tau} := \frac{c_\lambda}{\pi^{r(r+1)/2}} \int_D \left(\tau((I - ww^*)^{-1})f(w), g(w)\right)_\tau \det(I - ww^*)^{\lambda-(r+1)} dw.$$

We return to the general case. Our goal is to compute the $\tilde{G}$-invariant inner product $\langle\cdot,\cdot\rangle_{\lambda,\tau}$. In order to achieve this, we want to compare this inner product with another fixed inner product on each K-type, instead of using Taylor expansion. So we define another inner product on $\mathcal{O}(\mathfrak{p}^+,V)$.

$$\langle f,g\rangle_{F,\tau} := \frac{1}{\pi^n}\int_{\mathfrak{p}^+}(f(w),g(w))_\tau e^{-|w|^2}dw \qquad (f,g\in\mathcal{O}(\mathfrak{p}^+,V)),$$

where $|w|$ is a suitable K-invariant norm on $\mathfrak{p}^+$. Let

$$\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+,V) = \bigoplus_i W_i$$

be an irreducible decomposition under K such that each subspace is orthogonal to other subspaces with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$. Then since $\|\cdot\|^2_{\lambda,\tau}$ and $\|\cdot\|^2_{F,\tau}$ are both K-invariant, the ratio of two norms are constant on W_i. We denote this ratio by $R_i(\lambda)$. Moreover, if we assume that "$W_i\perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$ implies $W_i\perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{\lambda,\tau}$" (for example, if $\mathcal{P}(\mathfrak{p}^+,V)$ is K-multiplicity free), then we have

$$\|f\|^2_{\lambda,\tau} = \sum_i R_i(\lambda)\|f_i\|^2_{F,\tau} \qquad (f\in\mathcal{O}(D,V))$$

where f_i is the orthogonal projection of f onto W_i. Accordingly, the reproducing kernel is expanded as

$$K_{\lambda,\tau}(z,w) = \sum_i R_i(\lambda)^{-1}K_i(z,w) \in \mathcal{O}(D\times\overline{D},\mathrm{End}(V))$$

where $K_i(z,w)$ is the reproducing kernel of W_i with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$. Then $R_i(\lambda)$, initially defined when $\operatorname{Re}\lambda$ is sufficiently large, is meromorphically continued on $\lambda\in\mathbb{C}$. Moreover, there exists a unitary subrepresentation $\mathcal{H}_\lambda(D,V)\subset\mathcal{O}(D,V)$ if and only if $R_i(\lambda)^{-1}\geq 0$ holds for all i. In this case, the underlying $(\mathfrak{g},K)$-module is given by

$$\mathcal{H}_\lambda(D,V)_K = \bigoplus_{i:R_i(\lambda)^{-1}\neq 0} W_i.$$

As mentioned above, this argument is available only if "$W_i\perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{F,\tau}$ implies $W_i\perp W_j$ with respect to $\langle\cdot,\cdot\rangle_{\lambda,\tau}$" holds (e.g., if $\mathcal{P}(\mathfrak{p}^+,V)$ is K-multiplicity free). Therefore the goal of this talk is to calculate this ratio $R_i(\lambda)$ for the cases in the following table.

G	K	V
$Sp(r,\mathbb{R})$	$U(r)$	$\bigwedge^k(\mathbb{C}^r)^\vee \quad (0\leq k\leq r-1)$
$SU(q,s)$	$S(U(q)\times U(s))$	$\mathbb{C}\boxtimes V' \quad (V'$: any irrep of $U(s))$
$SO^*(2s)$	$U(s)$	$S^k(\mathbb{C}^s)^\vee$ $S^k(\mathbb{C}^s)\otimes\det^{-k/2} \quad (k\in\mathbb{N})$
$Spin_0(2,n)$	$(Spin(2)\times Spin(n))/\mathbb{Z}_2$	$\mathbb{C}_{-k}\boxtimes V_{(k,\dots,k,\pm k)} \quad (k\in\frac{1}{2}\mathbb{N}) \quad (n:\text{ even})$ $\mathbb{C}_{-k}\boxtimes V_{(k,\dots,k)} \quad (k\in\{0,\frac{1}{2}\}) \quad (n:\text{ odd})$
$(E_{6(-14)})$	$(SO(2)\times Spin(10))$	$(\mathbb{C}_{-k/2}\boxtimes\mathcal{H}^k(\mathbb{R}^{10}) \quad (k\in\mathbb{N}))$
$(E_{7(-25)})$	$(SO(2)\times E_6)$	$(\mathbb{C})$

Here, when $G=E_{6(-14)}$, we only state the conjecture later, and when $G=E_{7(-25)}$, this assumption holds only when scalar type case, and in this case the norm is already computed by Faraut-Korányi [6].

Remark 2.2. *(1) The question of when the analytic continuation of the holomorphic discrete series representation is unitarizable is studied by e.g. Berezin [1], Clerc [2], Vergne-Rossi [22], and Wallach [23], and completely classified by Enright-Howe-Wallach [3] and Jakobsen [12] by different methods.*

(2) The results on norm computation are already proved for several settings.

- *B. Ørsted (1980) [16] for $G = SU(r,r)$, scalar type.*
- *J. Faraut and A. Korányi (1990) [5] for G any Hermitian Lie group, scalar type.*
- *B. Ørsted and G. Zhang (1994, 1995) [17, 18] for $G = Sp(r,\mathbb{R})$, $V = (\mathbb{C}^r)^\vee$, $G = SU(r,r)$, $V = \mathbb{C}\boxtimes\mathbb{C}^r$, $G = SO^*(4r)$, $V = (\mathbb{C}^{2r})^\vee$.*
- *S. Hwang, Y. Liu and G. Zhang (2004) [10] for $G = SU(n,1)$, $V = \bigwedge^p(\mathbb{C}^n)^\vee \boxtimes \mathbb{C}$, $\bigwedge^q \mathbb{C}^n \boxtimes \mathbb{C}$.*

3 Main results

First we state the theorem on the norm computation for $Sp(r,\mathbb{R})$.

Theorem 3.1. *When $(G,K,V) = (Sp(r,\mathbb{R}), U(r), \bigwedge^k(\mathbb{C}^r)^\vee)$ $(0 \le k \le r-1)$, $\|\cdot\|^2_{\lambda,\tau}$ converges if $\operatorname{Re}\lambda > r$, the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by*

$$\mathcal{P}(\mathfrak{p}^+, V) = \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r \\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\ |\mathbf{k}|=k \\ m_j+k_j\le m_{j-1}}} V^\vee_{(2m_1+k_1, 2m_2+k_2,\dots,2m_r+k_r)},$$

and for $f \in V^\vee_{(2m_1+k_1,\dots,2m_r+k_r)}$, the ratio of norms is given by

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}} &= \frac{\prod_{j=1}^k\left(\lambda-\frac{1}{2}(j-1)\right)}{\prod_{j=1}^r\left(\lambda-\frac{1}{2}(j-1)\right)_{m_j+k_j}} \\ &= \frac{1}{\prod_{j=1}^k\left(\lambda-\frac{1}{2}(j-1)+1\right)_{m_j+k_j-1}\prod_{j=k+1}^r\left(\lambda-\frac{1}{2}(j-1)\right)_{m_j+k_j}}.\end{aligned}$$

From this result we can determine when the analytic continuation of the holomorphic discrete series representation becomes unitarizable.

Corollary 3.2. *When $(G,K,V) = (Sp(r,\mathbb{R}), U(r), \bigwedge^k(\mathbb{C}^r)^\vee)$ $(0 \le k \le r-1)$, $(\tau_\lambda, \mathcal{O}(D,V))$, originally unitarizable if $\lambda > r$, has a unitary subrepresentation $\mathcal{H}_\lambda(D,V) \subset \mathcal{O}(D,V)$ if and only if*

$$\lambda \in \left\{\frac{k}{2}, \frac{k+1}{2}, \dots, \frac{r-1}{2}\right\} \cup \left(\frac{r-1}{2}, \infty\right),$$

and when $\lambda = l/2$ $(l = k,\dots,r-1)$, the underlying $(\mathfrak{g},\tilde{K})$-module is given by

$$\mathcal{H}_\lambda(D,V) = \bigoplus_{\mathbf{m},\mathbf{k}:\ m_{k+1}+k_{k+1}=\cdots=m_r+k_r=0} V^\vee_{(2m_1+k_1,2m_2+k_2,\dots,2m_r+k_r)}.$$

Proof. This is because the reproducing kernel is given by

$$\det(I_r - zw^*)^{-\lambda}\tau(I_r - zw^*) = \sum_{\mathbf{m},\mathbf{k}} \frac{\prod_{j=1}^r\left(\lambda-\frac{1}{2}(j-1)\right)_{m_j+k_j}}{\prod_{j=1}^k\left(\lambda-\frac{1}{2}(j-1)\right)} K_{\mathbf{m},\mathbf{k}}(z,w),$$

and is positive definite if and only if λ is as above. □

For other classical groups, similar results also holds.

