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General integral transforms by the concept of generalized reproducing kernels (preliminaries report)

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1 Introduction

In order to fix our background in this note, following [6, 7, 8], we recall a general theory for linear mappings in the framework of Hilbert spaces using the general theory of reproducing kernels.

Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be a Hilbert $\mathcal{H}$-valued function on E. Then, we will consider the linear transform

$$f(x) = (\mathbf{f}, \mathbf{h}(x))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{1}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ consisting of all complex-valued functions on E. In order to investigate the linear mapping (1), we form a positive definite quadratic form function $K(x, y)$ on $E \times E$ defined by

$$K(x, y) = (\mathbf{h}(y), \mathbf{h}(x))_{\mathcal{H}} \quad \text{on} \quad E \times E.$$

A complex-valued function $k : E \times E \to \mathbb{C}$ is called a **positive definite quadratic form function** on the set E, or shortly, **positive definite function**, when it satisfies the property that, for an arbitrary function $X : E \to \mathbb{C}$ and any finite subset F of E,

$$\sum_{x,y \in F} \overline{X(x)} X(y) k(x, y) \geq 0. \tag{2}$$

By the fundamental theorem, we know that for any positive definite quadratic form function K, there exists a uniquely determined reproducing kernel Hilbert space admitting the reproducing property.

Then, we obtain the following fundamental result.

Proposition 1.1

(I) The range of the linear mapping (1) by $\mathcal{H}$ is characterized as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel $K(x,y)$ whose characterization is given by the two properties: (i) $K(\cdot,y) \in H_K(E)$ for any $y \in E$ and, (ii) for any $f \in H_K(E)$ and for any $x \in E$, $(f(\cdot), K(\cdot .x))_{H_K(E)} = f(x)$.

(II) In general, we have the inequality

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}.$$

Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ satisfying

$$f(x) = (\mathbf{f}^*, \mathbf{h}(x))_{\mathcal{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}.$$

(III) In general, we have the inversion formula in (1) in the form

$$f \mapsto \mathbf{f}^* \tag{3}$$

in (II) by using the reproducing kernel Hilbert space $H_K(E)$.

However, this formula (3) is, in general, involved and delicate. Consequently, case-by-case, we need different arguments. See [7] and [8] for the details and applications. Recently, however, we obtained a very general inversion formula based on the Aveiro Discretization Method in Mathematics ([2]) by using the ultimate realization of reproducing kernel Hilbert spaces. In this note, however, in order to include the prototype example Fourier integral transform with the analytical nature, and in order to give a general framework of Proposition 1.1, we will consider the following general inversion formula in the general situation with natural assumptions.

Here we consider a concrete case of Proposition 1.1. In order to derive a general inversion formula that is widely applicable in analysis, we assume that $\mathcal{H} = L^2(I, dm)$ and that $H_K(E)$ is a closed subspace of $L^2(E, d\mu)$. For a simplicity statement we assume that I is an interval on the real line. Furthermore, below we assume that $(I, \mathcal{I}, dm)$ and $(E, \mathcal{E}, d\mu)$ are both σ-finite measure spaces and that

$$H_K(E) \hookrightarrow L^2(E, d\mu). \tag{4}$$

Suppose that we are given a measurable function $h : I \times E \to \mathbb{C}$ satisfying $h_y = h(\cdot, y) \in L^2(I, dm)$ for all $y \in E$. Let us set

$$K(x,y) \equiv \langle h_y, h_x \rangle_{L^2(I,dm)}. \tag{5}$$

As we have established in Proposition 1.1, we have

$$H_K(E) \equiv \{f \in \mathcal{F}(E) \,:\, f(x) = \langle F, h_x \rangle_{L^2(I,dm)} \text{ for } F \in \mathcal{H}\}. \tag{6}$$

Let us now define

$$L : \mathcal{H} \to H_K(E)(\hookrightarrow L^2(E, d\mu)) \tag{7}$$

by

$$LF(x) \equiv \langle F, h_x \rangle_{L^2(I,dm)} = \int_I F(\lambda)\overline{h(\lambda, x)}\, dm(\lambda), \quad x \in E \tag{8}$$

for $F \in \mathcal{H} = L^2(I, dm)$, keeping in mind (4). Observe that $LF \in H_K(E)$.
The next result will give the inversion formula.

