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General initial value problems using eigenfunctions and reproducing kernels (preliminaries report)

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1 Introduction

To clearify our problem, we will start with a prototype example. Let K_t, $(t > 0)$ be the positive definite quadratic form function on the real line defined by:

$$K_t(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i(x-y)\xi}e^{-t\xi^2}d\xi = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}} \quad (x, y \in \mathbb{R}\times\mathbb{R}). \tag{1}$$

The function K_t is known as the heat kernel of the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0 & x \in \mathbb{R}, t > 0 \\ u(\cdot, 0) = f & x \in \mathbb{R}. \end{cases} \tag{2}$$

Denote by u_f the solution of (2) when we are given $f \in L^2(\mathbb{R})$. Then we can consider the uniquely determined reproducing kernel Hilbert space $H_{K_t}(\mathbb{R})$ admitting the kernel K_t, $(t > 0)$. Observe that

$$H_{K_t}(\mathbb{R}) = \{u_f(\cdot, t) \,:\, f \in L^2(\mathbb{R})\}$$

and that

$$\|u_f(\cdot, t)\|_{H_{K_t}(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}.$$

Therefore, for any $0 < t_1 < t_2$,

$$K_{t_2} \ll K_{t_1}; \tag{3}$$

that is, $K_{t_1} - K_{t_2}$ is a positive definite quadratic form function as we can see from (1). Hence we have

$$H_{K_{t_2}}(\mathbb{R}) \subset H_{K_{t_1}}(\mathbb{R})$$

and

$$\|f\|_{H_{K_{t_2}}(\mathbb{R})} \downarrow \|f\|_{H_{K_{t_1}}(\mathbb{R})} \quad (t_2 \downarrow t_1)$$

for any function $f \in H_{K_{t_2}}(\mathbb{R})$ in the sense of the non-decreasing norm convergence; see [2]. In [2] N. Aronzajn discussed such a property in detail for nondecreasing family of reproducing kernels $\{K_t\}_{t>0}$ satisfying (3) when the limit

$$\lim_{t_1 \downarrow t} K_{t_1}(x, y) \tag{4}$$

of functions converges in some set.

However, in the present case (1), the limit $t_1 \downarrow 0$ fails to converge in the usual sense. However, we claim that we have a formal representation;

$$\delta(x-y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i(x-y)\xi} d\xi. \tag{5}$$

In this case $\delta(x - y)$ is not a usual function, but from the above calculation we learn that it is determined as an increasing limit in the above sense of reproducing kernels. Aronszajn did not treat such a case in [2]. Denote by $K|\mathrm{diag}$ the restriction of K to the diagonal:$K|\mathrm{diag}(x) = K(x,x)$ for $x \in E$. He established a natural theory on the point set where $\lim\limits_{t_1 \downarrow 0} K_{t_1}|\mathrm{diag}$ converges. In our model case, the limit diverges everywhere on diag as the explicit formula (1) implies.

We wish to establish the fact corresponding to divergent nondecreasing sequences of reproducing kernels under a natural condition. We will obtain some generalized delta functions which may be considered as reproducing kernels in a reasonable sense. We will give the fundmental applications to some general initial value problems using eigenfunctions.

We organize the remaining part of this note as follows: First, we recall an important result on the range of the integral transform in Section 2. In Section 3, we move on to our concrete setting of $L^2(I, e^{-t\lambda^2}dm)$. We apply our result to initial value problems in Section 4. Our main theorem is given in Section 5, which is stated in full generality. Further examples are given in Sections 6 and 7. Section 6 considers applications to Szegö spaces. We pass to a discrete case in Section 7.

2 Preliminaries on linear mappings and inversions

In order to analyze the integral transform and in order to fix the basic background for our purpose, we review the essence of the theory of reproducing kernels.

We are interested in the integral transforms in the framework of Hilbert spaces. Of course, we hope to characterize the image functions, the isometric identity like the Parseval identity and the inversion formula, basically. For these general and fundamental problems, we have a unified and fundamental method and concept in the general situation as follows:

Following [14, 15, 16], we recall a general theory for linear mappings in the framework of Hilbert spaces. Let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space. Let E be an abstract set and $\mathbf{h}$ be an $\mathcal{H}$-valued function on E. Thenwe will consider the linear transform

$$f = L\mathbf{f} = \langle \mathbf{f}, \mathbf{h}(\cdot)\rangle_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{6}$$

from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ consisting of all complex-valued functions on E. In order to investigate the linear mapping (6), we form a positive definite quadratic form function $K : E \times E \to \mathbb{C}$ defined by:

$$K(x,y) = \langle \mathbf{h}(y), \mathbf{h}(x)\rangle_{\mathcal{H}} \quad \text{on} \quad E \times E.$$

A complex-valued function $k : E \times E \to \mathbb{C}$ is called a **positive definite quadratic form function** on the set E, or shortly, **positive definite function**, when

$$\sum_{x,y \in F} \overline{X(x)} X(y) k(x,y) \geq 0 \tag{7}$$

for an arbitrary function $X : E \to \mathbb{C}$ and any finite subset F of E.

