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Existence and uniqueness of entropy solutions to strongly degenerate parabolic equations with variable coefficients

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Abstract

In this paper, we consider the one dimensional initial value problem for strongly degenerate parabolic equations with variable coefficients. This equation has both properties of parabolic equation and those of hyperbolic equation. Moreover, the convection and diffusion coefficients depend on the spatial variable x. In particular, we consider the case that convective coefficients are the functions of bounded variation with respect to x. Then, we prove the strong precompactness of a family of approximate solution to the problem and characterize the limit function as an entropy solution. Moreover, we give a proof of the uniqueness of entropy solutions to the problem using the methods of Karlsen-Ohlberger [6] and Karlsen-Risebro-Towers [10].

1 Introduction

We consider the initial value problem for a degenerate parabolic equation of the form

$$\text{(P)} \qquad \begin{cases} u_t + \partial_x A(x,u) = \partial_x^2 \beta(x,u), & (x,t) \in \Pi_T = \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \quad u_0 \in BV(\mathbb{R}). \end{cases}$$

Here, $[0,T]$ is a fixed time interval. $A(x,\xi)$ and $\beta(x,\xi)$ are $\mathbb{R}$-valued functions defined on $\mathbb{R} \times \mathbb{R}$. In particular, the function $\beta(x,\xi)$ is supposed to be monotone nondecreasing and locally Lipschitz continuous with respect to ξ for fixed x. From the assumptions of β, the set of points ξ where $\partial_\xi \beta(x,\xi) = 0$ may have a positive measure. In this sense, we say that the equation posed in (P) is a *strongly degenerate parabolic equation*.

This equation is an one dimensional version of the following multi-dimensional equations:

$$(1) \qquad u_t + \nabla \cdot A(x,u) = \Delta \beta(x,u).$$

The equation (1) can be applied to several mathematical models; hyperbolic conservation laws, porous medium, Stefan problem, filtration problem, sedimentation process, traffic flow, blood flow, etc. Moreover, (1) is regarded as a linear combination of the time dependent conservation laws (quasilinear hyperbolic equation) and the porous medium equation (nonlinear degenerate parabolic equation). Thus, (1) has both properties of hyperbolic equations and those of parabolic equations. Moreover, by the assumptions on β, (1) has the following properties:

. If β is strictly increasing, then "parabolicity" is majorant to "hyperbolicity".

. If β is monotone nondecreasing, then "parabolicity" and "hyperbolicity" are not necessarily comparable.

In our research, we consider (P) in the case that $A(x,\xi)$ is discontinuous with respect to x for $\xi \in \mathbb{R}$. In particular, our aim is to prove the well-posedness of (P) in the case that $A(\cdot,\xi) \in BV(\mathbb{R})$. In this paper, we prove the strong precompactness of a family of approximate solution to (P) and characterize the limit function as an entropy solution to (P). Moreover, we show the uniqueness of entropy solutions.

The mathematical analysis of strongly degenerate parabolic equations was given by Vol'pert-Hudjaev [16], Carrillo [3], Karlsen-Ohlberger [6] and Karlsen-Risebro [8]. In the discontinuous convective coefficient case, it is difficult to show that approximate solutions have bounded total variation. Hence, we may not directly apply the classical Kružkov's theory [11]. One of the methods to overcome this difficulty is the compensated compactness method which was introduced by Tartar [14]. To apply this method, we necessitate the following estimates:

. $||u_\varepsilon(\cdot,t)||_{L^\infty} \leq C$,

. $||\sqrt{\partial_\xi\beta(x,u_\varepsilon)+\varepsilon}\ \partial_x u_\varepsilon||_{L^2} \leq C$.

In fact, Karlsen-Risebro-Towers [9] proved the existence of weak solutions and the uniqueness of the constructed weak solutions to the one dimensional Cauchy problem with variable separation flux:

$$\partial_t u + \partial_x(\gamma(x)f(u)) = \partial_x^2\beta(u),$$

where $\gamma(x) \in BV(\mathbb{R})$ and $f(\xi) \in C^2(\mathbb{R})$ is a genuinely nonlinear function satisfying several conditions. Moreover, Karlsen-Risebro-Towers [10] proved L^1 stability and uniqueness of entropy solutions to the similar problems, provided that the flux function satisfies a so called crossing condition. On the other hand, Watanabe [18] proved the same results of Karlsen-Risebro-Towers [9] under the more general form than [9] using the compactness results of Panov [13]. Also, Watanabe [20, 21] considered the same setting for one dimensional zero-flux boundary problems.

