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Applications of the Laplace-Carson Transform to Option Pricing: A Tutorial*

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1 Introduction

We provide a brief tutorial how to apply the *Laplace-Carson transform* (LCT) to option pricing. The LCT is a variant of the *Laplace transform* (LT), and it is named after John Renshaw Carson (1886–1940), a telecommunication engineer at AT&T Bell Labs. The LCT has been applied particularly in the fields of physics and railway engineering. Integral transforms such as Fourier, Fourier-Bessel, Mellin, Hilbert and Laplace are powerful tools for analyzing dynamical systems formulated by ordinary/partial differential equations. Option pricing problem formulated by a partial differential equation (PDE) is a natural application of the integral transforms.

Among those integral transforms, we primarily focus on the LCT and its applications to pricing various American options in this tutorial, because the LCT is equivalent to the price of an American option with random maturity; see Carr [7]. Carr's procedure is referred to as the *randomization* approach, which is a special case in the general framework of randomization of Feller [14, Chapter II]. From the view point of a tutorial, we particularly emphasize the importance of an elementary European vanilla option in pricing more complex options.

This tutorial is organized as follows: In Section 2, we define the LCT formally and summarize its basic properties. In Section 3, we introduce a basic stochastic framework of the underlying asset process. In Section 4, the main section of this tutorial, we describe the LCT approach, starting from European vanilla options and going to American vanilla options, exchange options, Russian options, and continuous-installment options. Further extensions to barrier options (Avram et al. [5], Petrella and Kou [38]), lookback options (Kimura [26]), Asian options (Geman and Yor [17]) and other derivatives (Hayashi et al. [19]) are possible, but they are omitted due to the page restriction. Finally, in Section 5, we give issues on deck for the LCT approach to option pricing.

2 Laplace-Carson transform

Let $f(x)$ be a continuous real-valued function for $x \in \mathbb{R}_+ = [0, \infty)$, and assume that $|f(x)| \leq Ae^{Bx}$ $(x \geq 0)$ for constants A and B. Then, for $\lambda \in \mathbb{C}$ $(\mathrm{Re}(\lambda) > B)$, define the LT of f as

$$\mathcal{L}[f(x)](\lambda) \equiv \int_0^\infty \mathrm{e}^{-\lambda x} f(x)\mathrm{d}x. \tag{1}$$

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For $f(x)$ specified above, define its LCT as

$$\mathcal{LC}[f(x)](\lambda) \equiv \lambda \mathcal{L}[f(x)](\lambda) = \int_0^\infty \lambda \mathrm{e}^{-\lambda x} f(x) \mathrm{d}x. \tag{2}$$

Denote $\mathcal{LC}[f(x)](\lambda) = f^*(\lambda)$ for simplicity. For a random variable $X \sim Exp(\lambda)$, the LCT can be interpreted as $f^*(\lambda) = \mathbb{E}[f(X)]$. For a constant A, the LCT is an identity map, i.e., $\mathcal{LC}[A](\lambda) = A$, whereas $\mathcal{L}[A](\lambda) = A/\lambda$. This invariant property of the LCT is important to generate much simpler formulas for option prices than the LT.

From the definition (2), we can derive some useful identities of the LCT, in the same manner as for the LT case; check the differences between the LT and LCT.

Proposition 2.1 (Useful Identities)

1. $\mathcal{LC}\left[\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right](\lambda) = \lambda\big(f^*(\lambda) - f(0+)\big)$
2. $\mathcal{LC}\left[f^{(n)}(x)\right](\lambda) = \lambda^n f^*(\lambda) - \sum_{k=0}^{n-1} \lambda^{n-k} f^{(k)}(0+), \quad n \geq 1$
3. $\mathcal{LC}\left[\int_0^x f(y)\mathrm{d}y\right](\lambda) = \frac{1}{\lambda} f^*(\lambda)$
4. $\mathcal{LC}\left[\frac{f(x)}{x}\right](\lambda) = \lambda \int_\lambda^\infty \frac{f^*(s)}{s} \mathrm{d}s$
5. $\mathcal{LC}\left[f(x-a)\mathbf{1}_{\{x \geq a\}}(x)\right](\lambda) = \mathrm{e}^{-a\lambda} f^*(\lambda), \quad a > 0$
6. $\mathcal{LC}\left[\mathrm{e}^{ax} f(x)\right](\lambda) = \frac{\lambda}{\lambda - a} f^*(\lambda - a)$
7. $\mathcal{LC}\left[f\left(\frac{x}{a}\right)\right](\lambda) = f^*(a\lambda), \quad a > 0$
8. $\mathcal{LC}\left[\int_0^x f_1(y) f_2(x-y)\mathrm{d}y\right](\lambda) = \frac{1}{\lambda} f_1^*(\lambda) f_2^*(\lambda)$

Proposition 2.2 (Bromwich Integral)

$$f(x) = \mathcal{LC}^{-1}[f^*(\lambda)](x) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{\lambda x} \frac{f^*(\lambda)}{\lambda} \mathrm{d}\lambda = \mathcal{L}^{-1}\left[\frac{f^*(\lambda)}{\lambda}\right](x), \quad x > 0, \tag{3}$$

where a is a real number such that the contour path of integration is in the region of convergence of $f^*(\lambda)$.

Proposition 2.3 (Abelian Theorem)

$$\lim_{x \to 0} f(x) = \lim_{\lambda \to \infty} f^*(\lambda), \tag{4}$$

$$\lim_{x \to \infty} f(x) = \lim_{\lambda \to 0} f^*(\lambda). \tag{5}$$

3 Basic Framework

For an economy with finite time period $[0, T]$, consider an underlying asset with price process $(S_t)_{t\geq 0}$. For given S_0, assume that $(S_t)_{t\geq 0}$ is a risk-neutralized diffusion process with the Black-Scholes-Merton dynamics

$$\frac{\mathrm{d}S_t}{S_t} = (r - \delta)\mathrm{d}t + \sigma \mathrm{d}W_t, \quad t \in [0, T], \tag{6}$$

where $r > 0$ is the risk-free rate of interest, $\delta \geq 0$ is the continuous dividend rate, and $\sigma > 0$ is the volatility coefficient of $(S_t)_{t\geq 0}$. Assume that all of these coefficients (r, δ, σ) are constant. $(W_t)_{t\geq 0}$ denotes a one-dimensional standard Brownian motion process defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to $(W_t)_{t\geq 0}$, the probability measure $\mathbb{P}$ is chosen so that each of assets has mean rate of return r, and the conditional expectation $\mathbb{E}_t[\,\cdot\,] \equiv \mathbb{E}[\,\cdot\,|\mathcal{F}_t]$ is calculated under the risk-neutral probability measure $\mathbb{P}$.

4 Vanilla Options [24]

4.1 European options

Consider a European call option written on $(S_t)_{t\geq 0}$. Let

$$c(t, S_t) \equiv c(t, S_t; K, r, \delta)$$

denote the value of the European call option at time $t \in [0, T]$, which has strike price K and maturity date T. Then, it has been well known that

$$\begin{aligned} c(t, S_t) &= \mathbb{E}_t[\mathrm{e}^{-(T-t)}(S_T - K)^+] \\ &= S_t \mathrm{e}^{-\delta\tau}\Phi\big(d_+(S_t, K, \tau)\big) - K\mathrm{e}^{-r\tau}\Phi\big(d_-(S_t, K, \tau)\big), \end{aligned}$$

where $\tau = T - t$, $\Phi(\,\cdot\,)$ is the standard normal cdf, and

$$d_\pm(x, y, \tau) = \frac{\log(x/y) + (r - \delta \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Let $\mathcal{D} = [0, T] \times \mathbb{R}_+$ and $S \equiv S_t$ for abbreviation. Then, the call value $c(t, S)$ satisfies the Black-Scholes-Merton PDE

$$\frac{\partial c}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r - \delta)S\frac{\partial c}{\partial S} - rc = 0, \quad (t, S) \in \mathcal{D}, \tag{7}$$

with the boundary conditions

$$\lim_{S\to 0} c(t, S) = 0 \quad \text{and} \quad \lim_{S\to\infty} \frac{\partial c}{\partial S} < \infty, \tag{8}$$

and the terminal condition

$$c(T, S) = (S - K)^+.$$

For the remaining time to maturity $\tau = T - t$, define the *time-reversed value*

$$\widetilde{c}(\tau, S) = c(T - \tau, S) = c(t, S),$$

and its LCT with respect to τ as

$$c^*(\lambda, S) = \mathcal{LC}[\widetilde{c}(\tau, S)](\lambda) = \int_0^\infty \lambda e^{-\lambda\tau}\widetilde{c}(\tau, S)d\tau.$$