Theorem 3.3. *When* $(G,K,V)=(U(q,s),U(q)\times U(s),\mathbb{C}\boxtimes V_{\mathbf{k}}^{(s)})$ ($\mathbf{k}\in\mathbb{N}^r$, $k_1\ge\cdots\ge k_s\ge 0$), $\|\cdot\|_{\lambda,\tau}^2$ *converges if* $\operatorname{Re}\lambda+k_s>q+s-1$, *the* K*-type decomposition of* $\mathcal{O}(D,V)_K$ *is given by*

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+,V)&=\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^{\min\{q,s\}}\\ m_1\ge\cdots\ge m_{\min\{q,s\}}\ge 0}} V_{\mathbf{m}}^{(q)\vee}\boxtimes(V_{\mathbf{m}}^{(s)}\otimes V_{\mathbf{k}}^{(s)})\\ &=\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^{\min\{q,s\}}\\ m_1\ge\cdots\ge m_{\min\{q,s\}}\ge 0}}\ \bigoplus_{\substack{\mathbf{n}\in\mathbb{N}^r\\ n_1\ge\cdots\ge n_r\ge 0}} c_{\mathbf{k},\mathbf{m}}^{\mathbf{n}} V_{\mathbf{m}}^{(q)\vee}\boxtimes V_{\mathbf{n}}^{(s)},\end{aligned}$$

and for $f\in V_{\mathbf{m}}^{(q)\vee}\boxtimes V_{\mathbf{n}}^{(s)}$, *the ratio of norms is given by*

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2}=\frac{\prod_{j=1}^s(\lambda-(j-1))_{k_j}}{\prod_{j=1}^s(\lambda-(j-1))_{n_j}}=\frac{1}{\prod_{j=1}^s(\lambda-(j-1)+k_j)_{n_j-k_j}}.$$

Theorem 3.4. *When* $(G,K,V)=(SO^*(2s),U(s),S^k(\mathbb{C}^s)^\vee)$ ($k\in\mathbb{N}$), $\|\cdot\|_{\lambda,\tau}^2$ *converges if* $\operatorname{Re}\lambda>2s-3$, *the* K*-type decomposition of* $\mathcal{O}(D,V)_K$ *is given by*

$$\mathcal{P}(\mathfrak{p}^+,V)=\begin{cases}\displaystyle\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}}\ \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^r,\ |\mathbf{k}|=k\\ m_j+k_j\le m_{j-1}}} V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r)} & (s=2r),\\ \displaystyle\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}}\ \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^{r+1},\ |\mathbf{k}|=k\\ m_j+k_j\le m_{j-1}}} V^\vee_{(m_1+k_1,m_1,m_2+k_2,m_2,\dots,m_r+k_r,m_r,k_{r+1})} & (s=2r+1),\end{cases}$$

and for $f\in V^\vee_{(m_1+k_1,m_1,\dots,m_r+k_r,m_r,(k_{r+1}))}$, *the ratio of norms is given by*

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2}=\begin{cases}\dfrac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}} & (s=2r),\\[2ex] \dfrac{(\lambda)_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j+k_j}(\lambda-2r)_{k_{r+1}}} & (s=2r+1).\end{cases}$$

Theorem 3.5. *When* $(G,K,V)=(SO^*(2s),U(s),S^k(\mathbb{C}^s)\otimes\det^{-k/2})$ ($k\in\mathbb{N}$), $\|\cdot\|_{\lambda,\tau}^2$ *converges if* $\operatorname{Re}\lambda>2s-3$, *the* K*-type decomposition of* $\mathcal{O}(D,V)_K$ *is given by*

$$\mathcal{P}(\mathfrak{p}^+,V)=\begin{cases}\displaystyle\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}}\ \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^r,\ |\mathbf{k}|=k\\ m_j-k_j\ge m_{j+1}}} V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r)+(\frac{k}{2},\dots,\frac{k}{2})} & (s=2r),\\ \displaystyle\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}}\ \bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^{r+1},\ |\mathbf{k}|=k\\ m_j-k_j\ge m_{j+1}}} V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\dots,m_r,m_r-k_r,-k_{r+1})+(\frac{k}{2},\dots,\frac{k}{2})} & (s=2r+1),\end{cases}$$

and for $f\in V^\vee_{(m_1,m_1-k_1,\dots,m_r,m_r-k_r,(-k_{r+1}))+(\frac{k}{2},\dots,\frac{k}{2})}$, *the ratio of norms is given by*

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2}=\begin{cases}\dfrac{\prod_{j=1}^{r-1}(\lambda-2(j-1))_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j-k_j+k}} & (s=2r),\\[2ex] \dfrac{\prod_{j=1}^{r}(\lambda-2(j-1))_k}{\prod_{j=1}^r(\lambda-2(j-1))_{m_j-k_j+k}(\lambda-2r+1)_{k-k_{r+1}}} & (s=2r+1).\end{cases}$$

Theorem 3.6. *When* $(G,K)=(Spin_0(2,n),(Spin(2)\times Spin(n))/\mathbb{Z}_2)$ *and*

$$V=\begin{cases}\mathbb{C}_{-k}\boxtimes V_{(k,\ldots,k,\pm k)} & \left(k\in\frac{1}{2}\mathbb{Z}_{\geq 0}\right)\quad (n:\textit{even}),\\ \mathbb{C}_{-k}\boxtimes V_{(k,\ldots,k)} & \left(k=0,\frac{1}{2}\right)\quad (n:\textit{odd}),\end{cases}$$

$\|\cdot\|_{\lambda,\tau}^2$ *converges if* $\operatorname{Re}\lambda>n-1$, *the* K*-type decomposition of* $\mathcal{O}(D,V)_K$ *is given by*

$$\mathcal{P}(\mathfrak{p}^+)\otimes V=\bigoplus_{\mathbf{m}\in\mathbb{Z}_{++}^2}\ \bigoplus_{\substack{-k\leq l\leq k\\ m_1-m_2+l\geq k}}\mathbb{C}_{-(m_1+m_2+k)}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,\pm l\ (|l|\ \mathit{resp.}))},$$

and for $\mathbb{C}_{m_1+m_2+k}\boxtimes V_{(m_1-m_2+l,k,\ldots,k,\pm l\ (|l|\ \mathit{resp.}))}$, *the ratio of norms is given by*

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2}=\frac{(\lambda)_{2k}}{(\lambda)_{m_1+k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}=\frac{1}{(\lambda+2k)_{m_1-k+l}\left(\lambda-\frac{n-2}{2}\right)_{m_2+k-l}}.$$

From these results, we can also determine when they are unitarizable, but we omit the detail.

4 Proof of main results

4.1 Preliminaries

Before starting the proof, we prepare some more notations. Let G be a Hermitian simple Lie group, with $\operatorname{rank}_{\mathbb{R}} G=r$. We denote its complexified Lie algebra by $\mathfrak{g}^{\mathbb{C}}=\mathfrak{p}^+\oplus\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{p}^-$ as before. We take a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}\subset\mathfrak{k}^{\mathbb{C}}$. Then it automatically becomes a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $\Delta=\Delta(\mathfrak{g}^{\mathbb{C}},\mathfrak{h}^{\mathbb{C}})$ be the root system, and decompose this into a union of subsets $\Delta=\Delta_{\mathfrak{p}^+}\cup\Delta_{\mathfrak{k}^{\mathbb{C}}}\cup\Delta_{\mathfrak{p}^-}$ in the obvious way. We take a suitable maximal set of mutually strongly orthogonal roots $\{\gamma_1,\ldots,\gamma_r\}\subset\Delta_{\mathfrak{p}^+}$, and fix $e_j\in\mathfrak{p}^+_{\gamma_j}$ such that $-[[e_j,\vartheta e_j],e_j]=2e_j$ holds for each j. We define

$$h_j:=-[e_j,\vartheta e_j]\in\mathfrak{h}^{\mathbb{C}},\qquad \mathfrak{a}_{\mathfrak{l}}:=\bigoplus_{j=1}^r\mathbb{R}h_j\subset\mathfrak{h}^{\mathbb{C}},$$

$$e:=\sum_{j=1}^r e_j\in\mathfrak{p}^+,\qquad h:=\sum_{j=1}^r h_j=-[e,\vartheta e]\in\mathfrak{a}_{\mathfrak{l}}.$$