In the variable diffusion coefficient case, Chen-Karlsen [4] and Wang-Wang-Li [17] obtained the well-posedness for the quasilinear anisotropic equations with time-space dependent diffusion coefficients.

In this paper, we consider the one dimensional Cauchy problem (P) for strongly degenerate parabolic equations with discontinuous convective and variable diffusion coefficients. At first, we prove the strong precompactness of a family of approximate solutions to (P) in the case that $A(\cdot,\xi) \in BV(\mathbb{R})$ for $\xi \in \mathbb{R}$. Moreover, it is confirmed that the constructed limit function is a distributional and an entropy solution to (P). We can obtain estimates for approximate solutions along the same method of Karlsen-Risebro-Towers [9]. Advantage of this paper is to apply the compactness result using H-measure (Panov [13]). Using the compensated compactness method for the type of equation (1), compactness results are only given in the case of $N = 1, 2$. However, there are possibility to get results in higher dimensional case using H-measure.

Secondly, it is shown that the uniqueness of entropy solutions to (P). Then, we draw a direct line with the methods of Karlsen-Ohlberger [6] and Karlsen-Risebro-Towers [10]. In particular, we use the definition of entropy solution and the crossing condition for the function $A(x,\xi)$ in Karlsen-Risebro-Towers [10].

Throughout this paper, we use the following notation:

$$\partial_x \alpha(x,u) = [\partial_x \alpha](x,u) + [\partial_\xi \alpha](x,u)\partial_x u,$$

for $\alpha(\cdot,\xi) \in W^{1,1}(\mathbb{R})$ for $\xi \in \mathbb{R}$, $\alpha(x,\cdot) \in \mathrm{Lip}(\mathbb{R})$ for $x \in \mathbb{R}$, $\alpha(x,0) = 0$ for $x \in \mathbb{R}$, and $u \in W^{1,1}(\mathbb{R})$ (see [2, 5]). Moreover, we suppose that u_ε^δ vanishes sufficiently fast as $|x| \to \infty$, if necessary.

2 Assumptions and the main results

In this section, we present some assumptions and the main results. At first, we assume that the initial function $u_0 \in BV(\mathbb{R})$ satisfies:

$$L_1 < u_0 < L_2,$$

where L_1 and L_2 are some real numbers with $L_1 < L_2$. In one dimensional case, it hold that $BV(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Thus, the assumption does not give a restriction to (P). Moreover, we suppose the following conditions:

$$\{\mathrm{A1}\} \begin{cases} A(\cdot,\xi) \in BV(\mathbb{R}) \;\; \text{for } \xi \in \mathbb{R}, \text{ and } A(x,\cdot) \in \mathrm{Lip}_{loc}(\mathbb{R}) \;\; \text{for } x \in \mathbb{R}, \\ A(x,0) = 0, \quad \text{for } x \in \mathbb{R}. \end{cases}$$

$$\{\mathrm{A2}\} \begin{cases} \beta(\cdot,\xi) \in C^2(\mathbb{R}) \cap W^{2,1}(\mathbb{R}), \;\; [\partial_\xi \beta](\cdot,\xi) \in C^1(\mathbb{R}) \;\; \text{for } \xi \in \mathbb{R}, \\ \beta(x,\cdot), \; [\partial_x \beta](x,\cdot), \; [\partial_\xi \beta](x,\cdot) \in \mathrm{Lip}_{loc}(\mathbb{R}) \;\; \text{for } x \in \mathbb{R}, \\ \beta(x,0) = [\partial_x \beta](x,0) = 0, \quad \text{for } x \in \mathbb{R}, \\ [\partial_\xi \beta](x,0) = 0 \text{ for } x \in \mathbb{R}, \text{ or } [\partial_\xi \beta](x,\xi) \equiv \text{const. for } (x,\xi) \in \mathbb{R} \times [L_1, L_2], \\ \beta(x,\xi) \text{ is nondecreasing with respect to } \xi \text{ for any } x \in \mathbb{R}. \end{cases}$$

$$\{\mathrm{A3}\} \;\; \partial_x A(x,L_1) - \partial_x^2 \beta(x,L_1) \le 0, \;\; \partial_x A(x,L_2) - \partial_x^2 \beta(x,L_2) \ge 0 \;\; \text{in } \mathbb{R}.$$