Then, the LCT $c^*(\lambda, S)$ satisfies the ordinary differential equation (ODE)

$$\tfrac{1}{2}\sigma^2 S^2 \frac{dc^*}{dS^2} + (r - \delta)S\frac{dc^*}{dS} - (\lambda + r)c^* + \lambda(S - K)^+ = 0, \tag{9}$$

together with the boundary conditions

$$\lim_{S\to 0} c^*(\lambda, S) = 0 \quad \text{and} \quad \lim_{S\to\infty} \frac{dc^*}{dS} < \infty. \tag{10}$$

Solving the ODE (9) with (10) and simplifying the terms, we have

Theorem 4.1 *The LCT $c^*(\lambda, S)$ for the European vanilla call option value is given by*

$$c^*(\lambda, S) = \begin{cases} \xi_1(S), & S < K \\ \xi_2(S) + \dfrac{\lambda S}{\lambda + \delta} - \dfrac{\lambda K}{\lambda + r}, & S \geq K, \end{cases}$$

where for $i = 1, 2$

$$\xi_i(S) = \frac{2}{\sigma^2} \frac{\lambda K}{\theta_i(\theta_i - 1)(\theta_1 - \theta_2)} \left(\frac{S}{K}\right)^{\theta_i},$$

and $\theta_i \equiv \theta_i(\lambda; r, \delta, \sigma^2)$ ($\theta_1 > 1$ and $\theta_2 < 0$) are two real roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\theta - (\lambda + r) = 0.$$

Proof. See Kimura [24, Theorem 3.2]. The expression for $\xi_i(S)$ given here is actually simpler than [24], using the relations $\lambda + r = -\frac{1}{2}\sigma^2\theta_1\theta_2$ and $r - \delta = -\frac{1}{2}\sigma^2(\theta_1 + \theta_2 - 1)$. □

Let $p(t, S)$ denote the European put option value associated with $c(t, S)$, and denote its time-reversed value by $\widetilde{p}(\tau, S)$. Then, we obtain

Theorem 4.2 *The LCT $p^*(\lambda, S) = \mathcal{LC}[\widetilde{p}(\tau, S)](\lambda)$ for the European vanilla put option value is given by*

$$p^*(\lambda, S) = \begin{cases} \xi_1(S) + \dfrac{\lambda K}{\lambda + r} - \dfrac{\lambda S}{\lambda + \delta}, & S < K \\ \xi_2(S), & S \geq K. \end{cases}$$

Corollary 4.3 *Between $c^*(\lambda, S)$ and $p^*(\lambda, S)$, there exists a relation such that*

$$c^*(\lambda, S) - p^*(\lambda, S) = \frac{\lambda S}{\lambda + \delta} - \frac{\lambda K}{\lambda + r} \tag{11}$$

for $S \geq 0$, which is the LCT version of the put-call parity relation

$$c(t, S) - p(t, S) = Se^{-\delta(T-t)} - Ke^{-r(T-t)}. \tag{12}$$

4.2 American options

Consider the associated American call option written on $(S_t)_{t\geq 0}$. Let

$$C(t,S_t) \equiv C(t,S_t;K,r,\delta)$$

denote the value of the American call option at time $t \in [0,T]$. Then, the value $C(t,S_t)$ is given by solving an *optimal stopping problem*

$$C(t,S_t) = \operatorname*{ess\,sup}_{\tau_e\in[t,T]} \mathbb{E}_t\left[\mathrm{e}^{-r(\tau_e-t)}(S_{\tau_e}-K)^+\right],$$

where τ_e is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the random variable $\tau_e^* \in [t,T]$ is called an *optimal stopping time* if it gives the supremum value of $C(t,S_t)$.

Solving the optimal stopping problem is equivalent to finding the points (t,S_t) for which early exercise is optimal. Let $\mathcal{E}$ and $\mathcal{C}$ denote the *exercise region* and *continuation region*, respectively. The exercise region $\mathcal{E}$ is defined by

$$\mathcal{E} = \left\{(t,S_t) \in \mathcal{D} \,|\, C(t,S_t) = (S_t-K)^+\right\}.$$

Of course, the continuation region $\mathcal{C}$ is the complement of $\mathcal{E}$ in $\mathcal{D}$. The boundary that separates $\mathcal{E}$ from $\mathcal{C}$ is referred to as the *early exercise boundary* (EEB), which is defined by

$$B_c(t) = \sup\left\{S_t \in \mathbb{R}_+ \,|\, C(t,S_t) = (S_t-K)^+\right\}, \quad t\in[0,T].$$

Mckean [35] showed that $C(t,S)$ and $B_c(t)$ can be jointly obtained by solving a *free boundary problem*, which is specified by the Black-Scholes-Merton PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + (r-\delta)S\frac{\partial C}{\partial S} - rC = 0, \quad (t,S)\in\mathcal{C}, \tag{13}$$

together with the boundary conditions

$$\left|\begin{array}{l} \lim\limits_{S\to 0} C(t,S) = 0 \\[2ex] \lim\limits_{S\to B_c(t)} C(t,S) = B_c(t) - K \\[2ex] \lim\limits_{S\to B_c(t)} \dfrac{\partial C}{\partial S} = 1, \end{array}\right. \tag{14}$$

and the terminal condition

$$C(T,S) = (S-K)^+. \tag{15}$$

Using (13) through (15), we can derive an ODE for the LCT $C^*(\lambda,S) = \mathcal{LC}[\widetilde{C}(\tau,S)](\lambda)$, from which we obtain

Theorem 4.4 *The LCT $C^*(\lambda,S)$ for the American vanilla call option value is given by*

$$C^*(\lambda,S) = \begin{cases} S-K, & S\geq B_c^* \\ c^*(\lambda,S)+\pi_c^*(\lambda,S), & S< B_c^*, \end{cases}$$

where

$$\pi_c^*(\lambda, S) = \frac{1}{\theta_1}\left\{\frac{\delta}{\lambda+\delta}B_c^* - \theta_2\xi_2(B_c^*)\right\}\left(\frac{S}{B_c^*}\right)^{\theta_1}, \quad S < B_c^*,$$

and $B_c^* \equiv B_c^*(\lambda) = \mathcal{LC}[\widetilde{B}_c(\tau)](\lambda)$ $(\geq K)$ *is a unique positive solution of the functional equation*

$$\lambda\left(\frac{B_c^*}{K}\right)^{\theta_2} + \delta\theta_2\frac{B_c^*}{K} + r(1-\theta_2) = 0. \tag{16}$$

Corollary 4.5 *For* $S < B_c^*$*,*

$$\Delta_{\pi_c}^* \equiv \mathcal{LC}\left[\frac{\partial\widetilde{\pi}_c}{\partial S}\right](\lambda) = \frac{1}{S}\left\{\frac{\delta}{\lambda+\delta}B_c^* - \theta_2\xi_2(B_c^*)\right\}\left(\frac{S}{B_c^*}\right)^{\theta_1} > 0,$$

$$\Gamma_{\pi_c}^* \equiv \mathcal{LC}\left[\frac{\partial^2\widetilde{\pi}_c}{\partial S^2}\right](\lambda) = \frac{\theta_1-1}{S^2}\left\{\frac{\delta}{\lambda+\delta}B_c^* - \theta_2\xi_2(B_c^*)\right\}\left(\frac{S}{B_c^*}\right)^{\theta_1} > 0,$$

$$\Theta_{\pi_c}^* \equiv -\mathcal{LC}\left[\frac{\partial\widetilde{\pi}_c}{\partial\tau}\right](\lambda) = -\frac{\lambda}{\theta_1}\left\{\frac{\delta}{\lambda+\delta}B_c^* - \theta_2\xi_2(B_c^*)\right\}\left(\frac{S}{B_c^*}\right)^{\theta_1} < 0.$$

For the put case, we also have

Theorem 4.6 *The LCT* $P^*(\lambda, S)$ *for the American vanilla put option value is given by*

$$P^*(\lambda, S) = \begin{cases} K - S, & S \leq B_p^* \\ p^*(\lambda, S) + \pi_p^*(\lambda, S), & S > B_p^*, \end{cases}$$

where

$$\pi_p^*(\lambda, S) = -\frac{1}{\theta_2}\left\{\frac{\delta}{\lambda+\delta}B_p^* + \theta_1\xi_1(B_p^*)\right\}\left(\frac{S}{B_p^*}\right)^{\theta_2}, \quad S > B_p^*,$$

and $B_p^* \equiv B_p^*(\lambda) = \mathcal{LC}[\widetilde{B}_p(\tau)](\lambda)$ $(\leq K)$ *is a unique positive solution of the functional equation*

$$\lambda\left(\frac{B_p^*}{K}\right)^{\theta_1} + \delta\theta_1\frac{B_p^*}{K} + r(1-\theta_1) = 0. \tag{17}$$