Proposition 1.2 *Assume that $\{E_N\}_{N=1}^{\infty}$ is an increasing sequence of measurable subsets in E such that*

$$\bigcup_{N=1}^{\infty} E_N = E \tag{9}$$

and that

$$\int_{I \times E_N} |h(\lambda, x)|^2\, dm(\lambda)\, d\mu(x) < \infty \tag{10}$$

for all $N \in \mathbb{N}$. Then we have

$$L^* f(\lambda) \left(= \lim_{N\to\infty} (L^*[\chi_{E_N} f])(\lambda)\right) = \lim_{N\to\infty} \int_{E_N} f(x) h(\lambda, x)\, d\mu(x) \tag{11}$$

for all $f \in L^2(I, d\mu)$ in the topology of $\mathcal{H} = L^2(I, dm)$. Here, $L^* f$ is the adjoint operator of L, but it represents the inversion with the minimum norm for $f \in H_K(E)$.

In this Proposition 1.2, we see that with the very natural way, the inversion formula may be given in the strong convergence in the space $\mathcal{H} = L^2(I, dm)$.

2 Formulation of a fundamental problem

Our basic assumption is that $h : I \times E \to \mathbb{C}$ satisfies $h_y = h(\cdot, y) \in L^2(I, dm)$ for all $y \in E$; that is, the integral kernel or linear mapping is in the framework of Hilbert spaces. In this paper, as stated in the abstract, we assume that the integral kernel $h_y = h(\cdot, y)$ does not belong to $L^2(I, dm)$, however, for any exhaustion $\{I_t\}_{t>0}$ such that $I_t \subset I_{t'}$ for $t \leq t'$, $\bigcup_{t>0} I_t = I$, $h_y = h(\cdot, y) \in L^2(I_t, dm)$ for all $y \in E$ and $\{h_y; y \in E\}$ is complete in $L^2(I_t, dm)$ for any $t > 0$.
In Proposition 1.2, as in (1), we consider the integral transform

$$f_t(x) = \langle F, h_x \rangle_{L^2(I_t, dm)} \text{ for } \quad F \in L^2(I, dm) \tag{12}$$

and the corresponding reproducing kernel

$$K_t(x, y) = \langle h_y, h_x \rangle_{L^2(I_t, dm)}. \tag{13}$$

Here, we assume that $\mathcal{H}_t$ is the Hilbert space $L^2(I_t, dm)$ and $h_x \in \mathcal{H}_t$ for any x. We assume as in stated in the introduction that the non-decreasing reproducing kernels $K_t(x,y)$, in the sense: for any $t' > t$, $K_{t'}(y,x) - K_t(y,x)$ is a positive definite quadratic form function, do, in general, not converge, when $\lim_{t\uparrow\infty} K_t(x,y)$. We write, however, the limit by $K_\infty(x,y)$ formally, that is,

$$K_\infty(x,y) := \lim_{t\uparrow\infty} K_t(x,y) \tag{14}$$

$$= \langle h_y, h_x \rangle_{L^2(I,dm)}.$$

This integral does, in general, not exist and the limit is a special meaning. We are interesting, however, in the relationship between the spaces $L^2(I_t, dm)$ and $L^2(I, dm)$ by associating the kernels $K_t(x,y)$ and $K_\infty(x,y)$, respectively.

First, for the space $\mathcal{H}_t$ and the reproducing kernel Hilbert space $H_{K_t}(E)$, we recall the isometric identity in (12), by assuming that $\{h_x : x \in E\}$ is complete in the space $\mathcal{H}_t$

$$\|f_t\|_{H_{K_t}(E)} = \|F\|_{L^2(I_t,dm)}. \tag{15}$$

Next note that for any $F \in L^2(I, dm)$,

$$\lim_{t\uparrow\infty} \|F\|_{L^2(I_t,dm)} = \|F\|_{L^2(I,dm)}. \tag{16}$$

Here, of course, the norms are nondecreasing.