Then $ad(h)|_{\mathfrak{p}^+}$ has eigenvalues 2 and 1. We define

$$\mathfrak{p}^+_{\mathrm{T}}:=\{x\in\mathfrak{p}^+:[h,x]=2x\},\qquad \mathfrak{p}^-_{\mathrm{T}}:=\vartheta(\mathfrak{p}^+_{\mathrm{T}}),$$
$$\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}:=[\mathfrak{p}^+_{\mathrm{T}},\mathfrak{p}^-_{\mathrm{T}}],\qquad \mathfrak{k}_{\mathrm{T}}:=\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}\cap\mathfrak{k},\qquad \mathfrak{g}^{\mathbb{C}}_{\mathrm{T}}:=\mathfrak{p}^+_{\mathrm{T}}\oplus\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}\oplus\mathfrak{p}^-_{\mathrm{T}},\quad \mathfrak{g}_{\mathrm{T}}:=\mathfrak{g}^{\mathbb{C}}_{\mathrm{T}}\cap\mathfrak{g}.$$

Let $K^{\mathbb{C}}_{\mathrm{T}}$, K_{T}, G_{T} be the connected subgroups of $G^{\mathbb{C}}$ corresponding to $\mathfrak{k}^{\mathbb{C}}_{\mathrm{T}}$, $\mathfrak{k}_{\mathrm{T}}$, $\mathfrak{g}_{\mathrm{T}}$ respectively, and we define

$${}^cG_{\mathrm{T}}:=\operatorname{Int}(e^{\frac{\pi i}{4}(e-\vartheta e)})G_{\mathrm{T}},\qquad L:={}^cG_{\mathrm{T}}\cap K^{\mathbb{C}}_{\mathrm{T}},\qquad K_L:=L\cap K_{\mathrm{T}}.$$

These groups are related as follows.

$$\begin{tikzcd}
 & G^{\mathbb{C}} & & G_{\mathrm{T}}^{\mathbb{C}} \arrow[ll, hook'] & \\
G \arrow[ur, hook] & & G_{\mathrm{T}} \arrow[ll, hook'] \arrow[ur, hook] \arrow[rr, "\sim"] & & {}^{c}G_{\mathrm{T}} \arrow[ul, hook'] \\
 & K^{\mathbb{C}} \arrow[uu] & & K_{\mathrm{T}}^{\mathbb{C}} \arrow[ll, hook'] \arrow[uu, hook] & \\
K \arrow[uu] \arrow[ur, hook] & & K_{\mathrm{T}} \arrow[ll, hook'] \arrow[uu] \arrow[ur, hook] & & L \arrow[uu] \arrow[ul, hook'] \\
 & & & K_L \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Also we define the integers

$$d := \dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}}_{\frac{1}{2}(\gamma_1+\gamma_2)|_{\mathfrak{a}_{\mathfrak{l}}}}, \qquad b := \frac{1}{2}\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}}_{\frac{1}{2}\gamma_1|_{\mathfrak{a}_{\mathfrak{l}}}}, \qquad p := 2 + (r-1)d + b.$$

Then $\dim_{\mathbb{C}} \mathfrak{p}^+$ is equal to $n := r + \frac{1}{2}r(r-1)d + br$. These Lie algebras and integers are given as follows.

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{p}^+$	$\mathfrak{g}_{\mathrm{T}}$	$\mathfrak{l}$	$\mathfrak{k}_{\mathfrak{l}}$	$\mathfrak{p}^+_{\mathrm{T}} \cap {}^{c}\mathfrak{g}_{\mathrm{T}}$
$\mathfrak{sp}(r,\mathbb{R})$	$\mathfrak{u}(r)$	$\mathrm{Sym}(r,\mathbb{C})$	$\mathfrak{sp}(r,\mathbb{R})$	$\mathfrak{gl}(r,\mathbb{R})$	$\mathfrak{o}(r)$	$\mathrm{Sym}(r,\mathbb{R})$
$\mathfrak{su}(q,s)$ $(q \geq s)$	$\mathfrak{s}(\mathfrak{u}(q)\oplus\mathfrak{u}(s))$	$M(q,s;\mathbb{C})$	$\mathfrak{su}(s,s)$	$\mathfrak{gl}(s,\mathbb{C})/\mathfrak{u}(1)$	$\mathfrak{su}(s)$	$\mathrm{Herm}(s,\mathbb{C})$
$\mathfrak{so}^*(2s)$	$\mathfrak{u}(s)$	$\mathrm{Skew}(s,\mathbb{C})$	$\mathfrak{so}^*(4\lfloor\frac{s}{2}\rfloor)$	$\mathfrak{gl}(\lfloor\frac{s}{2}\rfloor,\mathbb{H})$	$\mathfrak{sp}(\lfloor\frac{s}{2}\rfloor)$	$\mathrm{Herm}(\lfloor\frac{s}{2}\rfloor,\mathbb{H})$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2)\oplus\mathfrak{so}(n)$	$\mathbb{C}^n$	$\mathfrak{so}(2,n)$	$\mathbb{R}\oplus\mathfrak{so}(1,n-1)$	$\mathfrak{so}(n-1)$	$\mathbb{R}^{1,n-1}$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(2)\oplus\mathfrak{so}(10)$	$M(2,1;\mathbb{O})_{\mathbb{C}}$	$\mathfrak{so}(2,8)$	$\mathbb{R}\oplus\mathfrak{so}(1,7)$	$\mathfrak{so}(7)$	$\mathbb{R}^{1,7}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(2)\oplus\mathfrak{e}_6$	$\mathrm{Herm}(3,\mathbb{O})_{\mathbb{C}}$	$\mathfrak{e}_{7(-25)}$	$\mathbb{R}\oplus\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	$\mathrm{Herm}(3,\mathbb{O})$

$\mathfrak{g}$	r	n	d	b	p
$\mathfrak{sp}(r,\mathbb{R})$	r	$\frac{1}{2}r(r+1)$	1	0	$r+1$
$\mathfrak{su}(q,s)$ $(q \geq s)$	s	qs	2	$q-s$	$q+s$
$\mathfrak{so}^*(2s)$	$\lfloor\frac{s}{2}\rfloor$	$\frac{1}{2}s(s-1)$	4	0 (s:even) / 2 (s:odd)	$2(s-1)$
$\mathfrak{so}(2,n)$	2	n	$n-2$	0	n
$\mathfrak{e}_{6(-14)}$	2	16	6	4	12
$\mathfrak{e}_{7(-25)}$	3	27	8	0	18

4.2 Proof for tube type case

In this subsection we deal with the following cases.

G	K	V
$Sp(r,\mathbb{R})$	$U(r)$	$\bigwedge^k(\mathbb{C}^r)^\vee \quad (0 \leq k \leq r-1)$
$SU(q,s)$ $(q \geq s)$	$S(U(q)\times U(s))$	$\mathbb{C}\boxtimes V'$ (V': any irrep of $U(s)$)
$SO^*(4r)$	$U(2r)$	$S^k(\mathbb{C}^{2r})^\vee$ $S^k(\mathbb{C}^{2r})\otimes\det^{-k/2} \quad (k\in\mathbb{N})$
$Spin_0(2,n)$	$(Spin(2)\times Spin(n))/\mathbb{Z}_2$	$\mathbb{C}_{-k}\boxtimes V_{(k,\ldots,k,\pm k)} \quad (k\in\frac{1}{2}\mathbb{N})$ (n : even) $\mathbb{C}_{-k}\boxtimes V_{(k,\ldots,k)} \quad (k\in\{0,\frac{1}{2}\})$ (n : odd)
$(E_{7(-25)})$	$(SO(2)\times E_6)$	$(\mathbb{C})$

These G, except for $SU(q,s)$ $(q>s)$, are of tube type, that is, $G = G_{\mathrm{T}}$ holds. Though $SU(q,s)$ $(q>s)$ is of non-tube type, the same proof is available. For these cases, each V remains irreducible even if restricted to $K_L = O(r)$, $SU(s)$, $Sp(r)$, $Pin(n-1)$ respectively, and this property is essentially used. For a $K_{\mathrm{T}}^{\mathbb{C}}$-module V, we denote by $\overline{V}$ the conjugate representation of $K_{\mathrm{T}}^{\mathbb{C}}$ with respect to the real form $L \subset K_{\mathrm{T}}^{\mathbb{C}}$. Then the following theorem holds.

Theorem 4.1. *Let (τ, V) be an irreducible representation of $K^{\mathbb{C}}$. Suppose (τ, V) has a restricted lowest weight $-\left(\frac{k_1}{2}\gamma_1 + \cdots + \frac{k_r}{2}\gamma_r\right)\Big|_{\mathfrak{a}_{\mathfrak{l}}}$. Let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be a $K^{\mathbb{C}}$-irreducible subspace. We assume*

(A1) $(\tau, V)|_{K_L}$ still remains irreducible.

(A2) All the K_L-spherical irreducible subspaces in $W|_{K_{\mathrm{T}}^{\mathbb{C}}} \otimes \overline{V|_{K_{\mathrm{T}}^{\mathbb{C}}}}$ have the same lowest weight $-(n_1\gamma_1 + \cdots + n_r\gamma_r)$.