Corollary 4.7 *For* $S > B_p^*$*,*

$$\Delta_{\pi_p}^* \equiv \mathcal{LC}\left[\frac{\partial\widetilde{\pi}_p}{\partial S}\right](\lambda) = -\frac{1}{S}\left\{\frac{\delta}{\lambda+\delta}B_p^* + \theta_1\xi_1(B_p^*)\right\}\left(\frac{S}{B_p^*}\right)^{\theta_2} < 0,$$

$$\Gamma_{\pi_p}^* \equiv \mathcal{LC}\left[\frac{\partial^2\widetilde{\pi}_p}{\partial S^2}\right](\lambda) = \frac{1-\theta_2}{S^2}\left\{\frac{\delta}{\lambda+\delta}B_p^* + \theta_1\xi_1(B_p^*)\right\}\left(\frac{S}{B_p^*}\right)^{\theta_2} > 0,$$

$$\Theta_{\pi_p}^* \equiv -\mathcal{LC}\left[\frac{\partial\widetilde{\pi}_p}{\partial\tau}\right](\lambda) = \frac{\lambda}{\theta_2}\left\{\frac{\delta}{\lambda+\delta}B_p^* + \theta_1\xi_1(B_p^*)\right\}\left(\frac{S}{B_p^*}\right)^{\theta_2} < 0.$$

McDonald [34] proved that a *symmetric relation* holds between the call and put values, *i.e.*,

$$C(t, S_t; K, r, \delta) = P(t, K; S_t, \delta, r). \tag{18}$$

Also, between the EEBs $B_c(t) \equiv B_c(t; r, \delta)$ and $B_p(t) \equiv B_p(t; r, \delta)$, Carr and Chesney [6] proved a symmetric relation

$$B_c(t; r, \delta) = \frac{K^2}{B_p(t; \delta, r)}, \quad t \in [0, T] \tag{19}$$

For the LCTs of the option values and EEBs, there exist symmetric relations similar to (18) and (19); see Kimura [29, Theorem 1] for the proof.

Theorem 4.8

1. *Between the LCTs* $C^*(\lambda, S) \equiv C^*(\lambda, S; K, r, \delta)$ *and* $P^*(\lambda, S) \equiv P^*(\lambda, S; K, r, \delta)$, *there exists a symmetric relation such that*

$$C^*(\lambda, S; K, r, \delta) = P^*(\lambda, K; S, \delta, r). \tag{20}$$

2. *Between the LCTs* $B_c^*(\lambda) \equiv B_c^*(\lambda; r, \delta)$ *and* $B_p^*(\lambda) \equiv B_p^*(\lambda; r, \delta)$, *there exists a symmetric relation such that*

$$B_c^*(\lambda; r, \delta) B_p^*(\lambda; \delta, r) = K^2. \tag{21}$$

With the aid of Abelian theorem in Proposition 2.3, we can derive some asymptotic results for the LCTs as follows:

Proposition 4.1 *For the EEBs of the American call and put options, we have*

$$B_c(T) = \max\left(\frac{r}{\delta}, 1\right) K \quad \textit{and} \quad B_p(T) = \min\left(\frac{r}{\delta}, 1\right) K.$$

Proposition 4.2

1. *For the time-reversed EEBs,*

$$\lim_{\tau\to\infty} \widetilde{B}_c(\tau) \equiv \overline{B}_c = \frac{r}{\delta}\frac{\theta_2^\circ - 1}{\theta_2^\circ} K = \frac{\theta_1^\circ}{\theta_1^\circ - 1} K, \tag{22}$$

$$\lim_{\tau\to\infty} \widetilde{B}_p(\tau) \equiv \underline{B}_p = \frac{r}{\delta}\frac{\theta_1^\circ - 1}{\theta_1^\circ} K = \frac{\theta_2^\circ}{\theta_2^\circ - 1} K, \tag{23}$$

where $\theta_i^\circ = \lim_{\lambda\to 0} \theta_i(\lambda)$.

2. *For the time-reversed values,*

$$\lim_{\tau\to\infty} \widetilde{C}(\tau, S) = \frac{\overline{B}_c}{\theta_1^\circ}\left(\frac{S}{\overline{B}_c}\right)^{\theta_1^\circ}, \quad S < \overline{B}_c, \tag{24}$$

$$\lim_{\tau\to\infty} \widetilde{P}(\tau, S) = -\frac{\underline{B}_p}{\theta_2^\circ}\left(\frac{S}{\underline{B}_p}\right)^{\theta_2^\circ}, \quad S > \underline{B}_p. \tag{25}$$

5 Exchange Options [30]

An exchange option is a simple contingent claim written on two assets, which gives its holder the right to exchange one asset for another. European prototypes of the exchange option were independently introduced by Fischer [15] and Margrabe [31], which are special cases of a general European exchange option (EEO) studied by McDonald and Siegel [32]. For the perpetual American exchange option (AEO), McDonald and Siegel [33] obtained explicit formulas for the

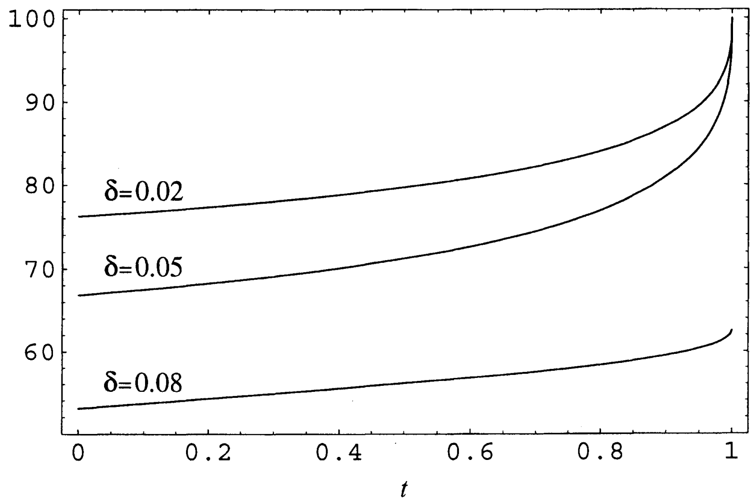

Figure 1: Early exercise boundaries $B_p(t)$ ($t \in [0,T]$) for American put options ($T = 1$, $K = 100$, $r = 0.05$, $\delta = 0.02, 0.05, 0.08$, $\sigma = 0.2$)

option value and the early exercise boundary. There has been, however, insufficient research on the finite-lived AEO, compared with the standard American option written on a single asset.

For an economy with finite time period $[0,T]$, consider a pair of assets with price processes $(S_t^1)_{t\geq 0}$ and $(S_t^2)_{t\geq 0}$. For S_0^i ($i = 1,2$) given, assume that $(S_t^i)_{t\geq 0}$ is a risk-neutralized diffusion process described by the SDE

$$\frac{\mathrm{d}S_t^i}{S_t^i} = (r - \delta_i)\mathrm{d}t + \sigma_i \mathrm{d}W_t^i, \quad t \in [0,T], \tag{26}$$

where $\delta_i \geq 0$ is the continuous dividend rate of asset i, and $\sigma_i > 0$ is the volatility coefficient of $(S_t^i)_{t\geq 0}$. Assume that all of these coefficients (r, δ_i, σ_i) are constant. In the SDE for $(S_t^i)_{t\geq 0}$, $(W_t^i)_{t\geq 0}$ ($i = 1,2$) denote one-dimensional standard Brownian motion processes with constant correlation ρ ($|\rho| < 1$), defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration corresponding to $(W_t^1, W_t^2)_{t\geq 0}$, and the probability measure $\mathbb{P}$ is chosen so that each of assets has mean rate of return r.

For these two assets, consider an option to exchange one asset for another with payoff

$$(S_t^2 - S_t^1)^+ = S_t^1(S_t - 1)^+, \quad \text{where} \quad S_t \equiv \frac{S_t^2}{S_t^1}$$

upon exercise. With numeraire $S_t^1 \mathrm{e}^{\delta_1 t}$, define the equivalent measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\tfrac{1}{2}\sigma_1^2 T + \sigma_1 W_T^1\right\}. \tag{27}$$

By Itô's lemma, $(S_t)_{t\geq 0}$ under $\mathbb{Q}$ has the dynamics

$$\frac{\mathrm{d}S_t}{S_t} = (\delta_1 - \delta_2 - \rho\sigma_1\sigma_2 + \sigma_1^2)\mathrm{d}t + \sigma_2 \mathrm{d}W_t^2 - \sigma_1 \mathrm{d}W_t^1.$$

By the Girsanov theorem, $(\hat{W}_t^i)_{t\geq 0}$ ($i = 1,2$) defined by

$$\mathrm{d}\hat{W}_t^1 = \mathrm{d}W_t^1 - \sigma_1 \mathrm{d}t, \qquad \mathrm{d}\hat{W}_t^2 = \mathrm{d}W_t^2 - \rho\sigma_1 \mathrm{d}t$$

are Brownian motion processes under $\mathbb{Q}$, and hence $(W_t)_{t\geq 0}$ defined by

$$\mathrm{d}W_t = \frac{1}{\sigma}\left(\sigma_2 \mathrm{d}\hat{W}_t^2 - \sigma_1 \mathrm{d}\hat{W}_t^1\right)$$

is also a Brownian motion under $\mathbb{Q}$, where $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$. Hence, under the measure $\mathbb{Q}$, we obtain the SDE

$$\frac{\mathrm{d}S_t}{S_t} = (\delta_1 - \delta_2)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{28}$$

which means that $(S_t)_{t\geq 0}$ is a geometric Brownian motion with drift $\delta_1 - \delta_2$ and volatility σ. From the relation $(S_t^2 - S_t^1)^+ = S_t^1(S_t - 1)^+$, the exchange option is equivalent to S^1 vanilla call options with unit strike, written on a single underlying asset with continuous dividend rate δ_2 and volatility σ, in a financial market with interest rate δ_1.