By the fundamental theorem, we know that for any positive definite quadratic form function K, there exists a uniquely determined reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing property. Here and below we always assume that $H_K(E)$ is separable, when we are given a positive definite kernel K.

The following result is fundamental.

Proposition 2.1.

(*I*) *We can characterize the range of the linear mapping* (6) *by $\mathcal{H}$ as the reproducing kernel Hilbert space $H_K(E)$ admitting the reproducing kernel K enjoying two properties: (i) $K(\cdot,y) \in H_K(E)$ for any $y \in E$ and, (ii) for any $f \in H_K(E)$ and for any $x \in E$, $\langle f, K(\cdot,x)\rangle_{H_K(E)} = f(x)$.*

(*II*) *In general, we have the inequality*

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}.$$

Here, for any member f of $H_K(E)$ there exists a uniquely determined $\mathbf{f}^* \in \mathcal{H}$ *satisfying*

$$f = \langle \mathbf{f}^*, \mathbf{h}(\cdot) \rangle_{\mathcal{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{8}$$

(*III*) *In general, we have the inversion formula in* (6) *in the form*

$$f \mapsto \mathbf{f}^* \tag{9}$$

in (*II*) *by using the reproducing kernel Hilbert space $H_K(E)$.*

However, this formula (9) is, in general, involved and delicate. Consequently, case-by-case we need different arguments; see [15, 16] for details and applications. Recently, however, we obtained a very general inversion formula based on the Aveiro Discretization Method in Mathematics [3] using the ultimate realization of reproducing kernel Hilbert spaces. In this note, however, to give prototype examples with the analytical nature, we will consider the following general inversion formula in the general situation with natural assumptions.

Here we consider a concrete case of Proposition 2.1. To derive a general inversion formula widely applicable in analysis, we assume that $\mathcal{H} = L^2(I, dm)$. To state our result simply, we will assume that I is an interval on the real line. Denote by $\mathcal{I}$ the Borel sigma algebra on I. Furthermore, below we assume that $(I, \mathcal{I}, dm)$ and $(E, \mathcal{E}, d\mu)$ are both σ-finite measure spaces and that

$$H_K(E) \hookrightarrow L^2(E, d\mu) \tag{10}$$

in the sense of continuous embeddings.

Suppose that we are given a measurable function $h : I \times E \to \mathbb{C}$ satisfying $h_y = h(\cdot, y) \in L^2(I, dm)$ for all $y \in E$. Let us set $K(x, y) \equiv \langle h_y, h_x \rangle_{L^2(I,dm)}$. As we have established in Proposition 2.1, we have

$$H_K(E) \equiv \{f \in \mathcal{F}(E) : f(x) = \langle F, h_x \rangle_{L^2(I,dm)} \text{ for } F \in \mathcal{H}\}. \tag{11}$$

Let us now define a linear mapping $L : \mathcal{H} \to H_K(E)(\hookrightarrow L^2(E, d\mu))$ by

$$LF(x) \equiv \langle F, h_x \rangle_{L^2(I,dm)} = \int_I F(\lambda)\overline{h(\lambda, x)}\, dm(\lambda), \quad x \in E \tag{12}$$

for $F \in \mathcal{H} = L^2(I, dm)$, keeping in mind (10). Observe that $LF \in H_K(E)$ since

$$LF \otimes \overline{LF} \ll K.$$

The next result will serve to the inversion formula.

Proposition 2.2. *Assume that $\{E_N\}_{N=1}^{\infty}$ is an increasing sequence of measurable subsets in E such that*

$$\bigcup_{N=1}^{\infty} E_N = E \tag{13}$$

and that

$$\iint_{I\times E_N} |h(\lambda,x)|^2\, dm(\lambda)\, d\mu(x) < \infty \tag{14}$$

for all $N \in \mathbb{N}$. Then we have

$$L^*f(\lambda) \left(= \lim_{N\to\infty} (L^*[\chi_{E_N} f])(\lambda)\right) = \lim_{N\to\infty} \int_{E_N} f(x)h(\lambda,x)\, d\mu(x) \tag{15}$$

for all $f \in L^2(I, d\mu)$ in the topology of $\mathcal{H} = L^2(I, dm)$. Here, L^*f is the adjoint operator of L and it represents the inversion with the minimum norm for $f \in H_K(E)$;

$$LL^*f = f \text{ and } \|L^*f\|_{\mathcal{H}} = \inf_{g\in\mathcal{H}, Lg=f} \|g\|_{\mathcal{H}}.$$

In this Proposition 2.2, we see that with the very natural way, the inversion formula may be given in the strong convergence in the space $\mathcal{H} = L^2(I, dm)$.