Then the integral $\|f\|_{\lambda,\tau}^2$ converges for any $f \in W$ if $\mathrm{Re}(\lambda) + k_r > p-1$, and for any $f \in W$, we have

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2} = \frac{\prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1)\right)_{k_j}}{\prod_{j=1}^r \left(\lambda - \frac{d}{2}(j-1)\right)_{n_j}}.$$

Example 4.2. *We apply this theorem for $G = Sp(r,\mathbb{R})$. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{u}(r) \subset \mathfrak{sp}(r,\mathbb{R})$, and take a basis $\{\varepsilon_1, \ldots, \varepsilon_r\} \subset (\sqrt{-1}\mathfrak{h})^\vee$ such that $\Delta_+(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) = \Delta_{\mathfrak{k}^{\mathbb{C}},+} \cup \Delta_{\mathfrak{p}^+}$ is given by*

$$\Delta_{\mathfrak{k}^{\mathbb{C}},+} = \{\varepsilon_j - \varepsilon_k : 1 \le j < k \le r\},$$
$$\Delta_{\mathfrak{p}^+} = \{\varepsilon_j + \varepsilon_k : 1 \le j \le k \le r\}.$$

Then we have $\gamma_j = 2\varepsilon_j$, $\mathfrak{a}_{\mathfrak{l}} = \sqrt{-1}\mathfrak{h}$. For any $K^{\mathbb{C}} = GL(r,\mathbb{C})$-module V, its conjugate representation $\overline{V}$ with respect to the real form $L = GL(r,\mathbb{R})$ is isomorphic to the original V. For $\mathbf{m} \in \mathbb{Z}^r$ with $m_1 \ge \cdots \ge m_r$, we denote by $V_{\mathbf{m}}^\vee$ the irreducible $K^{\mathbb{C}} = GL(r,\mathbb{C})$-module with lowest weight $-m_1\varepsilon_1 - \ldots - m_r\varepsilon_r$.

Let $V := V^\vee_{(\underbrace{1,\ldots,1}_{k},0,\ldots,0)} = \bigwedge^k(\mathbb{C}^r)^\vee$. Then this remains irreducible when restrected to $K_L = O(r)$, that is, the assumption (A1) holds. The K-type decomposition of $\mathcal{O}(D,V)_K = \mathcal{P}(\mathfrak{p}^+, V)$ is given by

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+, V) = \mathcal{P}(\mathrm{Sym}(r,\mathbb{C}), \textstyle\bigwedge^k(\mathbb{C}^r)^\vee) &= \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} V_{2\mathbf{m}}^\vee \otimes V^\vee_{(\underbrace{1,\ldots,1}_{k},0,\ldots,0)}\\ &= \bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\cdots\ge m_r\ge 0}} \bigoplus_{\substack{\mathbf{k}\in\{0,1\}^r,\ |\mathbf{k}|=k\\ m_j+k_j\le m_{j-1}}} V_{2\mathbf{m}+\mathbf{k}}^\vee.\end{aligned}$$

For each K-type $V_{2\mathbf{m}+\mathbf{k}}^\vee$, the only K_L-spherical submodule in $V_{2\mathbf{m}+\mathbf{k}}^\vee \otimes \overline{V} \simeq V_{2\mathbf{m}+\mathbf{k}}^\vee \otimes V$ is $V_{2\mathbf{m}+2\mathbf{k}}^\vee$, because an irreducible $GL(r,\mathbb{C})$-module is $O(r)$-spherical if and only if each component of its lowest weight is even. That is, the assumption (A2) holds with $\mathbf{n} = \mathbf{m}+\mathbf{k}$. By the theorem, for $f \in V_{2\mathbf{m}+\mathbf{k}}^\vee$ we have

$$\frac{\|f\|_{\lambda,\tau}^2}{\|f\|_{F,\tau}^2} = \frac{\prod_{j=1}^k \left(\lambda - \frac{1}{2}(j-1)\right)}{\prod_{j=1}^r \left(\lambda - \frac{1}{2}(j-1)\right)_{m_j+k_j}}.$$

From now we prove the key theorem for $G = Sp(r, \mathbb{R})$ case. Let (τ, V) be an irreducible representation of $K^{\mathbb{C}} = GL(r, \mathbb{C})$, and let $W \subset \mathcal{P}(\mathfrak{p}^+, V)$ be an irreducible subrepresentation of $K^{\mathbb{C}}$. Assume

(1) V has the lowest weight $-\mathbf{k} = (-k_1, \dots, -k_r)$.

(2) $V|_{K_L} = V|_{O(r)}$ still remains irreducible.

(3) All the $K_L = O(r)$-spherical irreducible subspaces in $W \otimes \overline{V} \simeq W \otimes V$ have the same lowest weight $-2\mathbf{n} = (-2n_1, \dots, -2n_r)$.

Our aim is to compute, for $f \in W$,

$$R_W(\lambda) := \frac{\dfrac{c_\lambda}{\pi^n} \displaystyle\int_D \left(\tau((I - ww^*)^{-1}) f(w), f(w)\right)_\tau \det(I - ww^*)^{\lambda - (r+1)} dw}{\dfrac{1}{\pi^n} \displaystyle\int_{\mathfrak{p}^+} |f(w)|_\tau^2 e^{-\operatorname{tr}(ww^*)} dw},$$

where $\mathfrak{p}^+ := \mathrm{Sym}(r, \mathbb{C})$, $D := \{w \in \mathfrak{p}^+ : I_r - ww^* \text{ is positive definite.}\}$.

Let $K_W(z, w) \in \mathcal{P}(\mathfrak{p}^+ \times \overline{\mathfrak{p}^+}, \mathrm{End}(V))$ be the reproducing kernel of W. Then we have

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_D \mathrm{Tr}_V \left(\tau((I - ww^*)^{-1}) K_W(w, w)\right) \det(I - ww^*)^{\lambda - (r+1)} dw}{\displaystyle\int_{\mathfrak{p}^+} \mathrm{Tr}_V(K_W(w, w)) e^{-\operatorname{tr}(ww^*)} dw}.$$

Let $\Omega := \{x \in \mathrm{Sym}(r, \mathbb{R}) : x \text{ is positive definite.}\}$, and recall $K = U(r)$, $\mathfrak{p}^+ = \mathrm{Sym}(r, \mathbb{C})$. Then we can consider the polar coordinate $K \times \Omega \to \mathfrak{p}^+$, $(k, x) \mapsto kx^{1/2}\, {}^t k$. By the $K^{\mathbb{C}}$-covariance of $K_W(z, w)$, we have

$$\begin{aligned} K_W(kx^{1/2}\, {}^t k, kx^{1/2}\, {}^t k) &= \tau(k) K_W(x^{-1/4} x x^{-1/4}, x^{1/4} I x^{1/4}) \tau(k^{-1}) \\ &= \tau(k) \tau(x^{-\frac{1}{4}}) K_W(x, I) \tau(x^{\frac{1}{4}}) \tau(k^{-1}). \end{aligned}$$

Thus we have

$$\begin{gathered} \mathrm{Tr}_V(\tau((I - kxk^*)^{-1}) K_W(kx^{1/2}\, {}^t k, kx^{1/2}\, {}^t k)) = \mathrm{Tr}_V(\tau((I - x)^{-1}) K_W(x, I)), \\ \mathrm{Tr}_V(K_W(kx^{1/2}\, {}^t k, kx^{1/2}\, {}^t k)) = \mathrm{Tr}_V(K_W(x, I)), \end{gathered}$$

and hence we can show

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_{\Omega \cap (I - \Omega)} \mathrm{Tr}_V \left(\tau((I - x)^{-1}) K_W(x, I)\right) \det(I - x)^{\lambda - (r+1)} dx}{\displaystyle\int_\Omega \mathrm{Tr}_V(K_W(x, I)) e^{-\operatorname{tr}(x)} dx}.$$

Now we regard $K_W(x, I) \in \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))$ as a function of x. We define the action $\tilde{\tau}$ of $K^{\mathbb{C}}$ on $\mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))$ by

$$(\tilde{\tau}(k) F)(x) := \tau(k) F(k^{-1} x {}^t k^{-1}) \tau({}^t k) \quad (k \in K^{\mathbb{C}},\ F \in \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V)),\ x \in \mathfrak{p}^+).$$

Then we have the isomorphism

$$\mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V)) \simeq \mathcal{P}(\mathfrak{p}^+, V) \otimes \overline{V}.$$

$K_W(x, I)$ is $K_L = O(r)$-invariant under $\tilde{\tau}$, i.e., $K_W(\cdot, I) \in (W \otimes \overline{V})^{K_L}$. By the assumption, we have

$$K_W(\cdot, I) \in (W \otimes \overline{V})^{K_L} \simeq (V_{2\mathbf{n}}^\vee)^{K_L}.$$