As the corresponding function to $f_t \in H_{K_t}(E)$, we consider the function, in the view point of (12)

$$f(x) = \langle F, h_x \rangle_{L^2(I,dm)} \text{ for } \quad F \in L^2(I, dm). \tag{17}$$

However, this function is not defined, because the above integral does, in general, not exist. So, we consider the function formally, tentatively. However, we are considering the correspondings

$$f_t \longleftrightarrow f \tag{18}$$

and

$$H_{K_t}(E) \longleftrightarrow H_{K_\infty}(E), \tag{19}$$

however, for the space $H_{K_\infty}(E)$, we have to give its meaning; here, when the kernel $K_\infty(x,y)$ exists by the condition $h_x \in L^2(I,dm), x \in E$, $H_{K_\infty}(E)$ is the reproducing kernel Hilbert space admitting the kernel $K_\infty(x,y)$.

We consider the formal calculations as follows: Following (11)

$$F(\lambda) \left(= \lim_{N\to\infty} (L^*[\chi_{E_N} f])(\lambda)\right) = \lim_{N\to\infty} \int_{E_N} f(y) h(\lambda, y)\, d\mu(y) \tag{20}$$

and (17), for $F \in L^2(I, dm)$

$$f(x) = \langle F, h_x \rangle_{L^2(I,dm)} \tag{21}$$

$$= \left\langle \lim_{N\to\infty} \int_{E_N} f(y) h(\lambda, y)\, d\mu(y), h_x \right\rangle_{L^2(I,dm)}$$

$$= \lim_{N\to\infty} \int_{E_N} f(y)\overline{K_\infty(y,x)}d\mu(y).$$

This formal calculation will show that $K_\infty(x,y)$ is like a reproducing kernel for the image space of (21) and we have the isometric identity, in (21)

$$\|f\|_{H_{K_\infty}} = \|F\|_{L^2(I,dm)}. \tag{22}$$

Then we obtain the norm convergence as follows:

$$\lim_{t\uparrow 0} \|f_t\|_{H_{K_t}} = \|f\|_{H_{K_\infty}} = \|F\|_{L^2(I,dm)}. \tag{23}$$

and the norms are nondecreasing.

Note that in (23), the first term and the last term have the real senses that their meanings do exist and they have the isometric relation. This will mean that the general L^2 norm is represented by a reproducing kernel Hilbert member and its norm.

We will catch the kernel $K_\infty(x,y)$ as a generalized reproducing kernel and the fundamental applications to some general initial value problems by using the related eigenfunctions are given in [9, 10].

In this note, we will give the natural and precise theory for the above formal treatment.

3 Completion property

We note the general and fundamental property.

We introduce a preHilbert space by

$$H_{K_\infty} := \bigcup_{t>0} H_{K_t}(E).$$

For any $f \in H_{K_\infty}$, there exists a space $H_{K_t}(E)$ containing the function f for some $t > 0$. Then, for any t' such that $t < t'$,

$$H_{K_t}(E) \subset H_{K_{t'}}(E)$$

and, for the function $f \in H_{K_\infty}$,

$$\|f\|_{H_{K_t}(E)} \geq \|f\|_{H_{K_{t'}}(E)}.$$

(Here, inequality holds, in general, however, in this case, equality, indeed, holds, for the sake of the completeness of the integral kernel.) Therefore, there exits the limit

$$\|f\|_{H_{K_\infty}} := \lim_{t'\uparrow\infty} \|f\|_{H_{K_{t'}}(E)}.$$

Denote by H_∞ the completion of H_{K_∞}.

Theorem 3.1 *For the general situation such that $K_t(x,y)$ exits for all $t>0$ and $K_\infty(x;y)$ does, in general, not exist, for any function $f \in H_\infty$*

$$\lim_{t\uparrow\infty}(f(x'),K_t(x',x))_{H_\infty}=f(x), \tag{24}$$

in the space H_∞.