3 Formulation of a fundamental problem

In Proposition 2.2, as in (1), we consider the integral transform $F \in \mathcal{H}_t \mapsto f_t \in \mathcal{F}(I)$ given by

$$f_t(x) = \langle F, h_x \rangle_{L^2(I, e^{-t\lambda^2}dm)} \quad (x \in E) \tag{16}$$

and the corresponding reproducing kernel K_t given by

$$K_t(x,y) = \langle h_y, h_x \rangle_{L^2(I, e^{-t\lambda^2}dm)} \quad (x, y \in I). \tag{17}$$

Here and below we assume that $\mathcal{H}_t$ is the Hilbert space $L^2(I, e^{-t\lambda^2}dm)$ and that $h_x \in \mathcal{H}_t$ for any $x \in E$. We assume as in stated in the introduction that the monotone family of reproducing kernels $\{K_t\}_{t>0}$ fail to converge in general, when $\lim_{t\downarrow 0} K_t(x,y)$. Nevertheless, we will write $K_0(x,y)$ for the limit formally as if it were the delta function, namely,

$$K_0(x,y) := \lim_{t\downarrow 0} K_t(x,y) = \langle h_y, h_x \rangle_{L^2(I,dm)}. \tag{18}$$

This integral fails to exist in general and the limit is understood as special one as in the introduction. We are interested, however, in the relationship beteween the spaces $L^2(I, e^{-t\lambda^2}dm)$ and $L^2(I, dm)$ by associating the kernels K_t and K_0, respectively.

We assume that $\{h_x : x \in E\}$ is complete in the space $\mathcal{H}_t$. At first, for the spaces $\mathcal{H}_t$ and the reproducing kernel Hilbert space $H_{K_t}(E)$, we recall the isometric identity (8);

$$\|f_t\|_{H_{K_t}(E)} = \|F\|_{L^2(I, e^{-t\lambda^2}dm)}. \tag{19}$$

Next note that for any $F \in L^2(I, dm)$,

$$\lim_{t\downarrow 0} \|F\|_{L^2(I, e^{-t\lambda^2}dm)} = \|F\|_{L^2(I,dm)} \tag{20}$$

by the momotone convergence theorem. Here, of course, the norms are nondecreasing.

Let $F \in L^2(I, dm)$. As the function corresponding to $f_t \in H_{K_t}(E)$, we will consider the function

$$f(x) = \langle F, h_x \rangle_{L^2(I,dm)} = \int_I F(\lambda)h(\lambda, x)\, dm(\lambda) \quad (x \in E) \tag{21}$$

in the view point of (16). However, this definition does not make sense, because the above integral fails to converge in general. So, we consider the function formally, tentatively. However, we are considering the correspondence

$$f_t \in H_{K_t}(E) \longleftrightarrow f \in H_{K_0}(E). \tag{22}$$

however, for the space $H_{K_0}(E)$, we have to make its meaning more precise; here, when the kernel K_0 exists by the condition $h_x \in L^2(I, dm)$, $x \in E$, $H_{K_0}(E)$ is the reproducing kernel Hilbert space admitting the kernel K_0.

We consider the formal calculations as follows: First assume (14). Following Proposition 2.2, we consider

$$F(\lambda) \left(= \lim_{N\to\infty} (L^*[\chi_{E_N} f])(\lambda)\right) = \lim_{N\to\infty} \int_{E_N} f(y)h(\lambda, y)\, d\mu(y) \tag{23}$$

for $F \in L^2(I, dm)$

$$\begin{aligned} f(x) &= \langle F, h_x \rangle_{L^2(I,dm)} \\ &= \left\langle \lim_{N\to\infty} \int_{E_N} f(y)h(\lambda, y)\, d\mu(y), h_x \right\rangle_{L^2(I,dm)} \\ &= \lim_{N\to\infty} \int_{E_N} f(y)\overline{K_0(y, x)} d\mu(y). \end{aligned}$$

This formal calculation will show that K_0 looks like a reproducing kernel for the image space of (21) and we have the isometric identity, in (21)

$$\|f\|_{H_{K_0}(E)} = \|F\|_{L^2(I,dm)}. \tag{24}$$

Then we obtain the norm convergence as follows:

$$\lim_{t\downarrow 0} \|f_t\|_{H_{K_t}(E)} = \|f\|_{H_{K_0}(E)} = \|F\|_{L^2(I,dm)}. \tag{25}$$

and the norms are nondecreasing.

Note that in (23), the first term and the last term make sense and they have the isometric relation. This will mean that the general L^2 norm is represented by a reproducing kernel Hilbert member and its norm. Indeed, in this note, we will grasp K_0 as a reproducing "kernel" together with a clear formulation.

We will take the kernel K_0 as a generalized reproducing kernel. We furthermore give the fundamental applications to some general initial value problems using the related eigenfunctions.

4 Applications to initial value problems

We first formulate a general initial value problem in the framework of reproducing kernel Hilbert spaces based on [5].

For some general linear operator L_x (and differential operator ∂_t), for some function space on a certain domain E, we will consider the initial value problem of the equation

$$(\partial_t + L_x)u_f(x,t) = 0, \qquad t > 0, \tag{26}$$

for an unknown u_f satisfying the initial value condition

$$u_f(x,0) = f(x). \tag{27}$$

Here we have to give a precise meaning of the equality in (27).

Having in mind the general framework of Section 3, we recall a general initial value problem based on [5, 6, 13]. For this purpose, we let I be an interval contained in $[0,\infty)$. Assume that the eigenvalues of L all belong to I. The parameter λ represents the eigenvalues for some linear operator L for functions on E satisfying

$$L[\overline{h(\lambda,\cdot)}] = \lambda\overline{h(\lambda,\cdot)}, \quad \lambda \in I. \tag{28}$$

Here, $\overline{h(\lambda,x)}$ is the eigenfunction and in order to set our notation in a consistent way, we take the complex conjugate there.