5.1 European option

Let

$$e(t, S_t^1, S_t^2) = \mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}(S_T^2 - S_T^1)^+\right]$$

denote the value of the EEO at time $t \in [0, T]$ with maturity date T. Then, by the change of numeraire, we have the McDonald and Siegel formula [32]:

$$\begin{aligned} e(t, S_t^1, S_t^2) &= S_t^1 \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\delta_1\tau}(S_T - 1)^+\right] = S_t^1 c(t, S_t; 1, \delta_1, \delta_2) \\ &= S_t^1\left\{S_t \mathrm{e}^{-\delta_2\tau}\Phi\big(d_+^e(S_t, 1, \tau)\big) - \mathrm{e}^{-\delta_1\tau}\Phi\big(d_-^e(S_t, 1, \tau)\big)\right\} \\ &= S_t^2 \mathrm{e}^{-\delta_2\tau}\Phi\big(d_+^e(S_t^2, S_t^1, \tau)\big) - S_t^1 \mathrm{e}^{-\delta_1\tau}\Phi\big(d_-^e(S_t^2, S_t^1, \tau)\big), \end{aligned} \tag{29}$$

where $\tau = T - t$, and

$$d_\pm^e(x, y, \tau) = \frac{\log(x/y) + (\delta_1 - \delta_2 \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

5.2 American option

Let $E(t, S_t^1, S_t^2)$ denote the value of the AEO at time $t \in [0, T]$ with maturity date T. In the absence of arbitrage opportunities, the value $E(t, S_t^1, S_t^2)$ is a solution of the optimal stopping problem

$$E(t, S_t^1, S_t^2) = \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}_t\left[\mathrm{e}^{-r(\tau_e - t)}(S_{\tau_e}^2 - S_{\tau_e}^1)^+\right].$$

Under the measure $\mathbb{Q}$, we have

$$E(t, S_t^1, S_t^2) = S_t^1 \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\delta_1(\tau_e - t)}(S_{\tau_e} - 1)^+\right] = S_t^1 C(t, S_t),$$

where $C(t, S_t) = C(t, S_t; 1, \delta_1, \delta_2)$ is the value of an American vanilla call option with unit strike, written on an underlying asset with dividend rate δ_2 and volatility σ, in a market with interest rate δ_1.

The value $C(t, S_t)$ can be represented as the sum of the European vanilla call value and the early exercise premium, i.e.,

$$C(t, S_t) = c(t, S_t; 1, \delta_1, \delta_2) + \pi(t, S_t), \tag{30}$$

where $c(t, S_t) = c(t, S_t; 1, \delta_1, \delta_2)$,

$$\pi(t, S_t) = \int_t^T \left\{\delta_2 S_t \mathrm{e}^{-\delta_2(u-t)}\Phi\big(d_+^e(S_t, B_u, u-t)\big) - \delta_1 \mathrm{e}^{-\delta_1(u-t)}\Phi\big(d_-^e(S_t, B_u, u-t)\big)\right\} \mathrm{d}u, \quad (31)$$

and $(B_t)_{t\in[0,T]}$ is the EEB of the American vanilla call, which is given by solving the integral equation

$$B_t - 1 = c(t, B_t) + \pi(t, B_t). \quad (32)$$

Proposition 5.1

$$C^*(\lambda, S) = \begin{cases} S - 1, & S \geq B^* \\ c^*(\lambda, S) + \pi^*(\lambda, S), & S < B^*, \end{cases}$$

where

$$\pi^*(\lambda, S) = \frac{1}{\theta_1}\left\{\frac{\delta_2 B^*}{\lambda + \delta_2} - \theta_2 \xi_2(B^*)\right\}\left(\frac{S}{B^*}\right)^{\theta_1}, \quad S < B^*,$$

$B^* \equiv B^*(\lambda) = \mathcal{LC}[\widetilde{B}_\tau](\lambda)$ (≥ 1) *is a unique positive solution of the functional equation*

$$\lambda B^{*\theta_2} + \delta_2\theta_2 B^* + \delta_1(1 - \theta_2) = 0,$$

and $\theta_i = \theta_i(\lambda; \delta_1, \delta_2, \sigma^2)$ $(i = 1, 2)$.

Theorem 5.1 *The LCT* $E^*(\lambda, S^1, S^2) = \mathcal{LC}[\widetilde{E}(\tau, S^1, S^2)](\lambda)$ *for the American exchange option value is given by*

$$E^*(\lambda, S^1, S^2) = \begin{cases} S^2 - S^1, & S^2 \geq S^1 B^* \\ S^1\left\{\xi_2\left(\frac{S^2}{S^1}\right) + \pi^*\left(\lambda, \frac{S^2}{S^1}\right)\right\} + \frac{\lambda S^2}{\lambda + \delta_2} - \frac{\lambda S^1}{\lambda + \delta_1}, & S^1 \leq S^2 < S^1 B^* \\ S^1\left\{\xi_1\left(\frac{S^2}{S^1}\right) + \pi^*\left(\lambda, \frac{S^2}{S^1}\right)\right\}, & S^2 < S^1. \end{cases} \quad (33)$$

Proposition 5.2

$$B_T = \max\left(\frac{\delta_1}{\delta_2}, 1\right). \quad (34)$$

Proposition 5.3

1. *For the time-reversed EEB,*

$$\lim_{\tau\to\infty} \widetilde{B}_\tau \equiv \overline{B} = \frac{\delta_1}{\delta_2}\frac{\theta_2^\circ - 1}{\theta_2^\circ} = \frac{\theta_1^\circ}{\theta_1^\circ - 1}. \quad (35)$$

2. *For the time-reversed value,*

$$\lim_{\tau\to\infty} \widetilde{E}(\tau, S^1, S^2) = \frac{\overline{B}}{\theta_1^\circ}\left(\frac{S^2}{S^1\overline{B}}\right)^{\theta_1^\circ}, \quad S^2 < S^1\overline{B}. \quad (36)$$

6 Russian Options [22]

Russian options are path-dependent contingent claims that give the holder the right to receive the realized *supremum* value of the underlying asset prior to his exercise time. The holder can exercise the option at any time, i.e., the option is of *American-style.* There exists an optimal threshold level of the asset price below which it is advantageous to exercise the option, provided that the asset pays dividends. Russian options are *not* genuine option contracts, because they pay the holder the supremum asset price, always finishing in-the-money. This means that high premiums are charged for Russian options in compensation for *reduced regret.*

For the underlying process $(S_t)_{t\geq 0}$ with $S_0 = s$ and a constant $m \geq s$, define the supremum process as

$$M_t = m \vee \sup_{0\leq u\leq t} S_u, \quad t \geq 0.$$

Given a finite time horizon $T > 0$, the arbitrage-free value of the Russian option at time $t \in [0,T]$ is given by

$$R(t,s,m) = \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}_{s,m}[\mathrm{e}^{-r(\tau_e - t)} M_{\tau_e}],$$

where the conditional expectation $\mathbb{E}_{s,m}[\,\cdot\,] \equiv \mathbb{E}[\,\cdot\,|\,S_0 = s, M_0 = m]$ is calculated under the risk-neutral probability measure $\mathbb{P}$.