Proof: Note that for any $t<t'$, and for any $f_t \in H_{K_t}(E)$, $f_t \in H_{K_{t'}}(E)$ and furthermore, for the sake of the completeness of the kernel h_x, in particular, that

$$\langle f,g\rangle_{H_{K_t(E)}}=\langle f,g\rangle_{H_{K_{t'}(E)}}$$

for all $t'>t$ and $f,g\in H_{K_t}(E)$.

Just observe that

$$\begin{aligned}|(f(x'),K_t(x',x))_{H_\infty}|^2 &\le \|f\|^2_{H_\infty}\|K_t(\cdot,x)\|^2_{H_\infty}\\ &\le \|f\|^2_{H_\infty}\|K_t(\cdot,x)\|^2_{H_{K_t}(E)}\\ &=\|f\|^2_{H_\infty}K_t(x,x).\end{aligned}$$

Therefore, we see that $(f(x'),K_t(x',x))_{H_\infty}\in H_{K_t}(E)$ and that

$$\|(f(x'),K_t(x',x))_{H_\infty}\|_{H_{K_t}(E)}\le\|f\|_{H_\infty}.$$

Indeed, for these, recall the identity

$$K_t(x,y)=\langle K_t(\cdot,y),K_t(\cdot,x)\rangle_{H_\infty}.$$

The mapping $f\mapsto(f(x'),K_t(x',x))_{H_\infty}$ being uniformly bounded, and so, we can assume that $f\in H_{K_t}(E)$ for any fixed $t>0$. However, in this case, the result is clear, since, $f\in H_{K_{t'}}(E)$ for $t<t'$

$$\lim_{t'\uparrow\infty}(f(x'),K_{t'}(x',x))_{H_\infty}=\lim_{t'\uparrow\infty}\langle f,K_{t'}(\cdot,x)\rangle_{H_\infty}=\lim_{t'\uparrow\infty}\langle f,K_{t'}(\cdot,x)\rangle_{H_{K_{t'}}(E)}=f(x).$$

■

Theorem 3.1 may be looked as a reproducing kernel in the natural topology and in the sense of Theorem 3.1, and the reproducing property may be written as follows:

$$f(x)=\langle f,K_\infty(\cdot,x)\rangle_{H_\infty},$$

with (24). Here **the limit $K_\infty(\cdot,x)$ does, in general, not need to exist**, however, the series are non-decreasing.

The completion space H_0 will be determined, in many concrete cases, from the realizations of the spaces $H_{K_t}(E)$, by case-by-case.

4 Convergence of $f_t(x) = \langle F, h_x \rangle_{L^2(I_t,dm)}$; $F \in L^2(I, dm)$

As in the case of Fourier integral, we will prove the convergence of (12) in the completion space H_∞. Indeed, for any $t, t' > 0, t < t'$, we have:

$$\begin{aligned}
&\lim_{t,t'\uparrow\infty} \|f_t - f_{t'}\|^2_{H_\infty} \\
&= \lim_{t,t'\uparrow\infty} \|f_t - f_{t'}\|^2_{H_{K_t(E)}} \\
&\le \lim_{t,t'\uparrow\infty} \left(\|f_t\|^2_{H_{K_t(E)}} + \|f_{t'}\|^2_{H_{K_{t'}(E)}} - 2\|f_t\|^2_{H_{K_t(E)}} \right) \\
&= 0.
\end{aligned}$$

In this sense, as in the Fourier integral of the cace $L^2(\mathbf{R}, dx)$ we will write, for

$$\lim_{t\uparrow\infty} f_t = f \quad \text{in} \quad H_\infty$$

as follows:

$$\begin{aligned}
f(x) &= \lim_{t\uparrow\infty} (F(\cdot), h(\cdot, x))_{L^2(I_t,dm)} \qquad (25) \\
&= (F(\cdot), h(\cdot, x))_{L^2(I,dm)} \, .
\end{aligned}$$