The conditions {A1} and {A2} are regularity assumptions for the functions $A(x,\xi)$ and $\beta(x,\xi)$. The condition {A3} is used to prove an uniform L^∞ estimate for approximate solutions to (P). Moreover, we assume a nondegenerate condition for $A(x,\xi)$ with respect to ξ in the sense of Aleksić-Mitrovic [1]:

{A4} There exists a function $h(x,\xi) \in C^1(\mathbb{R}_\xi; L^\infty(\mathbb{R}))$ such that for a.e. $x \in \mathbb{R}$ and for all $\lambda \in S^1$, there is no interval on which $\lambda_0 h(x,\xi) + \lambda_1 (A(x,\xi) - [\partial_x \beta](x,\xi))$ is constant in ξ.

Throughout this paper, we usually assume the conditions {A1}-{A4}. On the other hand, we impose the initial function u_0 to additional regularity assumption:

{A5} $|-A(x,u_0)+\partial_x\beta(x,u_0)|_{BV(\mathbb{R})}<\infty$.

Under the assumptions, we formulate the regularized problem for (P) as follows:

$$(RP)\begin{cases}\partial_t u_\varepsilon^\delta+\partial_x A^\delta(x,u_\varepsilon^\delta)=\partial_x^2\beta_\varepsilon(x,u_\varepsilon^\delta),\ (x,t)\in\Pi_T,\\ u_\varepsilon^\delta(x,0)=u_0^\delta(x),\end{cases}$$

where $A^\delta(x,\xi)$ is mollification of $A(x,\xi)$ with respect to x, that is, for $\xi\in\mathbb{R}$,

$$A^\delta(x,\xi)=(1/\delta)\omega(x/\delta)*A(x,\xi),$$

where $\omega:\mathbb{R}\to\mathbb{R}$ is an arbitrary smooth function such that $\omega(x)=\omega(-x)$, $\omega(x)=0$ for $|x|\geq 1$, and $\int_{\mathbb{R}}\omega(x)dx=1$. Moreover, we set

$$u_0^\delta(x)=(1/\delta)\omega(x/\delta)*u_0(x).$$

Here, $*$ stands for the convolution operator. In addition, we put $\beta_\varepsilon(x,\xi)=\beta(x,\xi)+\varepsilon\xi$ for $\varepsilon>0$. Therefore, we use the following notation:

$$\partial_x\beta_\varepsilon(x,u)=[\partial_x\beta](x,u)+[\partial_\xi\beta_\varepsilon](x,u)\partial_x u, \tag{2}$$

for $u\in BV(\mathbb{R})$, where $[\partial_\xi\beta_\varepsilon](x,u)=[\partial_\xi\beta](x,u)+\varepsilon$.

Remark 1. In the case that $A(x,\xi)=\gamma(x)f(\xi)$, the condition $\{A3\}$ is closed to the condition: $f(L_1)=f(L_2)=0$ which is used in Karlsen-Risebro-Towers [9].

We may prove the strong convergence of u_ε^δ in $L^1(\Pi_T)$ as ε, $\delta\to 0$. In fact, we get the following results:

Theorem 2.1. *We assume the conditions $\{A1\}$-$\{A4\}$. If $\delta=c\varepsilon$, for a constant $c>0$, then the family of approximate solutions $\{u_\varepsilon\}_{\varepsilon>0}\equiv\{u_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ to (P) is strongly precompact in $L^1_{loc}(\Pi_T)$. Moreover, the limit function u is an entropy solution to (P).*

Here, we define entropy solutions to (P) as follows:

Definition 2.2. Let $u_0\in BV(\mathbb{R})$. A function $u\in L^1(\mathbb{R}\times(0,T))\cap L^\infty(\mathbb{R}\times(0,T))$ is called an *entropy solution* to the problem (P), if it satisfies the following conditions:

(1) $\partial_x\beta(x,u)\in L^2(0,T;L^2(\mathbb{R}))$.