Let $\mathcal{D} = \{(t,s,m) \in [0,T] \times \mathbb{R}_+ \times [s,+\infty)\}$ be the whole domain, and $\mathcal{C}$ continuation region. In terms of the value function $R(t,s,m)$, the continuation region $\mathcal{C}$ is defined by $\mathcal{C} = \{(t,s,m)\,|\,R(t,s,m) > m\}$. Since R is nondecreasing in s, $(t,s,m) \in \mathcal{C}$ implies $(x,m,t) \in \mathcal{C}$ for all $x \in [s,m]$. Hence, there exists an EEB $B \equiv B(t,m)$ $(\leq m)$ such that

$$B(t,m) = \inf\{s \in [0,m]\,|\,(t,s,m) \in \mathcal{C}\}.$$

In terms of the EEB $B(t,m)$, the continuation region $\mathcal{C}$ can be represented as

$$\mathcal{C} = \{(t,s,m)\,|\,B(t,m) < s \leq m\}.$$

The value $R(t,s,m)$ satisfies the Black-Scholes-Merton PDE

$$\frac{\partial R}{\partial t} + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 R}{\partial s^2} + (r-\delta)s\frac{\partial R}{\partial s} - rR = 0, \quad (t,s,m) \in \mathcal{C}, \tag{37}$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim\limits_{s\to B(t,m)} R(t,s,m) = m \\ \lim\limits_{s\to B(t,m)} \dfrac{\partial R}{\partial s} = 0 \\ \lim\limits_{m\to s} \dfrac{\partial R}{\partial m} = 0, \end{array} \right. \tag{38}$$

and the terminal condition

$$R(T,s,m) = m. \tag{39}$$

Theorem 6.1 *The LCT $R^*(\lambda, s, m)$ for the finite-lived Russian option value is given by*

$$R^*(\lambda, s, m) = \begin{cases} \dfrac{r}{\lambda + r}\dfrac{m}{\theta_2 - \theta_1}\left\{\theta_2\left(\dfrac{s}{B^*}\right)^{\theta_1} - \theta_1\left(\dfrac{s}{B^*}\right)^{\theta_2}\right\} + \dfrac{\lambda m}{\lambda + r}, & B^* < s \le m \\ m, & 0 < s \le B^*, \end{cases}$$

where the LCT $B^* \equiv B^*(\lambda, m) = \mathcal{LC}[\widetilde{B}(\tau, m)](\lambda)$ $(\le m)$ is a unique positive solution of the functional equation

$$\frac{\theta_1(1-\theta_2)}{\theta_2(1-\theta_1)}\left(\frac{B^*}{m}\right)^{\theta_1 - \theta_2} + \frac{\lambda}{r}\frac{\theta_1 - \theta_2}{\theta_2(1-\theta_1)}\left(\frac{B^*}{m}\right)^{\theta_1} = 1.$$

Corollary 6.2 *The LCTs of the time-reversed Greeks*

$$\Delta_R^* \equiv \mathcal{LC}\left[\frac{\partial \widetilde{R}}{\partial s}\right](\lambda), \quad \Gamma_R^* \equiv \mathcal{LC}\left[\frac{\partial^2 \widetilde{R}}{\partial s^2}\right](\lambda) \quad \text{and} \quad \Theta_R^* \equiv -\mathcal{LC}\left[\frac{\partial \widetilde{R}}{\partial \tau}\right](\lambda)$$

for $s \in (B^*, m]$ are, respectively, given by

$$\Delta_R^* = \frac{r}{\lambda + r}\frac{\theta_1\theta_2}{\theta_2 - \theta_1}\frac{m}{s}\left\{\left(\frac{s}{B^*}\right)^{\theta_1} - \left(\frac{s}{B^*}\right)^{\theta_2}\right\} > 0,$$

$$\Gamma_R^* = \frac{r}{\lambda + r}\frac{\theta_1\theta_2}{\theta_2 - \theta_1}\frac{m}{s^2}\left\{(\theta_1 - 1)\left(\frac{s}{B^*}\right)^{\theta_1} - (\theta_2 - 1)\left(\frac{s}{B^*}\right)^{\theta_2}\right\} > 0,$$

$$\Theta_R^* = -\frac{\lambda r m}{\lambda + r}\frac{1}{\theta_2 - \theta_1}\left\{(\theta_2 - 1)\left(\frac{s}{B^*}\right)^{\theta_1} - (\theta_1 - 1)\left(\frac{s}{B^*}\right)^{\theta_2}\right\} < 0.$$

Proposition 6.1 *For the EEB of the finite-lived Russian option, we have*

$$B(T, m) = m.$$

Proposition 6.2

1. *For the time-reversed EEB,*

$$\lim_{\tau \to \infty} \widetilde{B}(\tau, m) \equiv \underline{B} = m\left(\frac{\theta_2^\circ(1-\theta_1^\circ)}{\theta_1^\circ(1-\theta_2^\circ)}\right)^{\frac{1}{\theta_1^\circ - \theta_2^\circ}}.$$

2. *For the time-reversed value,*

$$\lim_{\tau \to \infty} \widetilde{R}(\tau, s, m) = \begin{cases} \dfrac{m}{\theta_2 - \theta_1}\left\{\theta_2\left(\dfrac{s}{\underline{B}}\right)^{\theta_1} - \theta_1\left(\dfrac{s}{\underline{B}}\right)^{\theta_2}\right\}, & \underline{B} < s \le m \\ m, & 0 < s \le \underline{B}. \end{cases}$$

Theorem 6.3 *For $r, \delta > 0$, denote $\underline{B} \equiv \underline{B}(r, \delta)$. Then, $\underline{B}(r, \delta)$ is a symmetric function of r and δ, i.e.,*

$$\underline{B}(r, \delta) = \underline{B}(\delta, r).$$

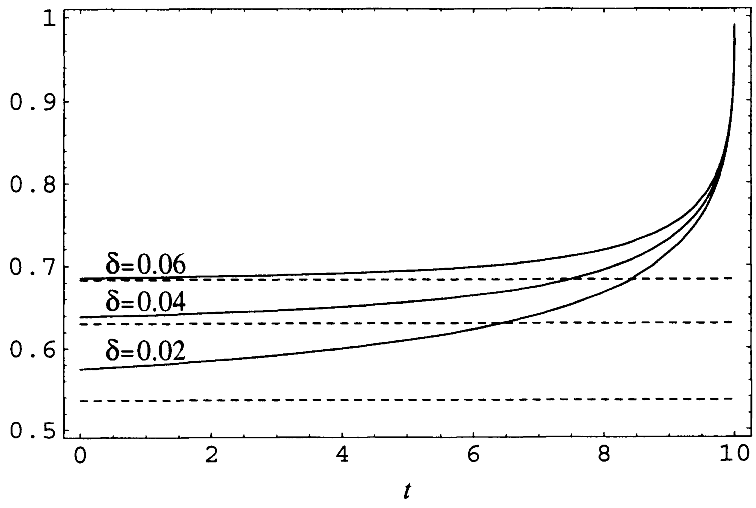

Figure 2: Normalized early exercise boundaries $B(t,m)/m$ ($t \in [0,T]$) for Russian options ($T = 10$, $r = 0.04$, $\delta = 0.02, 0.04, 0.06$, $\sigma = 0.2$)

7 Continuous-Installment Options

Installment options are contingent claims in which a small amount of up-front premium instead of a lump sum is paid at the time of purchase, and then a sequence of *installments* are paid up to a fixed maturity to keep the contract alive. The holder has the right of stopping payments at any time, thereby terminating the option contract. If the option is not worth the *Net Present Value* (NPV) of the remaining payments, he does not have to continue to pay further installments. Hence, an optimal stopping problem arises for the installment option even in European style. The option can be exercised only if all installments are paid until maturity. For brevity, we only deal with the *call* case.

7.1 European options [9, 10, 25, 36]

Consider a European continuous-installment option written on $(S_t)_{t\geq 0}$. Assume that the option holder pays his installments *continuously* with rate q (> 0), i.e., the holder pays an amount of $q\mathrm{d}t$ in time $\mathrm{d}t$. Let

$$c(t, S_t; q) \equiv c(t, S_t; q, K, r, \delta)$$

denote the value of the European continuous-installment call option at time $t \in [0,T]$, which has strike price K and maturity date T. Then, the value $c(t, S_t; q)$ is given by solving an optimal stopping problem

$$c(t, S_t; q) = \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}_t \left[\mathbf{1}_{\{\tau_e \geq T\}} \mathrm{e}^{-r(T-t)} (S_T - K)^+ - \frac{q}{r} \left(1 - \mathrm{e}^{-r(\tau_e \wedge T - t)} \right) \right]. \tag{40}$$

Solving the optimal stopping problem is equivalent to finding the points (t, S_t) for which the termination of the contract is optimal. Let $\mathcal{D} = [0,T] \times \mathbb{R}_+$, and $\mathcal{S}$ and $\mathcal{C}$ denote the *stopping*

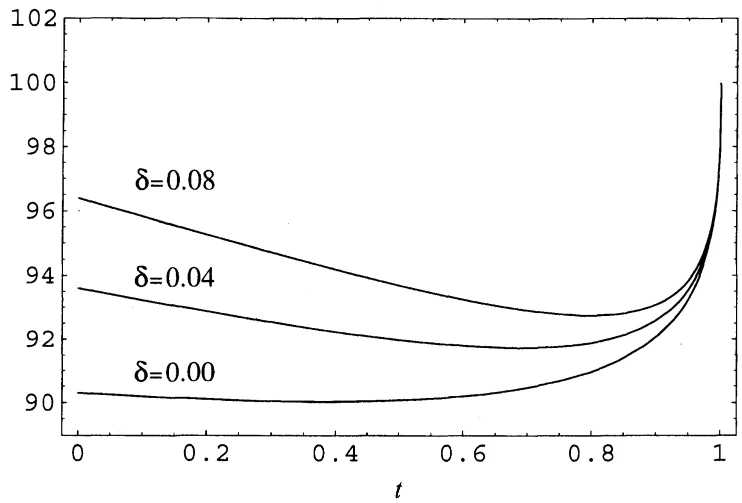

Figure 3: Stopping boundaries A_t ($t \in [0,T]$) for European continuous-installment call options ($T=1$, $K=100$, $q=10$, $r=0.05$, $\delta=0, 0.04, 0.08$, $\sigma=0.2$)

region and *continuation region*, respectively. The stopping region $\mathcal{S}$ is defined by

$$\mathcal{S} = \{(t, S_t) \in \mathcal{D} \,|\, c(t, S_t; q) = 0\}.$$

The continuation region $\mathcal{C}$ is the complement of $\mathcal{S}$ in $\mathcal{D}$. The EEB that separates $\mathcal{S}$ from $\mathcal{C}$ is referred to as a *stopping boundary*, which is defined by

$$A_t = \inf\{S_t \in \mathbb{R}_+ \,|\, c(t, S_t; q) > 0\}, \quad t \in [0,T].$$

Since $c(t, S_t; q)$ is nondecreasing in S_t, the stopping boundary is a *lower* critical asset price below which it is advantageous to terminate the option contract by stopping the payments, and it vanishes when $q \leq 0$.