Let $F(x) \in V_{2\mathbf{n}}^\vee \subset \mathcal{P}(\mathfrak{p}^+, \mathrm{End}(V))$ be the lowest weight vector. Then by averaging $F(x)$ on K_L, we get $K_W(\cdot, I)$, and thus we have

$$R_W(\lambda) = \frac{c_\lambda \displaystyle\int_{\Omega\cap(I-\Omega)} \mathrm{Tr}_V \left(\tau((I-x)^{-1})F(x)\right) \det(I-x)^{\lambda-(r+1)} dx}{\displaystyle\int_\Omega \mathrm{Tr}_V(F(x)) e^{-\mathrm{tr}(x)} dx}.$$

We define

$$B_W(\lambda) = \int_{\Omega\cap(I-\Omega)} \mathrm{Tr}_V \left(\tau((I-x)^{-1})F(x)\right) \det(I-x)^{\lambda-(r+1)} dx,$$

$$\Gamma_W = \int_\Omega \mathrm{Tr}_V(F(x)) e^{-\mathrm{tr}(x)} dx$$

so that $R_W(\lambda) = c_\lambda B_W(\lambda)/\Gamma_W$ holds. Also, we recall the generalized Gamma function which was introduced by Gindikin [7] (see also [6, Chapter VII]), which is defined as, for $\mathbf{s} \in \mathbb{C}^r$,

$$\Gamma_\Omega(\mathbf{s}) := \int_\Omega \Delta_{\mathbf{s}}(x) \det(x)^{-\frac{r+1}{2}} e^{-\mathrm{tr}(x)} dx$$

where

$$\Delta_{\mathbf{s}}(x) := \prod_{l=1}^{r-1} \det((x_{ij})_{1\le i,j\le l})^{s_l - s_{l+1}} \det(x)^{s_r}.$$

We want to show

$$B_W(\lambda) = \frac{\Gamma_\Omega\left(\lambda + \mathbf{k} - \frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})} \Gamma_W$$

where λ is the abbreviation of $(\lambda, \ldots, \lambda)$, so that

$$R_W(\lambda) = c_\lambda \frac{B_W(\lambda)}{\Gamma_W} = c_\lambda \frac{\Gamma_\Omega\left(\lambda + \mathbf{k} - \frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})}.$$

This is an analogue of the well-known formula

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt, \qquad \Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt.$$

In order to compute $B_W(\lambda)$, we compute

$$J := \int_{y\in\Omega} e^{-\mathrm{tr}(y)} \int_{x\in\Omega\cap(y-\Omega)} \mathrm{Tr}_V \left(\tau((y-x)^{-1})F(x)\right) \det(y-x)^{\lambda-(r+1)} dx dy$$

in two ways. First, by taking the lower triangular matrix b such that $y = b{}^tb$ and letting $x = bz{}^tb$, we get

$$\begin{aligned}
J =& \int_{y\in\Omega} e^{-\operatorname{tr}(y)} \int_{z\in\Omega\cap(I-\Omega)} \operatorname{Tr}_V \left(\tau((b(I-z){}^tb)^{-1})F(bz{}^tb)\right) \det(b(I-z){}^tb)^{\lambda-(r+1)} \\
&\qquad \times \det(b)^{r+1} dzdy \\
=& \int_{y\in\Omega} e^{-\operatorname{tr}(y)} \int_{z\in\Omega\cap(I-\Omega)} \operatorname{Tr}_V \left(\tau((I-z)^{-1})\tau(b^{-1})F(bz{}^tb)\tau({}^tb^{-1})\right) \det(I-z)^{\lambda-(r+1)} \\
&\qquad \times \det(b)^{2\lambda-(r+1)} dzdy \\
=& \int_{z\in\Omega\cap(I-\Omega)} \operatorname{Tr}_V \left(\tau((I-z)^{-1})F(z)\right) \det(I-z)^{\lambda-(r+1)} dz \\
&\qquad \times \int_{y\in\Omega} e^{-\operatorname{tr}(y)} \Delta_{2\mathbf{n}}(b) \det(b)^{2\lambda-(r+1)} dzdy \\
=& B_W(\lambda) \int_\Omega e^{-\operatorname{tr}(y)} \Delta_{\mathbf{n}}(y) \det(y)^{\lambda-\frac{r+1}{2}} dy = B_W(\lambda)\Gamma_\Omega(\lambda+\mathbf{n}).
\end{aligned}$$

Here we used

$$\Delta_{2\mathbf{n}}(b) = \Delta_{\mathbf{n}}(b{}^tb) = \Delta_{\mathbf{n}}(y).$$

Second, by putting $y - x =: z$, we get

$$J = \operatorname{Tr}_V \left(\int_\Omega e^{-\operatorname{tr}(z)}\tau(z^{-1}) \det(z)^{\lambda-(r+1)} dz \int_\Omega e^{-\operatorname{tr}(x)} F(x) dx \right).$$

Since V is irreducible under $K_L = O(r)$ by assumption, and the integral

$$\int_\Omega e^{-\operatorname{tr}(z)}\tau(z^{-1}) \det(z)^{\lambda-(r+1)} dz$$

commutes with $O(r)$-action, this is proportional to the identity map I_V. Moreover, for the lowest weight vector $v \in V$, by taking the lower triangular matrix b such that $z = b{}^tb$, we get

$$(\tau(z^{-1})v, v)_\tau = (\tau({}^tb^{-1}b^{-1})v, v)_\tau = |\tau(b^{-1})v|_\tau = \Delta_{\mathbf{k}}(b)^2|v|_\tau^2 = \Delta_{\mathbf{k}}(z)|v|_\tau^2$$

from the assumption that V has the lowest weight $-\mathbf{k}$, and hence

$$\begin{aligned}
\left(\int_\Omega e^{-\operatorname{tr}(z)}\tau(z^{-1}) \det(z)^{\lambda-(r+1)} dz v, v \right) =& \int_\Omega e^{-\operatorname{tr}(z)} \Delta_{\mathbf{k}}(z) \det(z)^{\lambda-(r+1)} dz |v|_\tau^2 \\
=& \Gamma_\Omega \left(\lambda + \mathbf{k} - \frac{r+1}{2} \right) |v|_\tau^2.
\end{aligned}$$

That is, we have

$$\int_\Omega e^{-\operatorname{tr}(z)}\tau(z^{-1}) \det(z)^{\lambda-(r+1)} dz = \Gamma_\Omega \left(\lambda + \mathbf{k} - \frac{r+1}{2} \right) I_V.$$

Thus we have

$$J = \Gamma_\Omega \left(\lambda + \mathbf{k} - \frac{r+1}{2} \right) \int_\Omega e^{-\operatorname{tr}(x)} \operatorname{Tr}_V(F(x)) dx = \Gamma_\Omega \left(\lambda + \mathbf{k} - \frac{r+1}{2} \right) \Gamma_W.$$

Comparing two expressions of J, we get the desired formula

$$B_W(\lambda) = \frac{\Gamma_\Omega\left(\lambda + \mathbf{k} - \frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})} \Gamma_W, \qquad R_W(\lambda) = c_\lambda \frac{B_W(\lambda)}{\Gamma_W} = c_\lambda \frac{\Gamma_\Omega\left(\lambda + \mathbf{k} - \frac{r+1}{2}\right)}{\Gamma_\Omega(\lambda+\mathbf{n})}.$$

By normalization assumption, the constant c_λ is determined as

$$c_\lambda = \frac{\Gamma_\Omega(\lambda + \mathbf{k})}{\Gamma_\Omega\left(\lambda + \mathbf{k} - \frac{r+1}{2}\right)},$$

and therefore

$$R_W(\lambda) = \frac{\Gamma_\Omega(\lambda + \mathbf{k})}{\Gamma_\Omega(\lambda + \mathbf{n})}.$$

The value of $\Gamma_\Omega(\mathbf{s})$ is well-known (see [6, Theorem VII.1.1]), and finally we get

$$R_W(\lambda) = \frac{\pi^{r(r-1)/4}\prod_{j=1}^r \Gamma\left(\lambda + k_j - \frac{j-1}{2}\right)}{\pi^{r(r-1)/4}\prod_{j=1}^r \Gamma\left(\lambda + n_j - \frac{j-1}{2}\right)} = \frac{\prod_{j=1}^r \left(\lambda - \frac{j-1}{2}\right)_{k_j}}{\prod_{j=1}^r \left(\lambda - \frac{j-1}{2}\right)_{n_j}},$$

and this completes the proof. □

4.3 Proof for non-tube type: Easy case

For following cases, we cannot apply the previous arguments.