5 Inversion of the integral transforms

We will consider the inversion of the integral transform (25) from the space H_∞ onto $L^2(I, dm)$. For any $f \in H_\infty$, we take functions $f_t \in H_{K_t}(E)$ such that

$$\lim_{t\uparrow\infty} f_t = f$$

in the space H_∞. This is possible, because the space H_∞ is the completion of the spaces $H_{K_t}(E)$. However, f_t may be constructed by Theorem 3.1, in the form

$$f_t(x) := (f(x'), K_t(x', x))_{H_\infty} \, .$$

For the functions $f_t \in H_{K_t}(E)$, by Proposition 1.2 - we are assuming the conditions in Propostioin 1.2 - , we can construct the inversion in the following way:

$$F_t(\lambda) = \lim_{N\to\infty} \int_{E_N} f_t(x) h(\lambda, x)\, d\mu_t(x) \qquad (26)$$

in the topology of $L^2(I, dm)$ satisfying

$$\begin{aligned}
f_t(x) &= (F_t(\cdot), h_x(\cdot))_{L^2(I,dm)} \qquad (27) \\
&= (F_t, h_x)_{L^2(I_t,dm)}.
\end{aligned}$$

Here, of course, the function F_t of $L^2(I, dm)$ is the zero extension of a function F_t of $L^2(I_t, dm)$. Note that the isometric relation that for any $t < t'$

$$\|f_t - f_{t'}\|_{H_0} = \|F_t - F_{t'}\|_{L^2(I,dm)}. \qquad (28)$$

Then, we see the desired result: The functions F_t converse to a function F in $L^2(I, dm)$ and

$$f(x) = (F, h_x)_{L^2(I,dm)} \tag{29}$$

in our sense. We can write down the inversion formula as follows:

$$F(\lambda) = \lim_{t\uparrow\infty} \lim_{N\to\infty} \int_{E_N} (f(x'), K_t(x', x))_{H_\infty} h(\lambda, x)\, d\mu_t(x), \tag{30}$$

where both limits $\lim_{N\to\infty}$ and $\lim_{t\uparrow\infty}$ are taken in the sense of the space $L^2(I, dm)$.

Of course, the correspondence $f \in H_\infty$ and $F \in L^2(I, dm)$ is one to one.

Indeed, we assume that $f \in H_\infty$, $f \equiv 0$, then

$$0 \equiv f(x) = \lim_{t\uparrow\infty} (f, K_t(\cdot, x) = \lim_{t\uparrow\infty} f_t(x) \tag{31}$$

in the space H_∞.; that is,

$$\lim_{t\uparrow\infty} \|f_t\|_{H_{K_t}} = 0 = \lim_{t\uparrow\infty} \|F\|_{L^2(I,dm)}; \tag{32}$$

that implies the desired result that $F = 0$ on the space $L^2(I, dm)$.

6 Fourier integral transform case

As a typical example, we shall examine the Fourier integral transform. For one dimensiona case, we consider the integral transform, for the functions F of $L_2(-\pi t, +\pi t)$, $t > 0$ as

$$f_t(z) = \frac{1}{2\pi} \int_{-\pi t}^{\pi t} F(t) e^{-iz\xi} d\xi. \tag{33}$$

In order to identify the image space following the theory of reproducing kernels, we form the reproducing kernel

$$\begin{aligned} K_t(z, \overline{u}) &= \frac{1}{2\pi} \int_{-\pi t}^{\pi t} e^{-iz\xi} \overline{e^{-iu\xi}} d\xi \\ &= \frac{1}{\pi(z - \overline{u})} \sin \pi t(z - \overline{u}). \end{aligned} \tag{34}$$

The image space of (31) is called the Paley Wiener space $W(\pi t)$ consisting of all analytic functions of exponential type satisfying, for some constant C and as $z \to \infty$

$$|f_t(z)| \leq C \exp(\pi |z| t)$$

and

$$\int_{\mathbf{R}} |f_t(\xi)|^2 d\xi < \infty.$$

From the identity

$$K_t\left(\frac{j}{t}, \frac{j'}{t}\right) = t\delta(j, j')$$

(the Kronecker's δ), since $\delta(j,j')$ is the reproducing kernel for the Hilbert space ℓ^2, from the general theory of integral transforms and the Parseval's identity we have the isometric identities in (31)

$$\frac{1}{2\pi}\int_{-\pi t}^{\pi t}|F(\xi)|^2 d\xi = \frac{1}{t}\sum_{j=-\infty}^{\infty}|f_t(j/t)|^2 = \int_{\mathbf{R}}|f_t(\xi)|^2 d\xi.$$