(2) For $\varphi\in C_0^\infty(\mathbb{R}\times(0,T))^+$ and $k\in\mathbb{R}$,

$$\int_0^T\int_{\mathbb{R}}\operatorname{sgn}(u-k)\{(u-k)\varphi_t-[\partial_x\beta(x,u)-\partial_x\beta(x,k)]\partial_x\varphi+[A(x,u)-A(x,k)]\partial_x\varphi\}dxdt$$
$$-\int_0^T\int_{\mathbb{R}\setminus\Omega_S}\operatorname{sgn}(u-k)\partial_x A(x,k)\varphi dxdt+\int_0^T\int_{\Omega_S}\varphi|D_x^s A(x,k)|dt$$
$$+\int_0^T\int_{\mathbb{R}}\operatorname{sgn}(u-k)\partial_x^2\beta(x,k)\varphi dxdt+\int_{\mathbb{R}}|u_0(x)-k|\varphi dx\geq 0,$$

where Ω_S is an area where the measure $D_x A(x,\xi)$ is singular with respect to x.

Our second purpose of this paper is to prove the uniqueness of entropy solutions. To see this, we introduce the following additional assumptions:

$$\{A6\} \qquad \beta(x,\xi) \equiv \gamma(x)\tilde{\beta}(u), \quad \gamma(x) > 0 \ \text{ for } x \in \mathbb{R}.$$

Notice that, the functions $\gamma(x)$ and $\tilde{\beta}(\xi)$ satisfy the conditions corresponding to {A2}.

{A7} $[\partial_x A](x,\cdot) \in Lip_{loc}(\mathbb{R})$ for $x \in \mathbb{R}$,

{A8} There exists a family of points $\{x_i\}_{i=1}^M$ such that $A(\cdot,\xi)$ is discontinuous at $x = x_i$ for all $\xi \in [L_1, L_2]$ and $i = 1,\cdots,M$. Here, M is a positive constant. That is, $A(\cdot,\xi)$ belongs to $SBV(\mathbb{R})$ and has finitely many jumps for all $\xi \in [L_1, L_2]$.

{A9} For any jump point $x \in \mathbb{R}$,

$$A(x_+,\xi) - A(x_-,\xi) < 0 < A(x_+,\eta) - A(x_-,\eta) \Rightarrow \xi < \eta.$$

The condition {A9} is called a crossing condition. The conditions {A8} and {A9} is used in Karlsen-Risebro-Towers [10] to prove the uniqueness of entropy solutions for strongly degenerate parabolic equations with discontinuous convective terms. Then, we get second main result.

Theorem 2.3. *We assume the conditions {A1}-{A4} and {A6}-{A9}, then an entropy solution u to (P) is uniquely determind.*

3 Estimates for the approximate solution u_ε^δ.

In this section, we prove several estimates for the approximate solution u_ε^δ. Throughout this section, we usually assume the conditions {A1}-{A4}. At first, we prove the following L^1 and L^∞-estimate:

Lemma 3.1 (L^1 bound). *For $t \geq s \geq 0$, it follows that*

$$||u_\varepsilon^\delta(\cdot,t)||_{L^1(\mathbb{R})} \leq ||u_\varepsilon^\delta(\cdot,s)||_{L^1(\mathbb{R})} \leq ||u_0^\delta||_{L^1(\mathbb{R})}.$$

Proof. Let us give the following approximate equation posed in (RP):

$$\partial_t u_\varepsilon^\delta + \partial_x A^\delta(x,u_\varepsilon^\delta) = \partial_x^2 \beta_\varepsilon(x,u_\varepsilon^\delta). \tag{3}$$

Multiplying both side on the above equality by the approximated signum function $\mathrm{sgn}_\rho(u_\varepsilon^\delta)$, $\rho > 0$, then it follows that

$$\begin{aligned}
\partial_t|u_\varepsilon^\delta| &= -\lim_{\rho\to 0} \mathrm{sgn}'_\rho(u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta[\partial_x\beta_\varepsilon(x,u_\varepsilon^\delta) - A^\delta(x,u_\varepsilon^\delta)] \\
&= -\lim_{\rho\to 0} \mathrm{sgn}'_\rho(u_\varepsilon^\delta)\{\partial_x u_\varepsilon^\delta([\partial_x\beta](x,u_\varepsilon^\delta) - A^\delta(x,u_\varepsilon^\delta)) + ([\partial_\xi\beta](x,u_\varepsilon^\delta) + \varepsilon)(\partial_x u_\varepsilon^\delta)^2\},
\end{aligned}$$