G	K	V
$SU(q,s)$ $(q<s)$	$S(U(q)\times U(s))$	$\mathbb{C}\boxtimes V'$ (V': any irrep of $U(s)$)
$SO^*(4r+2)$	$U(2r+1)$	$S^k(\mathbb{C}^{2r+1})^\vee$ $(k\in\mathbb{N})$
$SO^*(4r+2)$	$U(2r+1)$	$S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2}$ $(k\in\mathbb{N})$

However, for $(G,V) = (SU(q,s), \mathbb{C}\boxtimes V')$ or $(SO^*(4r+2), S^k(\mathbb{C}^{2r+1})^\vee)$, we can easily compute the norm by using the embedding

$$\begin{aligned} U(p)\times U(q,s) &\hookrightarrow U(p+q,s), & V_\mathbf{k}^{(p)\vee}\boxtimes\mathcal{P}(M(q,s,\mathbb{C}),V_\mathbf{k}^{(s)}) &\hookrightarrow \mathcal{P}(M(p+q,s,\mathbb{C})),\\ SO^*(2s) &\hookrightarrow SO^*(2s+2), & \mathcal{P}(\mathrm{Skew}(s,\mathbb{C}),\mathcal{P}_k(\mathbb{C}^s)) &\hookrightarrow \mathcal{P}(\mathrm{Skew}(s+1,\mathbb{C})). \end{aligned}$$

On the other hand, for $(G,V) = (SO^*(4r+2), S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2})$, computing the norm is more difficult, and we postpone this case to the next subsection. In this section we deal with $G = U(q,s)$ case. We set

$$\begin{aligned} G &= U(q,s), & K &= U(q)\times U(s), & \mathfrak{p}^+ &= M(q,s;\mathbb{C}),\\ G' &= U(p)\times U(q,s), & K' &= U(p)\times U(q)\times U(s), & &\\ G'' &= U(p+q,s), & K'' &= U(p+q)\times U(s), & \mathfrak{p}^{+\prime\prime} &= M(p+q,s;\mathbb{C}). \end{aligned}$$

Then $G/K = G'/K'$, G''/K'' are diffeomorphic to some bounded domains $D\subset\mathfrak{p}^+$, $D''\subset\mathfrak{p}^{+\prime\prime}$ respectively. We set $V := \mathbb{C}\boxtimes V_\mathbf{k}^{(s)}$, and consider the representation $(\tau_{\lambda,\mathbf{k}}, \mathcal{O}(D,\mathbb{C}\boxtimes V_\mathbf{k}^{(s)}))$ of $\tilde{G}$. We assume p is greater than or equal to the leg length of $\mathbf{k}$. Then we can embed the representation $V_\mathbf{k}^{(p)\vee}\boxtimes V_\mathbf{k}^{(s)}$ of $U(p)\times U(s)$ into the polynomial space $\mathcal{P}(M(p,s,\mathbb{C}))$. Accordingly, we can embed the representation $V_\mathbf{k}^{(p)\vee}\boxtimes\mathcal{O}(D,\mathbb{C}\boxtimes V_\mathbf{k}^{(s)})$ of $\tilde{G}'$ into $\mathcal{O}(D\times M(p,s;\mathbb{C}))$. We denote this embedding by ι. Then under this embedding the action of $\tilde{G}'$ on $\iota\left(V_\mathbf{k}^{(p)\vee}\boxtimes\mathcal{O}(D,\mathbb{C}\boxtimes V_\mathbf{k}^{(s)})\right)\subset\mathcal{O}(D\times M(p,s;\mathbb{C}))$ is given by

$$\tau'_\lambda\left(u, \begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1}\right) f(w,z) := \det(cw+d)^{-\lambda} f\left((aw+d)(cw+d)^{-1}, u^{-1}z(cw+d)^{-1}\right).$$

We embed G' into G'' block diagonally, and identify $\mathfrak{p}^+ \oplus M(p,s;\mathbb{C}) = M(p,s;\mathbb{C}) \oplus M(q,s;\mathbb{C})$ with $\mathfrak{p}^{+\prime\prime} = M(p+q,s;\mathbb{C})$ in a standard way. Then we can show that the restriction of the scalar type holomorphic discrete series representation $(\tau''_\lambda, \mathcal{O}(D''))$ of $\tilde{G}''$ to $\tilde{G}'$ coincides with τ'_λ, and the embedding

$$\iota : V_{\mathbf{k}}^{(p)\vee} \boxtimes \mathcal{O}(\mathfrak{p}^+, \mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}) \to \mathcal{O}(\mathfrak{p}^{+\prime\prime})$$

preserves the norm $\|\cdot\|_F$. Now we consider the K'-type decomposition of $\mathcal{O}(D'')_{K''} = \mathcal{P}(\mathfrak{p}^{+\prime\prime})$ and $V_{\mathbf{k}}^{(p)\vee} \boxtimes \mathcal{O}(D, \mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)})_K = V_{\mathbf{k}}^{(p)\vee} \boxtimes (\mathcal{P}(\mathfrak{p}^+) \otimes (\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)}))$.

$$\begin{aligned} \mathcal{P}(\mathfrak{p}^{+\prime\prime})\big|_{K'} &= \bigoplus_{\mathbf{n}} V_{\mathbf{n}}^{(p+q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}\Big|_{K'} \\ &= \bigoplus_{\mathbf{n}} \bigoplus_{\mathbf{k}',\mathbf{m}} c_{\mathbf{k}',\mathbf{m}}^{\mathbf{n}} V_{\mathbf{k}'}^{(p)\vee} \boxtimes V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}, \\ V_{\mathbf{k}}^{(p)\vee} \boxtimes (\mathcal{P}(\mathfrak{p}^+) \otimes (\mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)})) &= \bigoplus_{\mathbf{m}} V_{\mathbf{k}}^{(p)\vee} \boxtimes V_{\mathbf{m}}^{(q)\vee} \boxtimes \left(V_{\mathbf{m}}^{(s)} \otimes V_{\mathbf{k}}^{(s)}\right) \\ &= \bigoplus_{\mathbf{m}} \bigoplus_{\mathbf{n}} c_{\mathbf{k},\mathbf{m}}^{\mathbf{n}} V_{\mathbf{k}}^{(p)\vee} \boxtimes V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}. \end{aligned}$$

Therefore we have

$$\iota(V_{\mathbf{k}}^{(p)\vee} \boxtimes V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(q)}) \subset V_{\mathbf{n}}^{(p+q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}.$$

Thus, for $f \in V_{\mathbf{k}}^{(p)\vee} \boxtimes V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$, using the result for the scalar type case, we have

$$\frac{\|\iota(f)\|_{\lambda;G''}^2}{\|f\|_{F;G'}^2} = \frac{\|\iota(f)\|_{\lambda;G''}^2}{\|\iota(f)\|_{F;G''}^2} = \frac{1}{\prod_{j=1}^s (\lambda - (j-1))_{n_j}}.$$

Since the norm of the tensor product representation of $V_{\mathbf{k}}^{(p)\vee}$ and $\mathcal{O}(D, \mathbb{C} \boxtimes V_{\mathbf{k}}^{(s)})$ is given by the product of each norm, and the norm is normalized so that $\|\cdot\|_{\lambda,\tau}$ and $\|\cdot\|_{F,\tau}$ coincide for constant functions, we get, for $f \in V_{\mathbf{m}}^{(q)\vee} \boxtimes V_{\mathbf{n}}^{(s)}$,

$$\frac{\|f\|_{\lambda,\tau;G}^2}{\|f\|_{F,\tau;G}^2} = \frac{\prod_{j=1}^s (\lambda - (j-1))_{k_j}}{\prod_{j=1}^s (\lambda - (j-1))_{n_j}},$$

and this proves the result on $G = U(q,s)$ case. The result for $(G,V) = (SO^*(4r+2), S^k(\mathbb{C}^{2r+1})^\vee)$ case is also proved similarly.

4.4 Proof for non-tube type: Difficult case

In this subsection we deal with the remaining case.

