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Assessing Capital Investment Strategy with Quadratic Adjustment Cost under Ambiguity*

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1 Introduction

The uncertainty of the business environment is increasing more and more. Firms' managers face complex business environments and the difficulty of predicting likely future outcomes. How they treat uncertainty is important in business decision making, such as the growth of a capital investment. In this paper, we consider a firm's investment problem under uncertainty. In particular, we focus on a certain type of uncertainty to incorporate the unpredictable business environment. We consider the firm's investment problem under ambiguity, which is also termed Knightian uncertainty. The probability of an outcome is not uniquely determined under ambiguity or Knightian uncertainty (Knight, 1921)[1]. A number of papers study decision making under ambiguity (Camerer and Weber, 1992; Etner et. al., 2012; Guidolin and Rinaldi, 2013).

Suppose that a firm produces a single output and sells it in a market. The firm's problem is to decide the production capital investment rate to maximize its profit as in Abel and Eberly (1997). Investing in the capital requires a quadratic-type adjustment cost in addition to the purchase price, which is assumed to be constant. In this paper, we consider the case in which the firm's manager cannot predict the future price of the output precisely. To be more precise, he cannot uniquely identify the probability distribution of the output price. Then, he has to determine the investment strategy under output price ambiguity. In Abel and Eberly (1997), the firm's manager can uniquely identify the distribution of the output price. This paper is an extension of the research of Abel and Eberly (1997) by incorporating ambiguity. In order to reflect the misspecification of the model, we use robust control techniques developed by Hansen and Sargent (2001), Hansen et al. (2002), and Hansen et al. (2006). These techniques are based on the multiple priors framework by Gilboa and Schmeidler (1989). We formulate the firm's problem as a robust control problem and show that the equation derives the optimal investment strategy.

This paper is also related to Tsujimura (2014, 2015). These papers examined investment problems under ambiguity in a two-period setting as in Miao (2004), which investigates optimal consumption under ambiguity. Tsujimura (2015) examines the pollutant abatement investment in a production economy by including investments in pollutant abatement capital into Tsujimura (2014), which examines capital investment.

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[1]Forty years later, in 1961, Ellsberg showed that decision-makers are not always able to derive a unique probability distribution (Ellsberg, 1961). Since Ellsberg's seminal paper, uncertain environments have become better known as being ambiguous.

The rest of the paper is organized as follows. In Section 2, we describe the setup of the firm's investment problem. In Section 3, we solve the firm's problem. Section 4 concludes the paper.

2 The Model

In this section, we set up a firm's investment problem. Suppose that a firm produces a single output by using capital K_t and labor L_t and sells it in a market. The firm's production function $F(L_t, K_t)$ takes the Cobb–Douglas form:

$$F(L_t, K_t) = L_t^{\gamma} K_t^{1-\gamma}, \tag{2.1}$$

where $\gamma \in (0,1)$ is the output elasticity of labor. The dynamics of the capital K_t is governed by:

$$\mathrm{d}K_t = (I_t - \delta K_t)\mathrm{d}t, \qquad K_0 = k, \tag{2.2}$$

where I_t is the investment rate at time t and $\delta \in (0,1)$ is the depreciation rate. When the firm invests in capital, it incurs the cost $C(I_t)$:

$$C(I_t) = c_0 I_t + \frac{1}{2} c_1 I_t^2, \tag{2.3}$$

where $c_0 > 0$ is the price of purchasing capital and $c_1 > 0$ is the quadratic adjustment cost parameter[2]. c_0 and c_1 are assumed to be constant. The output price, P_t, is governed by the following stochastic differential equation:

$$\mathrm{d}P_t = \mu P_t \mathrm{d}t + \sigma P_t \mathrm{d}W_t, \quad P_0 = p > 0, \tag{2.4}$$

where $\mu > 0$ and $\sigma > 0$ are constants. W_t is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, where $\mathcal{F}_t$ is generated by W_t.

In this paper, we consider the case in which the firm's manager does not have perfect confidence about the distribution of the output price. He is concerned about the robustness of his decisions to misspecification of the model. Then, he considers a set of possible probability measures, $\mathcal{P}$, on $(\Omega, \mathcal{F})$. The size of $\mathcal{P}$ is determined by a relative entropy[3]. Every element in $\mathcal{P}$ is equivalent to $\mathbb{P}$. Let $\mathbb{Q} \in \mathcal{P}$ be the distorted measure chosen by the firm's manager. Then, the measure $\mathbb{P}$ is replaced by the probability measure $\mathbb{Q}$.