as $\rho \to 0$ in the sense of distribution by $A(\cdot,\xi) \in BV(\mathbb{R}) \subset L^1(\mathbb{R})$ and $\beta(\cdot,\xi) \in W^{1,1}(\mathbb{R})$ for all $\xi \in \mathbb{R}$. The first term of right-hand side on the above equality is equal to zero by the property $\lim_{\rho \to 0} \mathrm{sgn}'_\rho(\xi)\xi = 0$ for all $\xi \in \mathbb{R}$ and $[\partial_x \beta](x,0) = A(x,0) = 0$ for all $x \in \mathbb{R}$. The second term of it is nonnegative by the property $\mathrm{sgn}'_\rho(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Hence, we have

$$\int_{\mathbb{R}} |u_\varepsilon^\delta(x,t)|dx \leq \int_{\mathbb{R}} |u_\varepsilon^\delta(x,s)|dx \leq \int_{\mathbb{R}} |u_0^\delta|dx,$$

for all $t \geq s \geq 0$. □

Lemma 3.2 (L^∞ bound). *There exists a positive constant c_1, independent of ε and δ, such that*

$$||u_\varepsilon^\delta(\cdot,t)||_{L^\infty(\mathbb{R})} < c_1,$$

for $t > 0$. In particular, $L_1 \leq u_\varepsilon^\delta \leq L_2$ hold in Π_T.

Proof. For all $\gamma > 0$, we consider the following auxiliary problem:

$$(RP)_\gamma \begin{cases} \partial_t v(x,t) + \partial_x A^\delta(x,v) = \partial_x^2 \beta_\varepsilon(x,v) + \gamma h(v), \\ v(x,0) = u_0^\delta, \quad L_1 < u_0 < L_2, \end{cases}$$

where $h(v) = L_1 + L_2 - 2v$. Then, there exists a unique $C^{2,1}$ classical solution v to $(RP)_\gamma$ with the initial function $v(x,0) \in (L_1, L_2)$ for all $x \in \mathbb{R}$ by the classical theory for uniformly parabolic equations [12]. By Lemma 3.1 and $u_0 \in BV(\mathbb{R})$, the classical solution v is $L^1(\Pi_T) \cap L^\infty(\Pi_T)$-function for sufficiently small γ. Moreover, v belongs to $BV(\mathbb{R})$ for a.e. $t \in (0,T)$ by the method of Vol'pert-Hudjaev [16].

We lead a contradiction to show the result. Here, we put a subset $K \subset \Pi_T$ such that $v(x,t) \geq L_2$ for all $(x,t) \in K$. By $v \in BV(\mathbb{R})(\cap L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, the set K is compact (i.e. closed bounded). If K is nonempty, then we put

$$\bar{t} = \inf\{\ t \in (0,T) \mid \text{there exists } \bar{x} \in \mathbb{R} \text{ such that } v(\bar{x},t) = L_2\}.$$

By the inequality $L_1 < u_0 < L_2$, $\bar{t}$ is positive. By compactness of K and the smoothness of v, there must be a point $\bar{x}$ such that $v(\cdot,\bar{t})$ has a local maximum at $\bar{x}$ and $v(\bar{x},\bar{t}) = L_2$. Because, if $v(x,\bar{t}) \neq L_2$ for all $x \in \mathbb{R}$, then it must be that $v(x,\bar{t}) > L_2$ or $v(x,\bar{t}) < L_2$ for all $x \in \mathbb{R}$. The former contradict the definition of $\bar{t}$ by continuity of v with respect to t and $L_1 < v(x,0) < L_2$. The latter also contradict compactness of K.

For $\bar{x} \in \mathbb{R}$, we have the following properties:

$$\partial_x v(\bar{x},\bar{t}) = 0, \quad \partial_x^2 v(\bar{x},\bar{t}) \leq 0 \quad \text{and} \quad \partial_t v(\bar{x},\bar{t}) \geq 0.$$

On the other hand, it holds that

$$h(v(\bar{x},\bar{t})) = h(L_2) < 0.$$

Therefore, we obtain

$$\begin{aligned} &\partial_t v(\bar{x},\bar{t}) + [\partial_x A^\delta](\bar{x}, v(\bar{x},\bar{t})) - [\partial_x^2 \beta](\bar{x}, v(\bar{x},\bar{t})) \\ &= [\partial_\xi \beta_\varepsilon](x, v(\bar{x},\bar{t}))\partial_x^2 v(\bar{x},\bar{t}) + \gamma h(v(\bar{x},\bar{t})) \leq \gamma h(L_2) < 0 \end{aligned}$$

by the equation in $(RP)_\gamma$ at $(\bar{x},\bar{t})$. By the condition {A3}, this is a contradiction.