We form the reproducing kernel

$$K_t(x,y) = \int_I h(\lambda,y)\overline{h(\lambda,x)}\exp(-\lambda t)\,dm(\lambda), \quad t>0, \tag{29}$$

and

$$K_0(x,y) = \int_I h(\lambda,y)\overline{h(\lambda,x)}\,dm(\lambda), \tag{30}$$

Note that (29) stands for

$$K_t(x,y) = \lim_{R\to\infty}\int_{R^{-1}}^{R} h(\lambda,y)\overline{h(\lambda,x)}\exp(-\lambda t)\,dm(\lambda)$$

We assume that

$$\int_I |h(\lambda,y)|^2\,dm(\lambda) < \infty \tag{31}$$

for all $x \in E$.

Consider the reproducing kernel Hilbert space $H_{K_t}(E)$ admitting the kernel K_t. In particular, note that

$$K_t(\cdot,y) \in H_K(E), \quad y \in E,$$

in the situation of Section 2 for $K_0 = K$. Then we have

Proposition 4.1. *For any element $f \in H_K(E)$, the solution u_f of the initial value problem* (26)–(27) *exists and it is given by*

$$u_f(x,t) = \langle f, K_t(\cdot,x)\rangle_{H_K(E)} \quad (t>0, x\in E). \tag{32}$$

Here the meaning of the boundary condition (27) *is given by*

$$\lim_{t\to+0} u_f(x,t) = \lim_{t\to+0}\langle f, K_t(\cdot,x)\rangle_{H_K(E)} = \langle f, K(\cdot,x)\rangle_{H_K(E)} = f(x), \tag{33}$$

whose existence is ensured and the limit is given in the sense of uniform convergence on any subset of E where $K|\mathrm{diag}$ is bounded.

The uniqueness property of the initial value problem depends on the completeness of the family of functions

$$\{K_t(\cdot,x); x\in E\} \tag{34}$$

in $H_K(E)$.

In Proposition 4.1, the properties of the solutions u_f of (26)–(27) satisfying the initial value f may be completely derived by the reproducing kernel Hilbert space admitting the kernel

$$k(x,t;y,\tau) := \langle K_\tau(\cdot,y), K_t(\cdot,x)\rangle_{H_K(E)}. \tag{35}$$

In our method, we see that the existence of the solution of the initial value problem is based on the eigenfunctions and we are constructing the desired solution satisfying the considered initial condition. In view of this, with broader knowledge for the eigenfunctions we can consider more general initial value problems. Furthermore, by considering the linear mapping (32) with various situations, we will be able to obtain various inverse problems which may be described by looking for the initial values f from the various output data of $u_f(x,t)$.

We can rephrase the main purpose of this paper; we seek to consider the reproducing property of $f \in H_{K_0}(E)$. To see this delicate property, we recall the proof of Proposition 4.1 .

Proof of Proposition 4.1. First, note that the kernel $K_t(\cdot, y)$ satisfies the operator equation (26) for any fixed y, because the functions

$$\exp(-\lambda t)\overline{h(\lambda, x)} \quad (\lambda > 0)$$

satisfy the operator equation. The condition (31) guarantees the change of the limit with respect to R and L. Similarly, the function $u_f(x,t)$ defined by (32) is the solution of the operator equation (26).

In order to see the initial value property, we note the important general property:

$$K_t \ll K; \tag{36}$$

and hence we have $H_{K_t}(E) \subset H_K(E)$. For any function $f \in H_{K_t}(E)$, it holds

$$\|f\|_{H_K(E)} = \lim_{t \to +0} \|f\|_{H_{K_t}(E)}$$

in the sense of non-decreasing norm convergence (cf. [2]). To verify the crucial point in (33), note that

$$\begin{aligned} \|K(\cdot, y) - K_t(x,y)\|^2_{H_K(E)} &= K(y,y) - 2K_t(y,y) + \|K_t(\cdot, y)\|^2_{H_K(E)} \\ &\leq K(y,y) - 2K_t(y,y) + \|K_t(\cdot, y)\|^2_{H_{K_t}(E)} \\ &= K(y,y) - K_t(y,y), \end{aligned}$$

that converges to zero as $t \to +0$. We thus obtain the desired limit property in the theorem.

The uniqueness of the initial value problem follows directly from (32). $\square$

Now, we shall consider the general situation such that K_t exists for all $t > 0$ and but that K does not exist in general.

From these considerations, we formulate a general and abstract result in the next section.