As in Kleshchelski and Vincent (2007), we derive the output price process under the probability measure $\mathbb{Q}$. Let h_t be the measurable drift distortion and assume that $\int_0^\infty h_s^2 \mathrm{d}s < \infty$, $h \in \mathcal{H}$, where $\mathcal{H}$ is the set of all h such that the process $\xi^{\mathbb{Q}}$ is defined by:

$$\xi_t^{\mathbb{Q}} = \exp\left\{ \int_0^t h_s \mathrm{d}W_s - \frac{1}{2}\int_0^t h_s^2 \mathrm{d}s \right\}. \tag{2.5}$$

$\xi^{\mathbb{Q}}$ is a $\mathbb{P}$-martingale. The drift distortion h defines the probability measure $\mathbb{Q} \in \mathcal{P}$. $\xi^{\mathbb{Q}}$ is also the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$:

$$\xi_t^{\mathbb{Q}} = \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right] \tag{2.6}$$

[2]In Abel and Eberly (1997), the cost function is formulated as $C(I) = c_0 + c_1 I^n$, $n = \{2, 4, 6, \cdots\}$.

[3]This is also termed the Kullback–Leibler divergence.

By Girsanov's theorem, for all $h \in \mathcal{H}$ a Brownian motion $W_t^{\mathbb{Q}}$ under $\mathbb{Q}$ is given by:

$$W_t^{\mathbb{Q}} = W_t - \int_0^t h_s \mathrm{d}s, \tag{2.7}$$

From (2.5) and (2.7) we obtain that:

$$\xi_t^{\mathbb{Q}} = \exp\left\{\int_0^t h_s \mathrm{d}W_s^{\mathbb{Q}} + \frac{1}{2}\int_0^t h_s^2 \mathrm{d}s\right\}. \tag{2.8}$$

Then, the output price dynamics under the probability $\mathbb{Q}$ is given by:

$$\mathrm{d}P_t = (\mu + \sigma h_t)P_t \mathrm{d}t + \sigma P_t \mathrm{d}W_t^{\mathbb{Q}}, \quad P_0 = p > 0, \tag{2.9}$$

As in Hansen et al. (2002), Skiadas (2003), and Hansen et al. (2006), the difference between $\mathbb{P}$ and $\mathbb{Q}$ is measured by the relative entropy[4]:

$$\begin{aligned} R(\mathbb{Q}) &= r\int_0^\infty \mathrm{e}^{-rt}\left(\int \log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\mathrm{d}\mathbb{Q}\right)\mathrm{d}t \\ &= \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty \mathrm{e}^{-rt}\frac{h_t^2}{2}\mathrm{d}t\right] \end{aligned} \tag{2.10}$$

The firm's operating profit at t is given by:

$$P_t F(L_t, K_t) - wL_t, \tag{2.11}$$

where $w > 0$ is a constant wage. Labor is assumed to be costlessly and instantaneously adjusted. Then, the firm's maximized instantaneous operating profit at t, $\pi(K_t, P_t)$, is calculated as:

$$\pi(K_t, P_t) = \eta P_t^\alpha K_t, \tag{2.12}$$

where $\alpha = 1/(1-\gamma) > 1$ and $\eta = \alpha^{-\alpha}(\alpha-1)^{\alpha-1}w^{1-\alpha} > 0$.

Therefore, the firm's problem is to choose the investment rate at each time so as to maximize the expected firm's net profit even though the worst possible drift distortion h occurs and is formulated as the multiplier robust control model[5]:

$$V(k,p) = \max_{\{I_t\}}\min_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty \mathrm{e}^{-rt}[\pi(K_t, X_t) - C(I_t)]\mathrm{d}t + \theta R(\mathbb{Q})\right], \tag{2.13}$$

where $\theta \geq 0$ is the multiplier on the relative entropy penalty. θ can measure how much the firm's manager weights the possibility of $\mathbb{P}$ not being the correct distribution. That is, θ implies the firm's manager's sensitivity to ambiguity. A lower value of θ means the manager is more

[4]See also Funke and Paetz (2011) for the relationship between $\mathbb{P}$ and $\mathbb{Q}$ in Hansen-Sargent robust control techniques

[5]The firm's problem can be also written as the constraint robust control model:

$$V(k,p) = \max_{\{I_t\}}\min_{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty \mathrm{e}^{-rt}[\pi(K_t, X_t) - C(I_t)]\mathrm{d}t\right],$$
$$\text{s.t.} \quad R(\mathbb{Q}) \leq \zeta,$$

where ζ is the maximum specification error that the firm's manager is willing to accept. See Hansen et al. (2002) for more detail.

fearful of model misspecification. So he chooses $\mathbb{Q}$ further away from $\mathbb{P}$ in the relative entropy sense, that is, the size of $\mathcal{P}$ increases as θ decreases.

Combining (2.10) and (2.13) the firm's problem can be written as:

$$V(k,p) = \max_{\{I_t\}} \min_{\{h_t\}\in\mathcal{H}} \mathbb{E}_{\mathbb{Q}} \left[\int_0^\infty \mathrm{e}^{-rt} \left[\eta P_t^\alpha K_t - \left(c_0 I_t + \frac{1}{2} c_1 I_t^2 \right) + \theta \frac{h_t^2}{2} \right] \mathrm{d}t \right]. \tag{2.14}$$

3 Optimal Capital Investment

In this section, we solve the firm's problem (2.14) and derive the optimal capital investment strategy.