Therefore, it follows that K is empty and $v \leq L_2$. It is similar to prove in the case that $v \geq L_1$.

Using the continuous dependence result in [4], we have $v \to u_\varepsilon^\delta$ pointwise as $\gamma \downarrow 0$. Hence, we get the claim of this Lemma. □

Secondly, we prove a Lipschitz regularity of u_ε^δ with respect to t. To use the Panov's compactness result, this regularity estimate is necessary. In fact, Karlsen-Rascle-Tadmor [7] and Aleksić-Mitrovic [1] used this regularity estimate to prove strongly precompactness for a sequence of approximate solutions to a two dimensional hyperbolic scalar conservation laws using this regularity estimate.

Lemma 3.3 (Lipschitz regularity in time). *We assume the condition $\{A5\}$. If $\delta = c\varepsilon$, for a constant $c > 0$, then there exists a constant c_2, independent of ε and δ, such that for all $t > 0$,*

$$\int_{\mathbb{R}} |\partial_t u_\varepsilon^\delta(\cdot, t)| dx \leq c_2.$$

Proof. Differentiate both side on the above equality (3) in Lemma 3.1 with respect to t and put $w_\varepsilon^\delta = \partial_t u_\varepsilon^\delta$, then we have

$$\partial_t w_\varepsilon^\delta + \partial_x([\partial_\xi A^\delta](x, u_\varepsilon^\delta) w_\varepsilon^\delta) = \partial_x^2([\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta) w_\varepsilon^\delta).$$

Multiplying both side on the above equality by the approximated signum function $\mathrm{sgn}_\rho(w_\varepsilon^\delta)$, $\rho > 0$, then it satisfies the following equality:

$$\begin{aligned} \partial_t |w_\varepsilon^\delta| = {} & \partial_x^2([\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)|w_\varepsilon^\delta|) - \lim_{\rho \downarrow 0} \mathrm{sgn}'_\rho(w_\varepsilon^\delta) \partial_x([\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta) w_\varepsilon^\delta) \partial_x w_\varepsilon^\delta \\ & - \partial_x([\partial_\xi A^\delta](x, u_\varepsilon^\delta)|w_\varepsilon^\delta|), \end{aligned} \tag{4}$$

as $\rho \to 0$ in the sense of distribution. Here, it is computed that

$$\begin{aligned} \mathrm{sgn}'_\rho(w_\varepsilon^\delta) \partial_x([\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta) w_\varepsilon^\delta) \partial_x w_\varepsilon^\delta = {} & \mathrm{sgn}'_\rho(w_\varepsilon^\delta)([\partial_x \partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta) w_\varepsilon^\delta \partial_x w_\varepsilon^\delta \\ & + [\partial_\xi^2 \beta_\varepsilon](x, u_\varepsilon^\delta) \partial_x u_\varepsilon^\delta w_\varepsilon^\delta \partial_x w_\varepsilon^\delta + [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)(\partial_x w_\varepsilon^\delta)^2) \equiv \sum_{i=1}^{3} B_i. \end{aligned} \tag{5}$$

Here, we see that

$$\lim_{\rho \to 0}(B_1 + B_2) = \lim_{\rho \to 0} \mathrm{sgn}'_\rho(w_\varepsilon^\delta) w_\varepsilon^\delta([\partial_x \partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta) \partial_x w_\varepsilon^\delta + [\partial_\xi^2 \beta_\varepsilon](x, u_\varepsilon^\delta) \partial_x u_\varepsilon^\delta \partial_x w_\varepsilon^\delta) = 0,$$

by $\lim_{\rho \to 0} \mathrm{sgn}'_\rho(\xi)\xi = 0$ for all $\xi \in \mathbb{R}$. Moreover, $B_3 \geq 0$ hold using $\mathrm{sgn}'_\rho(\xi) \geq 0$ and $[\partial_\xi \beta_\varepsilon](x, \xi) \geq 0$ for all $(x, \xi) \in \mathbb{R}^2$. Therefore, we obtain the following estimate:

$$\int_{\mathbb{R}} |w_\varepsilon^\delta(x, t)| dx \leq \int_{\mathbb{R}} |w_\varepsilon^\delta(x, 0)| dx,$$