5 The main results

Let E be a set. Assume that we are given a family of reproducing kernel $\{K_t\}_{t>0}$ satisfying $K_{t'} \gg K_t$ for $t' < t$. We wish to introduce a preHilbert space by

$$H_{K_0} := \bigcup_{t>0} H_{K_t}(E).$$

For any $f \in H_{K_0}$, there exists a space $H_{K_t}(E)$ containing the function f for some $t > 0$. Then, for any $t' \in (0, t)$,

$$H_{K_t}(E) \subset H_{K_{t'}}(E)$$

and, for the function $f \in H_{K_t}$,

$$\|f\|_{H_{K_t}(E)} \geq \|f\|_{H_{K_{t'}}(E)}.$$

Therefore, the limit exists :

$$\|f\|_{H_{K_0}} := \lim_{t \downarrow 0} \|f\|_{H_{K_t}(E)}.$$

Denote by H_0 the completion of H_{K_0}. Due to the fact that the normed space H_0 satisfies the parallelogram law, we see that H_0 is a Hilbert space.

Now we give a general application that is our main purpose in this paper and has many concrete applications in L^2 version initial value problems (see many concrete examples in [5, 6, 13]). However, in order to apply Theorem 5.1, we use nondecreasing kernels like (1), (17) and (29) in the sequel.

For the general situation such that K_t exists for all $t > 0$ but that K may fail to exist, Proposition 4.1 is still valid for any function $f \in H_0$.

Theorem 5.1. *Let E be a set and suppose that we are given a family of positive definite functions $\{K_t\}_{t>0}$ such that $K_{t_1} \leq K_{t_2}$ for all $0 < t_2 < t_1$. Then, for all $f \in H_0$, we have*

$$u_f(x, t) := \langle f, K_t(\cdot, x)\rangle_{H_0} \quad (x \in H_0, t > 0) \tag{37}$$

and

$$\lim_{t \to +0} u_f(\cdot, t) := f, \tag{38}$$

in the space H_0.

Proof. Let us check $f_t^* = u_f(\cdot, t) \in H_{K_t}(E)$ for $f \in H_0$. We can check

$$f_t^* \otimes \overline{f_t^*} \ll \|f\|_{H_0}^2 K_t$$

by using (37). Indeed, as we did in [15, page 45],

$$
\begin{aligned}
&\|f\|_{H_0}^2 \sum_j \sum_{j'} C_j \overline{C_{j'}} K_t(x_{j'}, x_j) - \left| \sum_j C_j f_t^*(x_j) \right|^2 \\
&= \|f\|_{H_0}^2 \sum_j \sum_{j'} C_j \overline{C_{j'}} K_t(x_{j'}, x_j) - \left| \left\langle f, \sum_j \overline{C_j} K_t(\cdot, x_j) \right\rangle_{H_0} \right|^2 \\
&\geq \|f\|_{H_0}^2 \sum_j \sum_{j'} C_j \overline{C_{j'}} K_t(x_{j'}, x_j) - \|f\|_{H_0}^2 \left\| \sum_j \overline{C_j} K_t(\cdot, x_j) \right\|_{H_0}^2 \\
&\geq \|f\|_{H_0}^2 \sum_j \sum_{j'} C_j \overline{C_{j'}} K_t(x_{j'}, x_j) - \|f\|_{H_0}^2 \left\| \sum_j \overline{C_j} K_t(\cdot, x_j) \right\|_{H_{K_t}}^2 = 0
\end{aligned}
$$

for any finite number of points $\{x_j\}$ of the set E and for any complex numbers $\{C_j\}$. Therefore $f_t^* \in H_{K_t}(E)$. From this calculation we see that $f_t^* \in H_{K_t}(E)$ and that

$$\|f_t^*\|_{H_{K_t}(E)} \leq \|f\|_{H_0}. \tag{39}$$

The mapping $f \mapsto f_t$ being uniformly bounded, we can assume that $f \in H_{K_r}(E)$ for some $r > 0$. Since $\{K_r(\cdot, q)\}_{q \in E}$ spans a dense subspace of $H_{K_r}(E)$, we may assume that $f = K_r(\cdot, q)$ for some $q \in E$. Let $0 < t < s < r$. Then we have

$$f_t^*(x) = \langle K_r(\cdot, q), K_t(\cdot, x) \rangle_{H_0}$$

and hence

$$\|f_t^*\|_{H_{K_s}(E)} \leq \|f_t^*\|_{H_{K_r}(E)} \leq \|K_r(\cdot, q)\|_{H_0(E)} \leq \|K_r(\cdot, q)\|_{H_{K_s}(E)},$$

where we used (39) for the second inequality.

Let $\{\varphi_\lambda^{(t)}\}_{\lambda \in \Lambda_t}$ be a CONS of $H_{K_t}(E)$, where Λ_t is at most countable. Then we have

$$K_t(\cdot, x) = \sum_{\lambda \in \Lambda_t} \overline{\varphi_\lambda^{(t)}(x)} \varphi_\lambda^{(t)}$$

with the convergence in $H_{K_t}(E)$ for any fixed $x \in E$. Therefore

$$f_t^*(x) = \sum_{\lambda \in \Lambda_t} \varphi_\lambda^{(t)}(x) \overline{\langle K_r(\cdot, q), \varphi_\lambda^{(t)} \rangle_{H_0}} \quad (x \in E).$$