It follows from the Bellman–Isaacs condition that the value function of the firm's problem (2.14) satisfies:

$$\begin{aligned} rV(k,p;\theta) = \max_I \min_h & \left[\left(\eta p^\alpha k - \left(c_0 I + \frac{1}{2} c_1 I^2 \right) + \theta \frac{h^2}{2} \right) \right. \\ & \left. + (I - \delta k) V_k(k,p;\theta) + (\mu + \sigma h) p V_p(k,p;\theta) + \sigma^2 p^2 \frac{1}{2} V_{pp}(k,p;\theta) \right]. \end{aligned} \tag{3.1}$$

See Fleming and Souganidis (1989) and Hansen et al. (2002) for more detail. The first-order conditions for I and h are:

$$I = \frac{1}{c_1}(V_k - c_0), \tag{3.2}$$

$$h = -\frac{\sigma p V_p}{\theta}. \tag{3.3}$$

It follows from (3.3) that h goes to 0 as θ goes to ∞. This implies that the firm's manager acts as if he knows the model with certainty and there are no robustness concerns, when θ goes to ∞ (Roseta-Palma and Xepapadeas, 2004).

As in Abel and Eberly (1997) and Chang (2004, §5.3), we assume that the value function is a linear function of the capital. Then, a guess solution to (3.1) is formulated as:

$$V(k,p) = G(p)k + H(p). \tag{3.4}$$

The guess solution implies that the expected firm's value is the sum of the expected value of the existing capital, $G(p)k$ and the expected value of the newly invested capital, $H(p)$. Note that the shadow price of the capital $V_k(k,p)$ is equal to $G(p)$

Substituting (3.4) into (3.1), we obtain that:

$$\begin{aligned} rG(p)k + rH(p) =& \eta p^\alpha k - (c_0 I + \frac{1}{2} c_1 I^2) + \theta \frac{h^2}{2} + IG(p) - \delta k G(p) \\ & + (\mu + \sigma h) p G'(p) k + (\mu + \sigma h) p H'(p) + \frac{1}{2} \sigma^2 p^2 G''(p) k + \frac{1}{2} \sigma^2 p^2 H''(p). \end{aligned} \tag{3.5}$$

Separating (3.5) into the terms with k and the terms without k, we obtain the following two differential equations:

$$\eta p^\alpha - (r+\delta) G(p) + (\mu + \sigma h) p G'(p) + \frac{1}{2} \sigma^2 p^2 G''(p) = 0, \tag{3.6}$$

$$IG(p) - \left(c_0 I + \frac{1}{2}c_1 I^2\right) + \theta\frac{h^2}{2} - rH(p) + (\mu + \sigma h)pH'(p) + \frac{1}{2}\sigma^2 p^2 H''(p) = 0. \tag{3.7}$$

A general solution to (3.6) is given by:

$$G(p) = A_1 p^{\beta_1} + A_2 p^{\beta_2} + B\eta p^{\alpha}. \tag{3.8}$$

The first two terms of the right-hand side are solutions to the homogeneous part of (3.6). We set $A_1 = A_2 = 0$ to rule out bubbles on the shadow price of installed capital. Then, the general solution is reduced to the particular solution:

$$G(p) = B\eta p^{\alpha}. \tag{3.9}$$

Substituting (3.9) into (3.6) yields:

$$B = \left[(r + \delta) + (\mu + \sigma h)\alpha - \frac{1}{2}\sigma^2 \alpha(\alpha - 1)\right]^{-1} \tag{3.10}$$

It follows from $B > 0$ that we obtain:

$$r + \delta > \frac{1}{2}\sigma^2 \alpha(\alpha - 1) - (\mu + \sigma h)\alpha \tag{3.11}$$

From (3.2) and (3.4), we obtain:

$$I = \frac{1}{c_1}(G(p) - c_0) \tag{3.12}$$

Then, substituting (3.3) and (3.12) into (3.7), we obtain:

$$\begin{aligned} &\frac{c_0}{c_1}\left(\frac{c_0}{2} - 1\right) G(p) + \frac{1}{c_1}(1 - c_0 + c_0 c_1)G(p)^2 - \frac{1}{2c_1}G(p)^3 \\ &+ \frac{\sigma^2}{2\theta}p^2 G'(p)^2 k^2 - rH(p) + \mu p H'(p) - \frac{\sigma^2}{2\theta}p^2 H'(p)^2 + \frac{1}{2}\sigma^2 p^2 H''(p) = 0 \end{aligned} \tag{3.13}$$

The optimal investment rate is derived from the nonlinear differential equation (3.13).

4 Conclusion

In this paper, we analyze capital investment strategy with the quadratic adjustment cost when the firm faced output price ambiguity. We obtain the differential equation, which derives the optimal investment strategy. Because the differential equation is nonlinear, it is solved numerically. We leave the numerical calculation for future work.

There are several ways to extend this paper. We could consider the firm's attitude to risk by using utility function as in Sandmo (1971). We also could investigate a social welfare by considering a production economy as in Tsujimura (2015). Furthermore, we could incorporate technological progress as well. These important topics are left to future research.

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