for all $t > 0$. Here, it follows that

$$\begin{aligned} \int_{\mathbb{R}} |w_\varepsilon^\delta(x, 0)| dx &= \int_{\mathbb{R}} |\partial_x^2 \beta_\varepsilon(x, u_0^\delta) - \partial_x A^\delta(x, u_0^\delta)| dx \\ &\leq C + \varepsilon \int_{\mathbb{R}} |\partial_x^2 u_0^\delta| dx \leq C + \frac{\varepsilon}{\delta} \int_{\mathbb{R}} |\partial_x u_0^\delta| dx < c_2, \end{aligned}$$

for some constant C and c_2 by the assumption $\{A5\}$, $\delta = c\varepsilon$ for a constant $c > 0$ and $u_0 \in BV(\mathbb{R})$. Therefore, we get the desired estimate. $\square$

Lemma 3.4 (Entropy dissipation bound). *There exists a constant $c_3 > 0$, independent of ε and δ, such that for all $t > 0$,*

$$\int_{\mathbb{R}} [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)(\partial_x u_\varepsilon^\delta(\cdot, t))^2 dx \leq c_3.$$

Proof. We begin with the approximate equation (3). Multiplying (3) by u_ε^δ and integrating the result on $\mathbb{R}$ with respect to x implies

$$\int_{\mathbb{R}} [u_\varepsilon^\delta \partial_t u_\varepsilon^\delta + u_\varepsilon^\delta \partial_x A^\delta(x, u_\varepsilon^\delta)] dx = \int_{\mathbb{R}} u_\varepsilon^\delta \partial_x([\partial_x \beta](x, u_\varepsilon^\delta) + [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta) dx.$$

We note that the second term of right-hand side in the above equation becomes

$$\int_{\mathbb{R}} u_\varepsilon^\delta \partial_x([\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta) dx = -\int_{\mathbb{R}} [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)(\partial_x u_\varepsilon^\delta)^2 dx.$$

Then, we have the following equality:

$$\int_{\mathbb{R}} [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)(\partial_x u_\varepsilon^\delta)^2 dx = -\int_{\mathbb{R}} u_\varepsilon^\delta [\partial_t u_\varepsilon^\delta + \partial_x A^\delta(x, u_\varepsilon^\delta) - \partial_x[\partial_x \beta](x, u_\varepsilon^\delta)] dx. \tag{6}$$

The second and third terms of the right-hand side in (6) imply

$$-\int_{\mathbb{R}} u_\varepsilon^\delta(\partial_x A^\delta(x, u_\varepsilon^\delta) - \partial_x[\partial_x \beta](x, u_\varepsilon^\delta)) dx = \int_{\mathbb{R}} \partial_x u_\varepsilon^\delta (A^\delta(x, u_\varepsilon^\delta) - [\partial_x \beta](x, u_\varepsilon^\delta)) dx$$

$$= \int_{\mathbb{R}} \left[\partial_x \left(\int_0^{u_\varepsilon^\delta} [A^\delta(x, \xi) - [\partial_x \beta](x, \xi)] d\xi \right) - \int_0^{u_\varepsilon^\delta} ([\partial_x A^\delta](x, \xi) - [\partial_x^2 \beta](x, \xi)) d\xi \right] dx.$$

Therefore, we have

$$\int_{\mathbb{R}} [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)(\partial_x u_\varepsilon^\delta)^2 dx$$
$$= -\int_{\mathbb{R}} u_\varepsilon^\delta \partial_t u_\varepsilon^\delta dx - \int_{\mathbb{R}} \left(\int_0^{u_\varepsilon^\delta} ([\partial_x A^\delta](x, \xi) - [\partial_x^2 \beta](x, \xi)) d\xi \right) dx,$$

by $A(\cdot, \xi) \in BV(\mathbb{R})$ and $\beta(\cdot, \xi) \in W^{1,1}(\mathbb{R})$ for all $\xi \in \mathbb{R}$. Hence, we have the following estimate:

$$\int_{\mathbb{R}} [\partial_\xi \beta_\varepsilon](x, u_\varepsilon^\delta)(\partial_x u_\varepsilon^\delta)^2 dx \leq ||u_\varepsilon^\delta||_{L^\infty(\Pi_T)} ||\partial_t u_\varepsilon^\delta||_{L^\infty(0,T;L^1(\mathbb{R}))}$$
$$+ \max\{|L_1|, |L_2|\}(\sup_{L_1 \leq \xi \leq L_2} |A^\delta(\cdot, \xi)|_{BV(\mathbb{R})} + \sup_{L_1 \leq \xi \leq L_2} |\partial_x^2 \beta(\cdot, \xi)|_{C(\mathbb{R})}),$$

by {A1} and {A2}. □

The method of compensated compactness and H-measure is usually used for hyperbolic conservation laws. In the case of degenerate parabolic equation, it is important to get several estimates about the degenerate diffusion term. At first, we can obtain the following regularity estimate.