Note that

$$\sum_{\lambda \in \Lambda_t} |\langle K_r(\cdot, q), \varphi_\lambda^{(t)} \rangle_{H_0}|^2 < \infty$$

thanks to the Bessel inequality. This implies

$$f_t^* = \sum_{\lambda\in\Lambda_t} \overline{\langle K_r(\cdot,q), \varphi_\lambda^{(t)}\rangle_{H_0}} \varphi_\lambda^{(t)},$$

where the convergence takes place in the topology of $H_{K_t}(E)$ for any $q \in E$. Inserting this expression into $\langle K_r(\cdot,q), f_t^*\rangle_{H_{K_t}(E)}$, we obtain

$$\langle K_r(\cdot,q), f_t^*\rangle_{H_{K_t}(E)} = \sum_{\lambda\in\Lambda_t} \langle K_r(\cdot,q), \varphi_\lambda^{(t)}\rangle_{H_0} \langle K_r(\cdot,q), \varphi_\lambda^{(t)}\rangle_{H_{K_t}(E)} \tag{40}$$

and

$$\|f_t^*\|_{H_{K_t}(E)} = \sqrt{\sum_{\lambda\in\Lambda_t} |\langle K_r(\cdot,q), \varphi_\lambda^{(t)}\rangle_{H_0}|^2}. \tag{41}$$

We also have

$$\|K_r(\cdot,q)\|_{H_{K_t}(E)} = \sqrt{\sum_{\lambda\in\Lambda_t} |\langle K_r(\cdot,q), \varphi_\lambda^{(t)}\rangle_{H_{K_t}(E)}|^2 (<\infty)} \tag{42}$$

for all $0 < r \le t$. By the Lebesgue convergence theorem, we obtain

$$0 = \limsup_{t\to 0} \|f - f_t^*\|_{H_{K_t}(E)} \ge \limsup_{t\to 0} \|f - f_t^*\|_{H_0} = 0.$$

In the correspondence (22), the space $H_{K_0}(E)$ corresponds to the space H_0 in Theorem 5.1 and the space H_0 is isometric to the space $L^2(I, dm)$. The integral in (29) exists and the function defined by the left hand side in (29) satisfies the partial differential equation (26), because the function $K_t(x', x)$ satisfies it for any fixed x' and it is the summation. Furthermore, the initial value is satisfied as in (38). Thus the proof of Thereom 5.1 is complete. □

The completion space H_0 will be determined, in many conclete cases, from the realizations of the spaces $H_{K_t}(E)$, by case-by-case.

6 Special example

For the simplest derivative operator $D = \dfrac{d}{dx}$, we have, of course,

$$De^{\lambda x} = \lambda e^{\lambda x}. \tag{43}$$

We will be able to see that we can consider initial value problems with various situations by considering consequent λ and the variable x. As typical cases we can handle the weighted Laplace transforms, the Paley-Wiener spaces and the Sobolev spaces depending on $\lambda > 0$, λ being on a symmetric interval or λ on the whole real space.

The Laplace transform may be taken into account in many situations by considering various weights; see [15]. So we consider the simplest case:

$$K(z,\overline{u}) = \int_0^\infty e^{-\lambda z} e^{-\lambda \overline{u}} d\lambda = \frac{1}{z+\overline{u}}, \quad z = x+iy, \tag{44}$$

on the right half complex plane. The reproducing kernel is the Szegö kernel and we have the image of the integral transform

$$f(z) = \int_0^\infty e^{-\lambda z} F(\lambda) d\lambda \quad (\mathrm{Re}(z) > 0), \tag{45}$$

for the $L^2(0,\infty)$ functions $F(\lambda)$. Thus, we obtain the isometric identity

$$\frac{1}{2\pi} \int_{-\infty}^\infty |f(iy)|^2 dy = \int_0^\infty |F(\lambda)|^2 d\lambda. \tag{46}$$

Here, $f(iy)$ stands for the Fatou's non-tangential boundary values of the Szegö space of analytic functions on the right hand-half complex plane.

Now, we will consider the reproducing kernel $K_t(z,\overline{u})$ and the corresponding reproducing kernel Hilbert space $H_{K_t}(\mathbb{C})$ by taking

$$K_t(z,\overline{u}) = \int_0^\infty e^{-\lambda t} e^{-\lambda z} e^{-\lambda \overline{u}} d\lambda. \tag{47}$$

Note that the reproducing kernel Hilbert space $H_{K_t}(\mathbb{C})$ is the Szegö space on the right hand complex plane $x > \frac{-t}{2}$.

For $f \in H_{K_t}(\mathbb{C})$ on the right-half complex plane, the function

$$U_f(t,z) = \langle f, K_t(\cdot,\overline{z})\rangle_{H_K} = \frac{1}{2\pi} \int_{-\infty}^\infty f(iy)\overline{K_t(iy,\overline{z})} dy$$

satisfies the partial differential equation

$$(\partial_t - D_z) U(t,z) = 0. \tag{48}$$

For the sake of the monotonicity of the reproducing kernels, it holds

$$K_t(z,\overline{u}) \ll K(z,\overline{u}); \tag{49}$$

we obtain the desired initial condition:

$$\lim_{t\to 0} U_f(t,z) = \lim_{t\to 0} \langle f, K_t(\cdot,\overline{z})\rangle_{H_{K_t}(\mathbb{C})} = \langle f, K(\cdot,\overline{z})\rangle_{H_K} = f(z)$$

in $H_{K_t}(\mathbb{C})$. From the general property of the reproducing kernels, we see that the above convergence is uniform on any compact subset of the right-half complex plane. Now, by the new Theorem 5.1, for any functions $f \in L^2(i\mathbb{R})$ on the pure imaginary axis we can obtain the corresponding result, and the general version results are valid for many situations; see, for example, [5, 6, 13].