That is, the reproducing kernel Hilbert space H_{K_t} with $K_t(z,\overline{u})$ is characterized as a space consisting of the Paley Wiener space $W(\pi t)$ and with the norm squares above. Here we used the well-known result that $\{j/t\}_{j=-\infty}^{\infty}$ is a unique set for the Paley Wiener space $W(\pi t)$; that is, $f_t(j/t) = 0$ for all j implies $f_t \equiv 0$. Then, the reproducing property of $K_t(z,\overline{u})$ states that

$$f_t(x) = (f_t(\cdot), K_t(\cdot, x))_{H_{K_t}} = \frac{1}{t}\sum_{j=-\infty}^{\infty} f_t(j/t)K_t(j/t, x)$$

$$= \int_{\mathbf{R}} f_t(\xi)K_t(\xi, x)d\xi.$$

In particular, on the real line x, this representation is the sampling theorem which represents the whole data $f_t(x)$ in terms of the discrete data $\{f_t(j/t)\}_{j=-\infty}^{\infty}$. For a general theory for the sampling theory and error estimates for some finite points $\{j/t\}_j$, see [7]. As this typical case, we note that all the reproducing kernel Hilbert spaces H_{K_t} may be realized in the space $L^2(\mathbf{R}, d\xi)$ which is now the completion H_∞ of the spaces H_{K_t}.

7 Discrete versions

We refer to a typical discrete version whose situation is very general. Let the family $\{U_n(x)\}_{n=0}^{\infty}$ be a complete orthonormal system in a Hilbert space with the norm

$$\|F\|^2 = \int_E |F(x)|^2 dm(x) \tag{35}$$

with a dm measurable set E in the usual form $L^2(E, dm)$. We consider the family of all the functions, for arbitrary complex numbers $\{C_n\}_{n=0}^{N}$

$$F(x) = \sum_{n=0}^{N} C_n U_n(x) \tag{36}$$

and we introduce the norm

$$\|F\|^2 = \sum_{n=0}^{N}|C_n|^2. \tag{37}$$

Then, the function space forms a Hilbert space $H_{K_N}(E)$ determined by the reproducing kernel $K_N(x,y)$:

$$K_N(x,y) = \sum_{n=0}^{N} U_n(x)\overline{U_n(y)} \tag{38}$$

with the inner product induced from the norm (35), as usual. Then, the functions in the Hilbert space $L^2(E, dm)$ and the norm (33) are realized as the completion $H_{K_\infty}(E)$ of the spaces $H_{K_N}(E)$. In this case, for the correspondence:

$$\ell^2 : \{C_n\} \leftrightarrow F(x) = \sum_{n=0}^{\infty} C_n U_n(x), \tag{39}$$

we obtain the same results in the classical analysis and in this note.

We can consider such linear mappings for arbitrary functions $\{U_n(x)\}$ which are linearly independent and by considering the kernel forms (36), however, the realization of the completion space H_∞ becomes the crucial problem, in our new approarch.

8 Conclusion

When we consider the integral transform

$$LF(x) = \int_I F(\lambda)\overline{h(\lambda, x)}\, dm(\lambda), \quad x \in E \tag{40}$$

for $F \in \mathcal{H} = L^2(I, dm)$, indeed, the integral kernel $h(\lambda, x)$ does not need to belong to the space $L^2(I, dm)$ and with the very general assumptions that for any exhausion $\{I_t\}$ of I,

$$h(\lambda, x) \quad \text{belongs to } L^2(I_t, dm) \text{ for any} \quad x \quad \text{of} \quad E$$

and

$$\{h(\lambda, x); x \in E\} \text{ is complete in } L^2(I_t, dm),$$

we can establish the isometric identity and inversion formula of the integral transform (38) by giving the natural interpretation of the integral transform (38), as in the Fourier transform.

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