The call value $c(t, S; q)$ satisfies the *inhomogeneous* PDE

$$\frac{\partial c}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r-\delta) S \frac{\partial c}{\partial S} - rc = q, \quad (t,S) \in \mathcal{C},$$

together with the boundary conditions

$$\left| \begin{array}{l} \lim\limits_{S \to A_t} c(t,S;q) = 0 \\[2ex] \lim\limits_{S \to A_t} \dfrac{\partial c}{\partial S} = 0 \\[2ex] \lim\limits_{S \to \infty} \dfrac{\partial c}{\partial S} < \infty, \end{array} \right.$$

and the terminal condition

$$c(T, S; q) = (S-K)^+.$$

Theorem 7.1 *The value function of the European continuous-installment call option has the integral representation*

$$c(t, S_t; q) = c(t, S_t) - q \int_t^T \mathrm{e}^{-r(u-t)} \Phi\big(d_-(S_t, A_u, u-t)\big) \mathrm{d}u,$$

where $c(t, S_t)$ is the value of the associated European vanilla call option.

Griebsch et al. [18] proved that the total premium has the decomposition

$$c(t, S_t; q) + K_t = c(t, S_t) + P_c(t, S_t; q),$$

where

$$K_t = \frac{q}{r}\left(1 - e^{-r(T-t)}\right)$$

is the NPV of the future payment stream at time t, and

$$\begin{aligned} P_c(t, S_t; q) &= \operatorname*{ess\,sup}_{\tau_s \in [t,T]} \mathbb{E}_t\left[e^{-r(\tau_s - t)}\left(K_{\tau_s} - c(\tau_s, S_{\tau_s})\right)^+\right] \\ &= q\int_t^T e^{-r(u-t)}\Phi\left(-d_-(S_t, A_u, u-t)\right)du \end{aligned}$$

represents the value of an *American compound put option* maturing in time T written on the vanilla call option.

Theorem 7.2 *The LCT $c^*(\lambda, S; q) = \mathcal{LC}[\widetilde{c}(\tau, S; q)]$ for the European continuous-installment call option is given by*

$$c^*(\lambda, S; q) = \begin{cases} c^*(\lambda, S) - \dfrac{q}{\lambda + r}\left\{1 - \dfrac{\theta_1}{\theta_1 - \theta_2}\left(\dfrac{S}{A^*}\right)^{\theta_2}\right\}, & S > A^* \\ 0, & S \leq A^*, \end{cases}$$

where $A^* \equiv A^*(\lambda) = \mathcal{LC}[\widetilde{A}_\tau]$ $(\leq K)$ is given by

$$A^*(\lambda) = \left[\frac{2(\lambda + \delta)q}{\lambda(1 - \theta_2)K\sigma^2}\right]^{\theta_1^{-1}} K.$$

Corollary 7.3 *The LCT $P_c^*(\lambda, S; q) = \widetilde{P}_c(\tau, S; q)$ for the American compound put option is given by*

$$P_c^*(\lambda, S; q) = \frac{q}{\lambda + r}\frac{\theta_1}{\theta_1 - \theta_2}\left(\frac{S}{A^*}\right)^{\theta_2}.$$

Corollary 7.4 *For $S > A^*$,*

$$\varDelta_{P_c}^* \equiv \mathcal{LC}\left[\frac{\partial \widetilde{P}_c}{\partial S}\right](\lambda) = \frac{q}{\lambda + r}\frac{\theta_1\theta_2}{\theta_1 - \theta_2}\frac{1}{S}\left(\frac{S}{A^*}\right)^{\theta_2} < 0,$$

$$\varGamma_{P_c}^* \equiv \mathcal{LC}\left[\frac{\partial^2 \widetilde{P}_c}{\partial S^2}\right](\lambda) = \frac{q}{\lambda + r}\frac{\theta_1\theta_2(\theta_2 - 1)}{\theta_1 - \theta_2}\frac{1}{S^2}\left(\frac{S}{A^*}\right)^{\theta_2} > 0,$$

$$\varTheta_{P_c}^* \equiv -\mathcal{LC}\left[\frac{\partial \widetilde{P}_c}{\partial \tau}\right](\lambda) = -\frac{\lambda q}{\lambda + r}\frac{\theta_1}{\theta_1 - \theta_2}\left(\frac{S}{A^*}\right)^{\theta_2} < 0.$$

Theorem 7.5 *For the stopping boundary,*

$$A_T = K \quad \textit{and} \quad \lim_{T \to \infty} A_t = \begin{cases} \infty, & \delta > 0 \\ \dfrac{2q}{2r + \sigma^2}, & \delta = 0. \end{cases}$$

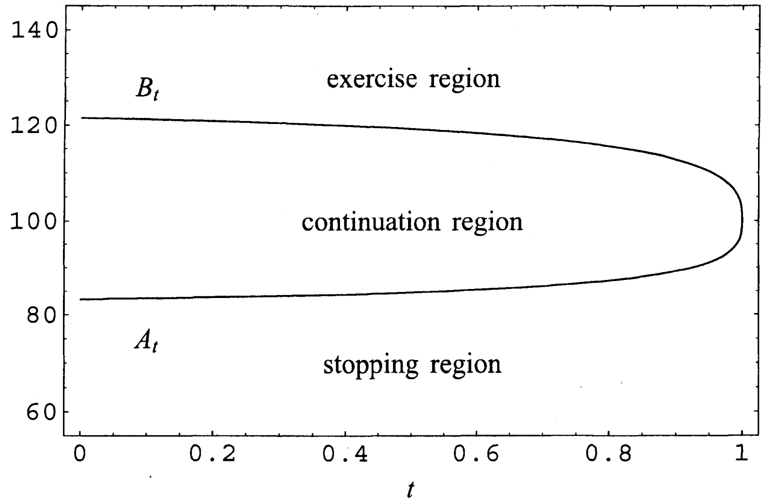

Figure 4: Three regions for American continuous-installment call option ($T = 1$, $K = 100$, $q = 5$, $r = 0.05$, $\delta = 0.04$, $\sigma = 0.2$)

7.2 American options [8, 11, 23]

Let

$$C(t, S_t; q) \equiv C(t, S_t; q, K, r, \delta)$$

denote the value of the American continuous-installment call option at time $t \in [0, T]$, which has strike price K and maturity date T. Then, the value $C(t, S_t; q)$ is given by solving an optimal stopping problem

$$\begin{aligned} C(t, S_t; q) = \operatorname*{ess\,sup}_{\tau_e, \tau_s} \mathbb{E}_t \Big[& \mathbf{1}_{\{\tau_e \wedge \tau_s \geq T\}} \mathrm{e}^{-r(T-t)} (S_T - K)^+ \\ & + \mathbf{1}_{\{\tau_e < \tau_s < T\}} \mathrm{e}^{-r(\tau_e - t)} (S_{\tau_e} - K)^+ - \frac{q}{r} \left(1 - \mathrm{e}^{-r(\tau_e \wedge \tau_s \wedge T - t)}\right) \Big] \end{aligned}$$

for $t \in [0, T]$, where τ_e and τ_s are stopping times of the filtration $(\mathcal{F}_t)_{t \geq 0}$. Solving the optimal stopping problem is equivalent to finding the points (t, S_t) for which termination of the contract or early exercise is optimal.

Let $\mathcal{S}$, $\mathcal{E}$ and $\mathcal{C}$ denote the *stopping region*, *exercise region* and *continuation region*, respectively. In terms of the value $C(t, S_t; q)$, these regions can be defined by

$$\begin{aligned} \mathcal{S} &= \{(t, S_t) \in \mathcal{D} \,|\, C(t, S_t; q) = 0\}, \\ \mathcal{E} &= \left\{(t, S_t) \in \mathcal{D} \,|\, C(t, S_t; q) = (S_t - K)^+\right\}, \\ \mathcal{C} &= \mathcal{D} \setminus \mathcal{S} \cup \mathcal{E}, \end{aligned}$$

among which there are two boundaries: the stopping boundary $(A_t)_{t \in [0,T]}$, which is a lower critical asset price, and the early exercise boundary $(B_t)_{t \in [0,T]}$, which is an upper critical asset price, due to the monotonicity of $C(t, S_t; q)$ in S_t.