7 Discrete versions

We refer to the discrete version as other typical situation. We will consider an Hermitian polynomial system as a typical case. For the differential operator $P(D) = D^2 - x^2 D$ we know the eigenfunctions u_n and the eigenvalues $\lambda_n = 2n+1$, $n \geq 0$, satisfying the property

$$P(D)u_n(x) = \lambda_n u_n(x). \tag{50}$$

In fact, these eigenfunctions are well known to be

$$u_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_n(x), \tag{51}$$

where we are using the Hermite polynomials

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2). \tag{52}$$

Moreover, the system $\{u_n\}$ is complete and orthonormal on the space $L^2(\mathbb{R})$ endowed with the norm $\|\cdot\|$ which satisfies

$$\|f\| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx} < \infty. \tag{53}$$

We will be able to consider the initial value problem $u_f(x,t)$,

$$(\partial_t + P(D))u_f(x,t) = 0 \quad (t > 0), \qquad u_f(0,x) = f(x),$$

and construct such a solution. Here, the important points are the characterization of the functions space $\{f\}$ and the precise meaning of the initial value

$$\lim_{t \to +0} u_f(x,t) = u_f(0,x) = f(x).$$

The crucial point is a realization of the reproducing kernels generated by the eigenfunctions. For such a concrete purpose, inspired by the interesting books [1, 8, 9, 10, 11], we find the following identity as the reproducing kernel which is generated by the eigenfunctions

$$\begin{aligned} K_r(x,x') &= e^{-\frac{x^2}{2}} e^{-\frac{x'^2}{2}} \sum_{n=0}^{\infty} \frac{H_n(x)H_n(x')}{2^n n!} r^n \\ &= e^{-\frac{x^2}{2}} e^{-\frac{x'^2}{2}} \frac{1}{\sqrt{1-r^2}} \exp\left(\frac{2xx'r - (x^2+x'^2)r^2}{1-r^2}\right) \end{aligned} \tag{54}$$

for $0 \leq r < 1$ (cf. [1, p. 280]). Now we are interested in idealizing the linear transform property, which is induced from the representation (54), for

$$f(x) = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} C_n \frac{H_n(x)}{2^n n!} r^n \tag{55}$$

where we are considering $\ell^2(\mathbb{N}_0)$ sequences $\{C_n\}_{n=0}^{\infty}$ satisfying

$$\sum_{n=0}^{\infty} \frac{|C_n|^2}{2^n n!} r^n < \infty. \tag{56}$$

Doing so, we obtain an isometric identity, because the system $\{H_n\}_{n=0}^{\infty}$ is linearly independent, for the reproducing kernel Hilbert space $H_{K_r}(\mathbb{C})$ admitting the reproducing kernel K_r,

$$\|f\|_{H_{K_r}(\mathbb{C})} = \sqrt{\sum_{n=0}^{\infty} \frac{|C_n|^2}{2^n n!} r^n} < \infty. \tag{57}$$

Meanwhile, we can realize the reproducing kernel Hilbert space $H_{K_r}(\mathbb{C})$ concretely, in a self-contained manner, as follows: At first, the reproducing kernel $K_r(x, x')$ is extended analytically onto the whole complex plane $z = x + iy$ in the form

$$K_r(z, \overline{u}) = e^{-\frac{z^2}{2}} e^{-\frac{\overline{u}^2}{2}} \frac{1}{\sqrt{1-r^2}} \exp\left(\frac{-z^2 r^2}{1-r^2} - \frac{\overline{u}^2 r^2}{1-r^2} + \frac{2rz\overline{u}}{1-r^2} \right). \tag{58}$$

Here, in particular, for any fixed $A > 0$, the kernel $e^{Az\overline{u}}$ is the reproducing kernel on the Fischer (Bergmann) Hilbert space consisting of the entire functions f with finite norms

$$\left(\frac{A}{\pi} \iint_{\mathbb{R}^2} |f(x+iy)|^2 e^{-A(x^2+y^2)} dxdy \right)^{1/2} < \infty.$$

From the basic properties of reproducing kernels about multiplications by positive constants and products of reproducing kernels, we are able to identify the reproducing kernel Hilbert space $H_{K_r}(\mathbb{C})$ admitting the kernel $K_r(z, \overline{u})$; the space $H_{K_r}(\mathbb{C})$ is composed of entire functions f with finite norms

$$\left(\frac{2r}{\pi\sqrt{1-r^2}} \iint_{\mathbb{R}^2} |f(x+iy)|^2 \exp\left(\frac{1-r}{1+r} x^2 + \frac{-(1+r)y^2}{1-r} \right) dxdy \right)^{1/2} < \infty. \tag{59}$$

Meanwhile, from (55), we obtain the representations of $\{C_n\}_{n=0}^{\infty}$, by using the orthogonality of the Hermite polynomials

$$C_n = \frac{1}{r^n \sqrt{\pi}} \int_{-\infty}^{\infty} f(x) H_n(x) e^{-\frac{x^2}{2}} dx. \tag{60}$$

Therefore, we see that the elements of the reproducing kernel Hilbert space $H_{K_r}(\mathbb{C})$ are characterized by the real valued functions satisfying (55) with (60) and this fact will give the analytic extension property of the elements of $H_{K_r}(\mathbb{C})$.