G	K	V
$SO^*(4r+2)$	$U(2r+1)$	$S^k(\mathbb{C}^{2r+1}) \otimes \det^{-k/2} \quad (k \in \mathbb{N})$

We compute the norm by combining

- The argument parallel to the proof for tube type cases.
- Embedding

$$\begin{aligned} &SU(2r,1) \hookrightarrow SO^*(4r+2), \\ &\mathcal{P}(\mathbb{C}^{2r}, \mathcal{P}(\mathrm{Skew}(2r,\mathbb{C})) \otimes (S^k(\mathbb{C}^{2r}) \otimes \det^{-k})) \\ &\hookrightarrow \mathcal{P}(\mathrm{Skew}(2r+1,\mathbb{C}), S^k(\mathbb{C}^{2r+1}) \otimes \det^{-k/2}). \end{aligned}$$

First we try to compute the norm by the argument parallel the tube type cases. The K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\begin{aligned}\mathcal{P}(\mathfrak{p}^+,V)&=\mathcal{P}(\mathrm{Skew}(2r+1,\mathbb{C}),S^k(\mathbb{C}^{2r+1})\otimes\det{}^{-k/2})\\&=\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\geq\cdots\geq m_r\geq 0}}\bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^{r+1},\ |\mathbf{k}|=k\\ m_j-k_j\geq m_{j+1}}}V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r,-k_{r+1})+(k/2,\ldots,k/2)}.\end{aligned}$$

We write $V^\vee_{(m_1,m_1-k_1,m_2,m_2-k_2,\ldots,m_r,m_r-k_r,-k_{r+1})+(k/2,\ldots,k/2)}=:V_{\mathbf{mk}}$ for short. Let $K_{\mathbf{mk}}(z,w)\in\mathcal{P}(\mathfrak{p}^+\times\overline{\mathfrak{p}^+},\mathrm{End}(V))$ be the reproducing kernel of $V_{\mathbf{mk}}$. Then for $f\in V_{\mathbf{mk}}$, $R_{\mathbf{mk}}(\lambda):=\|f\|^2_{\lambda,\tau}/\|f\|^2_{F,\tau}$ is equal to

$$R_{\mathbf{mk}}(\lambda)=\frac{c_\lambda\displaystyle\int_D\mathrm{Tr}_V\left(\tau((I-ww^*)^{-1})K_{\mathbf{mk}}(w,w)\right)\det(I-ww^*)^{\frac{1}{2}(\lambda-4r)}dw}{\displaystyle\int_{\mathfrak{p}^+}\mathrm{Tr}_V(K_{\mathbf{mk}}(w,w))e^{-\frac{1}{2}\mathrm{tr}(ww^*)}dw}.$$

Next, by using the $K=U(2r+1)$-invariance of $K_{\mathbf{mk}}(z,w)$, we can reduce the integral on $\mathfrak{p}^+=\mathrm{Sym}(2r+1,\mathbb{C})$ to the integral on $\mathfrak{p}^+_{\mathrm{T}}=\mathrm{Sym}(2r,\mathbb{C})$. Let $\mathrm{rest}:\mathcal{P}(\mathfrak{p}^+,V)\to\mathcal{P}(\mathfrak{p}^+_{\mathrm{T}},V)$ be the restriction map. Then we can show

$$\mathrm{rest}(V_{\mathbf{mk}})\subset\bigoplus_{l=k-k_{r+1}}^{k}\bigoplus_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l\\ k_j\leq l_j\leq m_{j+1}-m_j}}V^{\mathrm{T}\vee}_{(m_1,m_1-l_1,\ldots,m_r,m_r-l_r)+(k/2,\ldots,k/2)},$$

where $V^{\mathrm{T}\vee}_{\ldots}$ is the $K_{\mathrm{T}}=U(2r)$-module. Accordingly, there exist $\tilde{a}_{\mathbf{mkl}}\geq 0$ such that the restriction of the reproducing kernel is expanded as

$$K_{\mathbf{mk}}(z,w)|_{\mathfrak{p}^+_{\mathrm{T}}\times\overline{\mathfrak{p}^+_{\mathrm{T}}}}=\sum_{l=k-k_{r+1}}^{k}\sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l\\ k_j\leq l_j\leq m_{j+1}-m_j}}\tilde{a}_{\mathbf{mkl}}K^{\mathrm{T}}_{\mathbf{ml}}(z,w).$$

Accordingly, we can show that there exist $a_{\mathbf{mkl}}\geq 0$ such that the ratio of norm is given by (we omit the detail)

$$R_{\mathbf{mk}}(\lambda)=c_\lambda\frac{\sum_{l=k-k_{r+1}}^{k}\sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l\\ k_j\leq l_j\leq m_{j+1}-m_j}}a_{\mathbf{mkl}}R^{\mathrm{T}}_{\mathbf{ml}}(\lambda)}{\sum_{l=k-k_{r+1}}^{k}\sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l\\ k_j\leq l_j\leq m_{j+1}-m_j}}a_{\mathbf{mkl}}}$$

where

$$R^{\mathrm{T}}_{\mathbf{ml}}(\lambda):=\frac{\prod_{j=1}^{r-1}\Gamma(\lambda+k-(2r+2j-1))\Gamma(\lambda+k-|\mathbf{l}|-(4r-1))}{\prod_{j=1}^{r}\Gamma(\lambda+k+m_j-l_j-2(j-1))}.$$

By normalization assumption, we can show

$$\begin{aligned}c_\lambda^{-1}&=\frac{\sum_{l=0}^{k}(\dim S^l(\mathbb{C}^{2r}))R^{\mathrm{T}}_{\mathbf{0},(0,\ldots,0,l)}(\lambda)}{\dim S^k(\mathbb{C}^{2r+1})}\\&=\frac{1}{\prod_{j=1}^{r-1}(\lambda+k-(2r+2j-1))_{2r+1}(\lambda-4r+1)_{2r}(\lambda+k-2r+1)}.\end{aligned}$$

Substituting c_λ, we get

$$R_{\mathbf{mk}}(\lambda)=\frac{1}{\sum_{\mathbf{l}} a_{\mathbf{mkl}}}\sum_{l=k-k_{r+1}}^{k}\sum_{\substack{\mathbf{l}\in\mathbb{N}^r,\ |\mathbf{l}|=l\\ k_j\le l_j\le m_{j+1}-m_j}}\frac{a_{\mathbf{mkl}}(\lambda-4r+1)_{k-l}}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-l_j}(\lambda-2r+1)_k}.$$

It is difficult to know the values of $a_{\mathbf{mkl}}$, but at least we have proved

Lemma 4.3.

$$R_{\mathbf{mk}}(\lambda)=\frac{(\textit{monic polynomial of degree } k_{r+1})}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_k}.$$

Second, we consider the embedding of a smaller subgroup into $SO^*(4r+2)$. We set

$$G_{\mathrm{A}}:=SU(2r,1),\qquad K_{\mathrm{A}}:=S(U(2r)\times U(1)),\qquad \mathfrak{p}_{\mathrm{A}}^+:=\mathbb{C}^{2r}.$$

Also we set

$$\begin{aligned}V_{\mathrm{A}}:=&(S^k(\mathbb{C}^{2r})\otimes\det{}^{-k})\boxtimes\mathbb{C}\simeq(S^k(\mathbb{C}^{2r})\otimes\det{}^{-k/2})\boxtimes\mathbb{C}_{-k/2}\\ \subset V=&S^k(\mathbb{C}^{2r+1})\otimes\det{}^{-k/2},\end{aligned}$$

and consider the (non-irreducible) representation

$$\mathcal{O}(D_{\mathrm{A}},((\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V_{\mathrm{A}})$$

of $\tilde{G}_{\mathrm{A}}$, where $D_{\mathrm{A}}\subset\mathfrak{p}_{\mathrm{A}}^+$ is the unit circle, which is diffeomorphic to $G_{\mathrm{A}}/K_{\mathrm{A}}$. Then we can show that the embedding

$$\iota:\mathcal{O}(D_{\mathrm{A}},((\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V_{\mathrm{A}})\to\mathcal{O}(D,V)$$

which corresponds to the decomposition of the base space

$$\mathfrak{p}^+=\mathrm{Skew}(2r+1,\mathbb{C})=\mathbb{C}^{2r}\oplus\mathrm{Skew}(2r,\mathbb{C})=\mathfrak{p}_{\mathrm{A}}^+\oplus\mathrm{Skew}(2r,\mathbb{C})$$

intertwines the $\tilde{G}_{\mathrm{A}}$-action, and is an isometry with respect to $\|\cdot\|_{F,\tau}$. Next we define

$$\begin{aligned}F_{\mathbf{ml}}:=&V^{\mathrm{A}\vee}_{(m_1,m_1-l_1,m_2,m_2-l_2,\dots,m_r,m_r-l_r;0)+(k,\dots,k;0)}\\ \subset&V^{\mathrm{A}\vee}_{(m_1,m_1,m_2,m_2,\dots,m_r,m_r;0)}\otimes V^{\mathrm{A}\vee}_{(k,\dots,k,0;0)}\\ \subset&(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V_{\mathrm{A}},\end{aligned}$$

so that

$$(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V_{\mathrm{A}}=\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\dots\ge m_r\ge0}}\bigoplus_{\substack{\mathbf{l}\in\mathbb{Z}^r_{\ge0},\ |\mathbf{l}|=k\\ 0\le l_j\le m_j-m_{j+1}}}F_{\mathbf{ml}},$$

$$\mathcal{O}(D_{\mathrm{A}},(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V_{\mathrm{A}})=\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^r\\ m_1\ge\dots\ge m_r\ge0}}\bigoplus_{\substack{\mathbf{l}\in\mathbb{Z}^r_{\ge0},\ |\mathbf{l}|=k\\ 0\le l_j\le m_j-m_{j+1}}}\mathcal{O}(D_{\mathrm{A}},F_{\mathbf{ml}}),$$

and we also define

$$\begin{aligned}W_{\mathbf{mk}}:=&V^{\mathrm{A}\vee}_{(m_1-k_1,m_2,m_2-k_2,m_3,\dots,m_{r-1}-k_{r-1},m_r,m_r-k_r,-k_{r+1};m_1)+(k,\dots,k;0)}\\ \subset&V^{\mathrm{A}\vee}_{(m_1,m_2,m_2,m_3,\dots,m_{r-1},m_r,m_r,0;m_1)}\otimes V^{\mathrm{A}\vee}_{(k,\dots,k,0;0)}\\ \subset&V^{\mathrm{A}\vee}_{(m_1,0,\dots,0;m_1)}\otimes V^{\mathrm{A}\vee}_{(m_2,m_2,m_3,m_3,\dots,m_r,m_r,0,0;0)}\otimes V^{\mathrm{A}\vee}_{(k,\dots,k,0;0)}\\ \subset&\mathcal{P}(\mathbb{C}^{2r})\otimes(\mathcal{P}(\mathrm{Skew}(2r,\mathbb{C}))\boxtimes\mathbb{C})\otimes V^{\mathrm{A}\vee}_{(k,\dots,k,0;0)}.\end{aligned}$$

Then $V_{\mathbf{mk}}$, $F_{\mathbf{ml}}$ and $W_{\mathbf{mk}}$ are related as follows.