The call value $C(t, S; q)$ satisfies the inhomogeneous PDE

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - \delta) S \frac{\partial C}{\partial S} - rC = q, \quad (t, S) \in \mathcal{C},$$

together with the boundary conditions

$$\left|\begin{array}{ll} \lim_{S\to A_t} C(t,S;q)=0, & \lim_{S\to B_t} C(t,S;q)=\overline{S}_t-K \\ \lim_{S\to A_t} \dfrac{\partial C}{\partial S}=0, & \lim_{S\to B_t} \dfrac{\partial C}{\partial S}=1, \end{array}\right.$$

and the terminal condition

$$C(T,S;q)=(S-K)^+.$$

Theorem 7.6 *The value function of the American continuous-installment call option has the integral representation*

$$\begin{aligned} C(t,S_t;q) &= c(t,S_t) \\ &+\int_t^T \Big\{\delta S_t \mathrm{e}^{-\delta(u-t)}\Phi\big(d_+(S_t,B_u,u-t)\big)-(rK-q)\mathrm{e}^{-r(u-t)}\Phi\big(d_-(S_t,B_u,u-t)\big)\Big\}\,\mathrm{d}u \\ &-q\int_t^T \mathrm{e}^{-r(u-t)}\Phi\big(d_-(S_t,A_u,u-t)\big)\mathrm{d}u. \end{aligned} \tag{41}$$

Theorem 7.7 *The terminal values of the stopping and early exercise boundaries at expiry are given by*

$$A_T=K, \tag{42}$$

$$B_T=\max\left(\frac{rK-q}{\delta},\,K\right). \tag{43}$$

Theorem 7.8 *The LCT $C^*(\lambda,S;q)$ for the American continuous-installment call option value is given by*

$$C^*(\lambda,S;q)=\begin{cases} 0, & S\in[0,A^*] \\ c^*(\lambda,S)+\pi_c^*(\lambda,S;q)-\dfrac{q}{\lambda+r}, & S\in(A^*,B^*) \\ S-K, & S\in[B^*,\infty), \end{cases}$$

where $\pi_c^*(\lambda,S;q)$ is defined by

$$\pi_c^*(\lambda,S;q)=\frac{q}{\lambda+r}\frac{\theta_1\theta_2}{\theta_1-\theta_2}\left\{\frac{1}{\theta_2}\left(\frac{S}{A^*}\right)^{\theta_2}-\frac{1}{\theta_1}\left(\frac{S}{A^*}\right)^{\theta_1}\right\}-\xi_1(S),$$

A^* and B^* are given by solving a pair of nonlinear equations

$$\left\{\begin{array}{l} \dfrac{q}{\lambda+r}\dfrac{\theta_1\theta_2}{\theta_1-\theta_2}\left\{\dfrac{1}{\theta_2}\left(\dfrac{B^*}{A^*}\right)^{\theta_2}-\dfrac{1}{\theta_1}\left(\dfrac{B^*}{A^*}\right)^{\theta_1}\right\}=\xi_1(B^*)-\xi_2(B^*)+\dfrac{\delta}{\lambda+\delta}B^*-\dfrac{rK-q}{\lambda+r}, \\ \dfrac{q}{\lambda+r}\dfrac{\theta_1\theta_2}{\theta_1-\theta_2}\left\{\left(\dfrac{B^*}{A^*}\right)^{\theta_2}-\left(\dfrac{B^*}{A^*}\right)^{\theta_1}\right\}=\theta_1\xi_1(B^*)-\theta_2\xi_2(B^*)+\dfrac{\delta}{\lambda+\delta}B^*. \end{array}\right. \tag{44}$$

Corollary 7.9 *The LCTs of the time-reversed Greeks*

$$\Delta^*_{\widetilde{\pi}_c}\equiv\mathcal{LC}\left[\frac{\partial\widetilde{\pi}_c}{\partial S}\right](\lambda),\quad \Gamma^*_{\widetilde{\pi}_c}\equiv\mathcal{LC}\left[\frac{\partial^2\widetilde{\pi}_c}{\partial S^2}\right](\lambda)\quad \textit{and}\quad \Theta^*_{\widetilde{\pi}_c}\equiv-\mathcal{LC}\left[\frac{\partial\widetilde{\pi}_c}{\partial\tau}\right](\lambda)$$

for $S \in (A^*, B^*)$ are, respectively, given by

$$\Delta^*_{\pi_c} = \frac{1}{S}\left[\frac{q}{\lambda + r}\frac{\theta_1\theta_2}{\theta_1 - \theta_2}\left\{\left(\frac{S}{A^*}\right)^{\theta_2} - \left(\frac{S}{A^*}\right)^{\theta_1}\right\} - \theta_1\xi_1(S)\right],$$

$$\Gamma^*_{\pi_c} = \frac{1}{S^2}\left[\frac{q}{\lambda + r}\frac{\theta_1\theta_2}{\theta_1 - \theta_2}\left\{(\theta_2 - 1)\left(\frac{S}{A^*}\right)^{\theta_2} - (\theta_1 - 1)\left(\frac{S}{A^*}\right)^{\theta_1}\right\} - \theta_1(\theta_1 - 1)\xi_1(S)\right],$$

$$\Theta^*_{\pi_c} = -\lambda\left[\frac{q}{\lambda + r}\frac{\theta_1\theta_2}{\theta_1 - \theta_2}\left\{\frac{1}{\theta_2}\left(\frac{S}{A^*}\right)^{\theta_2} - \frac{1}{\theta_1}\left(\frac{S}{A^*}\right)^{\theta_1}\right\} - \xi_1(S)\right].$$

Theorem 7.10 *Let $C_\infty(S;q)$ denote the* perpetual *continuous-installment call option value. Then, for $S \in (A_\infty, B_\infty)$,*

$$\begin{aligned} C_\infty(S;q) &= \frac{-\frac{1}{\theta_1^\circ}(A_\infty)^{\theta_2^\circ}S^{\theta_1^\circ} + \frac{1}{\theta_2^\circ}(A_\infty)^{\theta_1^\circ}S^{\theta_2^\circ}}{(A_\infty)^{\theta_1^\circ}(B_\infty)^{\theta_2^\circ - 1} - (A_\infty)^{\theta_2^\circ}(B_\infty)^{\theta_1^\circ - 1}} - \frac{q}{r} \\ &= \frac{q}{r}\frac{\theta_1^\circ\theta_2^\circ}{\theta_1^\circ - \theta_2^\circ}\left\{-\frac{1}{\theta_1^\circ}\left(\frac{S}{A_\infty}\right)^{\theta_1^\circ} + \frac{1}{\theta_2^\circ}\left(\frac{S}{A_\infty}\right)^{\theta_2^\circ}\right\} - \frac{q}{r}, \end{aligned}$$

A_∞ and B_∞ are the optimal threshold levels given by

$$\begin{cases} A_\infty = \dfrac{q}{r}\dfrac{\theta_1^\circ\theta_2^\circ}{\theta_1^\circ - \theta_2^\circ}\left(\gamma^{\theta_2^\circ - 1} - \gamma^{\theta_1^\circ - 1}\right), \\ B_\infty = \gamma A_\infty = \dfrac{q}{r}\dfrac{\theta_1^\circ\theta_2^\circ}{\theta_1^\circ - \theta_2^\circ}\left(\gamma^{\theta_2^\circ} - \gamma^{\theta_1^\circ}\right), \end{cases}$$

in terms of $\gamma > 1$, which is the unique solution of the equation

$$\theta_2^\circ(\theta_1^\circ - 1)\gamma^{\theta_1^\circ} - \theta_1^\circ(\theta_2^\circ - 1)\gamma^{\theta_2^\circ} = (\theta_1^\circ - \theta_2^\circ)\left(1 - \frac{rK}{q}\right).$$

8 Issues on Deck

8.1 Numerical inversion methods

By virtue of the Bromwich integral in Proposition 2.2, we see that numerical inversion methods developed for inverting LTs also can be used for LCTs, i.e., for a LCT $f^*(\lambda) = \mathcal{LC}[f(x)](\lambda)$ ($\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$),

$$f(x) = \mathcal{LC}^{-1}[f^*(\lambda)](x) = \mathcal{L}^{-1}\left[\frac{f^*(\lambda)}{\lambda}\right](x), \quad x > 0$$

There have been so many numerical inversion methods developed for LT with an explicit form; see Table 1. For option pricing problems, however, the values in target are often given in implicit forms such as functions of the LCT of its EEB, which is a solution of a functional equation. Some numerical methods are required to obtain the value of $f^*(\lambda)$ for an arbitrary $\lambda \in \mathbb{C}$. Under the existing circumstances, the Gaver-Stehfest method is only one choice, because it depends only on the value of $f^*(\lambda)$ for $\lambda \in \mathbb{R}_+$. In fact, Figures 1–4 in this tutorial are drawn by using the

Table 1: Classification of numerical LT-inversion methods

Methods	References
Post-Widder method	Post [39], Widder [45], Jagerman [20, 21], Donolato [13]
Laguerre method	Weeks [44], Abate et al. [1, 2]
Gaver-Stehfest method	Gaver [16], Stehfest [42], Valkó and Abate [43]
Euler method	Abate and Whitt [3, 4], den Iseger [12], O'Cinneide [37], Sakurai [41]

Gaver-Stehfest method. Unfortunately, numerical experiments shows that the Gaver-Stehfest method is sometimes less stable than other methods. It is necessary to extend/modify the previous numerical LT-inversion methods, so that they can handle implicit LCTs.