Therefore any member $f \in H_{K_r}(\mathbb{C})$ is represented in the form (51) satisfying (51) and (57), and the function f is extended analytically as an entire function f satisfying (53) as the norm. Then, furthermore, we obtain the isometric identities (53) and (56). In addition, by using these isometric identities, we can obtain the corresponding inversion formulas.

Now, we form the reproducing kernel

$$K_r(x', x; t) = e^{-\frac{x^2}{2}} e^{-\frac{x'^2}{2}} \sum_{n=0}^{\infty} \frac{H_n(x')H_n(x)}{2^n n! \exp(\lambda_n t)} r^n, \quad t > 0, \tag{61}$$

and let us consider the reproducing kernel Hilbert space $H_{K_r(t)}(\mathbb{C})$ admitting the kernel $K_r(\cdot, \cdot; t)$. In particular, for each fixed x, $K_r(\cdot, x; t) \in H_{K_r}(\mathbb{C})$ (however, it is a symmetric function in the first and the second variables) Then, we can obtain the following result.

Proposition 7.1. *For any element $f \in H_{K_r}(\mathbb{C})$, the solution $u_f(x,t)$ of the differential equation*

$$(\partial_t + P(D))u_f(x,t) = 0 \quad (t > 0) \tag{62}$$

satisfying the initial value condition

$$u_f(0, x) = f(x), \tag{63}$$

exists uniquely and it is given by

$$u_f(x,t) = \langle f, K_r(\cdot, x; t)\rangle_{H_{K_r}(\mathbb{C})}. \tag{64}$$

Here, the meaning of the initial value (63) *is given by*

$$\lim_{t\to+0} u_f(x,t) = \lim_{t\to+0} \langle f, K_r(\cdot, x; t)\rangle_{H_{K_r}(\mathbb{C})} = \langle f, K_r(\cdot, x)\rangle_{H_{K_r}(\mathbb{C})} = f(x) \tag{65}$$

(*whose existence is ensured and the limit is considered in the uniform convergence sense on any subset of* $\mathbb{R}$ *such that $K_r(x,x)$ is bounded*).

In our Proposition 7.1, we naturally assume that the initial value function f belongs to the naturally determined reproducing kernel Hilbert space $H_{K_r}(\mathbb{C})$. However, the space may be extended to a naturally determined Hilbert space.

At first, recall the reproducing kernel Hilbert space $H_{K_r}(\mathbb{C})$ and its structure. We will consider the limit $r \uparrow 1$ in (50). Note that

$$K_1(x', x) := e^{-\frac{x^2}{2}} e^{-\frac{x'^2}{2}} \sum_{n=0}^{\infty} \frac{H_n(x')H_n(x)}{2^n n!} \tag{66}$$

is not a usual function, however, this is an expansion in terms of the complete orthonormal system

$$\left\{e^{-\frac{x^2}{2}} H_n(x)\right\}_{n=0}^{\infty}$$

in the Hilbert space $L^2(\mathbb{R})$ with the norm (53) in the symmetric form (as in reproducing kernel forms). Recall the Parseval identity and the inversion formula in the representation of the functions in the Hilbert space framework. This means that the given kernel form $K_1(\cdot, x)$ looks like the distribution $\delta(\cdot - x)$ and it is a reproducing kernel in the sense that

$$f(x) = \langle f, \delta(\cdot - x)\rangle_{L^2(\mathbb{R})} \tag{67}$$

in the Hilbert space $L^2(\mathbb{R})$.

Furthermore, for any $r \leq r' < 1$,

$$K_r(x, x') \ll K_{r'}(x, x'); \tag{68}$$

and hence $H_{K_r}(\mathbb{C}) \subset H_{K_{r'}}(\mathbb{C})$. For any function $f \in H_{K_r}(\mathbb{C})$,

$$\|f\|_{H_{K_r}(\mathbb{C})} = \lim_{r \uparrow r'} \|f\|_{H_{K_{r'}}(\mathbb{C})}$$

in the sense of non-decreasing norm-convergence. However, at the present case, K_1 is not a usual function, but it is determined as an increasing limit in the above sense of reproducing kernels.

From these considerations, we have

Theorem 7.2. *Proposition 7.1 is also valid for any function $f \in L^2(\mathbb{R})$ in the sense that*

$$u_f(x, t) := \langle f, K(\cdot, x; t)\rangle_{L^2(\mathbb{R})} \tag{69}$$

and

$$\lim_{t \to +0} u_f(\cdot, t) = f, \tag{70}$$

in $L^2(\mathbb{R})$.

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