Lemma 4.4. *(1)* $\iota(W_{\mathbf{mk}}) \subset V_{\mathbf{mk}}$.

(2) $W_{\mathbf{mk}} \subset \bigoplus_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=k \\ l_j\leq k_{j+1},\ l_r\geq k_{r+1}}} \mathcal{O}(D_{\mathrm{A}}, F_{(m_2,\ldots,m_r,0),\mathbf{l}})$.

(3) $\iota(F_{\mathbf{ml}}) \subset \bigoplus_{\substack{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^{r+1},\ |\mathbf{n}|=k \\ n_j\leq l_j,\ n_{r+1}\geq l_r-m_r}} V_{\mathbf{mn}}$.

For the proof of this Lemma see [15, Lemma 5.7]. Using Lemma 4.3 and 4.4, we want to show

$$R_{\mathbf{mk}}(\lambda) = \frac{1}{\prod_{j=1}^r (\lambda+k-2(j-1))_{m_j-k_j} (\lambda-2r+1)_{k-k_{r+1}}}$$

by induction on $\min\{j : m_j = 0\}$. When $\mathbf{m} = \mathbf{0}$, i.e., for $R_{\mathbf{0},(0,\ldots,0,k)}$, this is clear by normalization assumption. So we assume this holds when $m_j = 0$, and prove when $m_{j+1} = 0$. By Lemma 4.4 (1), it suffices to compute $\|\iota(f)\|^2_{\lambda,\tau}/\|\iota(f)\|^2_{F,\tau}$ for $f \in W_{\mathbf{mk}}$.

By Lemma 4.4 (2), we can write

$$f = \sum_{\substack{\mathbf{l}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{l}|=k \\ l_j\leq k_{j+1},\ l_r\geq k_{r+1}}} f_{\mathbf{l}} \qquad (f_{\mathbf{l}} \in \mathcal{O}(D_A, F_{\mathbf{m}',\mathbf{l}}),\quad \mathbf{m}' = (m_2,\ldots,m_r,0).) \tag{4.1}$$

Let $v_{\mathbf{l}}$ be any non-zero element in the minimal K_{A}-type $F_{\mathbf{m}',\mathbf{l}}$. Then by the result on $SU(2r,1)$,

$$\frac{\|\iota(f_{\mathbf{l}})\|^2_{\lambda,\tau}}{\|\iota(f_{\mathbf{l}})\|^2_{F,\tau}} \Bigg/ \frac{\|\iota(v_{\mathbf{l}})\|^2_{\lambda,\tau}}{\|\iota(v_{\mathbf{l}})\|^2_{F,\tau}}$$

is computable. Moreover, By Lemma 4.4 (3), we can write

$$\iota(v_{\mathbf{l}}) = \sum_{\substack{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^r,\ |\mathbf{n}|=k \\ n_j\leq l_j,\ n_r\geq l_r}} v_{\mathbf{ln}} \qquad (v_{\mathbf{ln}} \in V_{\mathbf{m}',\mathbf{n}}), \tag{4.2}$$

and by the induction hypothesis, $\|v_{\mathbf{ln}}\|^2_{\lambda,\tau}/\|v_{\mathbf{ln}}\|^2_{F,\tau}$ is also computable. Also, by (4.1) and (4.2), there exist numbers $b_{\mathbf{l}}, c_{\mathbf{ln}} \geq 0$ such that $\|\iota(f_{\mathbf{l}})\|^2_{F,\tau} = b_{\mathbf{l}}\|\iota(f)\|^2_{F,\tau}$ and $\|v_{\mathbf{ln}}\|^2_{F,\tau} = c_{\mathbf{ln}}\|\iota(v_{\mathbf{l}})\|^2_{F,\tau}$ holds. By these data we can show, for $f \in W_{\mathbf{mk}}$,

$$\frac{\|\iota(f)\|^2_{\lambda,\tau}}{\|\iota(f)\|^2_{F,\tau}} = \frac{(\text{monic polynomial of degree } k_2+\cdots+k_r)}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}\prod_{j=2}^r(\lambda+k+m_j-k_j-(2j-3))_{k_j}(\lambda-2r+1)_{k-k_{r+1}}}.$$

On the other hand, by Lemma 4.3 we have

$$\frac{\|\iota(f)\|^2_{\lambda,\tau}}{\|\iota(f)\|^2_{F,\tau}} = \frac{(\text{monic polynomial of degree } k_{r+1})}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_k}.$$

Combining these two formulas, we get

$$\frac{\|\iota(f)\|^2_{\lambda,\tau}}{\|\iota(f)\|^2_{F,\tau}} = \frac{1}{\prod_{j=1}^r(\lambda+k-2(j-1))_{m_j-k_j}(\lambda-2r+1)_{k-k_{r+1}}},$$

and the induction continues. Thus we have proved the result for $(G,V) = (SO^*(4r+2), S^k(\mathbb{C}^{2r+1})\otimes\det^{-k/2})$.

5 Conjecture on exceptional case

In this section we set $(G,K,V)=(E_{6(-14)},SO(2)\times Spin(10),\chi^{-\frac{k}{2}}\boxtimes\mathcal{H}^k(\mathbb{R}^{10}))$. Then the K-type decomposition of $\mathcal{O}(D,V)_K$ is given by

$$\mathcal{P}(\mathfrak{p}^+,V)=\bigoplus_{\substack{\mathbf{m}\in\mathbb{N}^2\\ m_1\geq m_2\geq 0}}\bigoplus_{\substack{\mathbf{k}\in\mathbb{N}^4,\ |\mathbf{k}|=k\\ k_2+k_4\leq m_2,\\ k_3\leq m_1-m_2}}\chi^{-\frac{3}{4}(m_1+m_2)-\frac{k}{2}}\boxtimes V_{\left(\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}.$$

Then for $f\in\chi^{-\frac{3}{4}(m_1+m_2)-\frac{k}{2}}\boxtimes V_{\left(\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}$, we can show by the method similar to Lemma 4.3 that the ratio of norms is given by

$$\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}}=\frac{(\text{monic polynomial of degree }2k_1+k_2+k_3)}{(\lambda+k)_{m_1+k_1+k_2-k}(\lambda+k-3)_{m_2+k_1+k_3-k}(\lambda-4)_k(\lambda-7)_k}.$$

So the author conjectures that

Conjecture 5.1. *For $f\in\chi^{-\frac{3}{4}(m_1+m_2)-\frac{k}{2}}\boxtimes V_{\left(\frac{m_1+m_2}{2}+k_1-k_4,\frac{m_1-m_2}{2}+k_2,\frac{m_1-m_2}{2},\frac{m_1-m_2}{2},-\frac{m_1-m_2}{2}+k_3\right)}$ the ratio of norms is given by*

$$\begin{aligned}\frac{\|f\|^2_{\lambda,\tau}}{\|f\|^2_{F,\tau}}&=\frac{(\lambda)_k(\lambda-3)_k}{(\lambda)_{m_1+k_1+k_2}(\lambda-3)_{m_2+k_1+k_3}(\lambda-4)_{k_2+k_3+k_4}(\lambda-7)_{k_4}}\\&=\frac{1}{(\lambda+k)_{m_1+k_1+k_2-k}(\lambda+k-3)_{m_2+k_1+k_3-k}(\lambda-4)_{k_2+k_3+k_4}(\lambda-7)_{k_4}}.\end{aligned}$$

We note that $m_1+k_1+k_2\geq m_2+k_1+k_3\geq k_2+k_3+k_4\geq k_4$ holds since $k_3\leq m_1-m_2$ and $k_2+k_4\leq m_2$ holds.

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