8.2 Future works

▷ Developing closed-form approximations for the EEB

From the view point of option holders, the EEB is more important than the option price. The EEB contains complete information on making a decision on the timing of exercise. If a simple and closed-form approximation for the EEB is available, it would be a powerful tool for the option holders. The LCT asymptotics may have potential to this issue (Kimura [27, 29]). Much work should be directed toward such approximations; see Kimura [28] for a survey.

▷ Generalizing the underlying asset process

No doubt, the geometric Brownian motion (GBM) defined in (6) is a standard process for underlying assets in option pricing. It has been, however, known that GBM is often inconsistent with actual data in capital markets. Better alternatives to GBM in such cases would be a CEV process or a Lévy process. For these processes, there are a few studies on the LCT approach; see Avram et al. [5], Pun and Wong [40], and Wong and Zhao [46]. We also need much work on such generalizations.

References

[1] Abate, J., G.L. Choudhury and W. Whitt, "On the Laguerre method for numerically inverting Laplace transforms," *INFORMS Journal on Computing*, **8** (1996), 413–427.

[2] Abate, J., G.L. Choudhury and W. Whitt, "Numerical inversion of multidimensional Laplace transforms by the Laguerre method," *Performance Evaluation*, **31** (1998), 216–243.

[3] Abate, J. and W. Whitt, "The Fourier-series method for inverting transforms of probability distributions," *Queueing Systems*, **10** (1992), 5–88.

[4] Abate, J. and W. Whitt, "Numerical inversion of Laplace transforms of probability distributions," *ORSA Journal on Computing*, **7** (1995), 36–43.

[5] Avram, F., T. Chan and M. Usabel, "On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts," *Stochastic Processes and Their Applications*, **100** (2002), 75–107.

[6] Carr, P. and M. Chesney, "American put call symmetry," Working paper, Morgan Stanley, 1997.

[7] Carr, P., "Randomization and the American put," *Review of Financial Studies*, **11** (1998), 597–626.

[8] Ciurlia, P. and C. Caperdoni, "A note on the pricing of perpetual continuous-installment options," *Mathematical Methods in Economics and Finance*, **4** (2009), 11–26.

[9] Ciurlia, P., "Valuation of European continuous-installment options," *Computers and Mathematics with Applications*, **62** (2011), 2518–2534.

[10] Ciurlia, P., "On a general class of free boundary problems for European-style installment options with continuous payment plan," *Communications on Pure and Applied Analysis*, **10** (2011), 1205–1224.

[11] Ciurlia, P., "An integral representation approach for valuing American-style installment options with continuous payment plan," *Nonlinear Analysis*, **74** (2011), 5506–5524.

[12] den Iseger, P., "Numerical transform inversion using Gaussian quadrature," *Probability in the Engineering and Informational Sciences*, **20** (2006), 1–44.

[13] Donolato, C., "Analytical and numerical inversion of the Laplace-Carson transform by a differential method," *Computer Physics Communications*, **145** (2002), 298–309.

[14] Feller, W., *An Introduction to Probability Theory and Its Applications*, Volume II, New York, Wiley, 1971.

[15] Fischer, S., "Call option pricing when the exercise price is uncertain, and the valuation of index bonds," *Journal of Finance*, **33**, 169–176, 1978.

[16] Gaver, D.P.Jr., "Observing stochastic processes and approximate transform inversion," *Operations Research*, **14** (1966), 444–459.

[17] Geman, H. and M. Yor, "Bessel processes, Asian options, and perpetuities," *Mathematical Finance*, **3** (1993), 207–216.

[18] Griebsch, S., U. Wystup and C. Kühn, "Installment options: A closed-form solution and the limiting case," Working Paper, Goethe University, 2007.

[19] Hayashi, D., M. Goto and T. Ohno, "Pricing convertible bonds: A Laplace-Carson transform approach," Working paper, Waseda University, 2006.

[20] Jagerman, D.L., "An inversion technique for the Laplace transform with applications,", *Bell System Technical Journal*, **57** (1978), 669–710.

[21] Jagerman, D.L., "An inversion technique for the Laplace transform", *Bell System Technical Journal*, **61** (1982), 1995–2002.

[22] Kimura, T., "Valuing finite-lived Russian options," *European Journal of Operational Research*, **189** (2008), 363–374.

[23] Kimura, T., "American continuous-installment options: Valuation and premium decomposition," *SIAM Journal on Applied Mathematics*, **70** (2009), 803–824.

[24] Kimura, T., "Alternative randomization for valuing American options," *Asia-Pacific Journal of Operational Research*, **27** (2010), 167–187.

[25] Kimura, T., "Valuing continuous-installment options," *European Journal of Operational Research*, **201** (2010), 222–230.

[26] Kimura, T., "American fractional lookback options: Valuation and premium decomposition," *SIAM Journal on Applied Mathematics*, **71** (2011), 517–539.

[27] Kimura, T., "Approximating the early exercise boundary for American-style options," *Lecture Note of the Research Institute for Mathematical Sciences*, Kyoto University, **1818** (2012), 1–16.

[28] Kimura, T., "Approximations for the early exercise boundary of American options: A review," [in Japanese], *Transactions of the Japan Society for Industrial and Applied Mathematics*, **24** (2014), 275–292.

[29] Kimura, T., "Asymptotic analysis on the early exercise boundary of American options," *Lecture Note of the Research Institute for Mathematical Sciences*, Kyoto University, **1886** (2014), 1–15.

[30] Kimura, T., "Dynamic investment decisions with an American exchange option," *Lecture Note of the Research Institute for Mathematical Sciences*, Kyoto University, **1933** (2015), 1–12.

[31] Margrabe, W., "The value of an option to exchange one asset for another," *Journal of Finance*, **33**, 177–186, 1978.

[32] McDonald, R.L. and D.R. Siegel, "Investment and the valuation of firms when there is an option to shut down," *International Economic Review*, **26**, 331–349, 1985.

[33] McDonald, R.L. and D.R. Siegel, "The value of waiting to invest," *Quarterly Journal of Economics*, **101**, 707–727, 1986.

[34] McDonald, R. and M. Schroder, "A parity result for American options," *Journal of Computational Finance*, **1** (1988), 5–13.

[35] McKean, H.P., "Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics," *Industrial Management Review*, **6** (1965), 32–39.

[36] Mezentsev, A., A. Pomelnikov and M. Ehrhardt, "Efficient numerical valuation of continuous installment options," *Advances in Applied Mathematics and Mechanics*, **3** (2011), 141–164.

[37] O'Cinneide, C.A., "Euler summation for Fourier series and Laplace transform inversion," *Stochastic Models*, **13** (1997), 315–337.

[38] Petrella, G. and S. Kou, "Numerical pricing of discrete barrier and lookback options via Laplace transforms," *Journal of Computational Finance*, **8** (2004), 1–37.

[39] Post, E.L., "Generalized differentiation," *Transactions of the American Mathematical Society*, **32** (1930), 723–781.

[40] Pun, C.S. and H.Y. Wong, "CEV asymptotics of American options," *Journal of Mathematical Analysis and Applications*, **403** (2013), 451–463.

[41] Sakurai, T., "Numerical inversion of Laplace transform of functions with discontinuities," *Advances in Applied Probability*, **20** (2004), 616–642.

[42] Stehfest, H., "Algorithm 368: Numerical inversion of Laplace transforms," *Communications of the ACM*, **13** (1970), 47–49.

[43] Valkó, P.P. and J. Abate, "Comparison of sequence accelerators for the Gaver method of numerical Laplace transform inversion," *Computers & Mathematics with Applications*, **48** (2004), 629–636.

[44] Weeks, W.T., "Numerical inversion of Laplace transforms using Laguerre functions," *Journal of the ACM*, **13** (1966), 419–426.

[45] Widder, DV. *The Laplace Transform*, Princeton University Press, Princeton, N.J., 1946.

[46] Wong, H.Y. and J. Zhao, "Valuing American options under the CEV model by Laplace-Carson transforms," *Operations Research Letters*, **38** (2010), 474–481.

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