Lemma 3.5. *There exists a positive constant C, depend on T but not on ε and δ, such that*

$$||\partial_x\beta(\cdot,u_\varepsilon^\delta)||_{L^2(\mathbb{R}\times(0,T))} < C,$$

and

$$||\beta(\cdot,u_\varepsilon^\delta(\cdot,\cdot+\tau)) - \beta(\cdot,u_\varepsilon^\delta(\cdot,\cdot))||_{L^2(\mathbb{R}\times(0,T-\tau))} \le C\sqrt{\tau},$$

for all $\tau \ge 0$. In particular, $\{\beta(x,u_\varepsilon^\delta)\}_{\varepsilon,\delta>0}$ is strongly compact in $L^2_{loc}(\Pi_T)$.

Proof. The first assertion is satisfied as follows:

$$\int_0^T\int_{\mathbb{R}}|\partial_x\beta(x,u_\varepsilon^\delta)|^2dxdt \le \frac{1}{2}\int_0^T\int_{\mathbb{R}}[\partial_x\beta](x,u_\varepsilon^\delta)^2dxdt$$
$$+\frac{1}{2}\max_{\xi\in[L_1,L_2]}||[\partial_\xi\beta](\cdot,\xi)||_{L^\infty(\mathbb{R})}\int_0^T\int_{\mathbb{R}}[\partial_\xi\beta](x,u_\varepsilon^\delta)|\partial_x u_\varepsilon^\delta|^2dxdt < C$$

by the assumption $\{A2\}$, the equality (2) and Lemma 3.4.

On the other hand, we prove the second assertion as follows:

$$\int_0^{T-\tau}\int_{\mathbb{R}}[\beta(x,u_\varepsilon^\delta(x,t+\tau)) - \beta(x,u_\varepsilon^\delta(x,t))]^2dxdt$$
$$\le ||\beta||_{Lip([L_1,L_2])}\int_0^{T-\tau}\int_{\mathbb{R}}(\int_t^{t+\tau}\partial_t u_\varepsilon^\delta(x,\xi)d\xi)(\beta(x,u_\varepsilon^\delta(x,t+\tau)) - \beta(x,u_\varepsilon^\delta(x,t)))dxdt$$
$$= ||\beta||_{Lip([L_1,L_2])}\int_0^{T-\tau}\int_{\mathbb{R}}(\int_t^{t+\tau}[-\partial_x A^\delta(x,u_\varepsilon^\delta(x,\xi)) + \partial_x^2\beta_\varepsilon(x,u_\varepsilon^\delta(x,\xi))]d\xi)$$
$$(\beta(x,u_\varepsilon^\delta(x,t+\tau)) - \beta(x,u_\varepsilon^\delta(x,t)))dxdt$$
$$= ||\beta||_{Lip([L_1,L_2])}\int_0^{\tau}[\int_0^{T-\tau}\int_{\mathbb{R}}[-\partial_x A^\delta(x,u_\varepsilon^\delta(x,t+s)) + \partial_x^2\beta_\varepsilon(x,u_\varepsilon^\delta(x,t+s))]$$
$$(\beta(x,u_\varepsilon^\delta(x,t+\tau)) - \beta(x,u_\varepsilon^\delta(x,t)))dxdt]ds$$
$$= ||\beta||_{Lip([L_1,L_2])}\int_0^{\tau}[\int_0^{T-\tau}\int_{\mathbb{R}}[A^\delta(x,u_\varepsilon^\delta(x,t+s))(\partial_x\beta(x,u_\varepsilon^\delta(x,t+\tau)) - \partial_x\beta(x,u_\varepsilon^\delta(x,t)))$$
$$-\partial_x\beta_\varepsilon(x,u_\varepsilon^\delta(x,t+s))(\partial_x\beta(x,u_\varepsilon^\delta(x,t+\tau)) - \partial_x\beta(x,u_\varepsilon^\delta(x,t)))dxdt]ds$$
$$\le ||\beta||_{Lip([L_1,L_2])}\int_0^{\tau}(||A(x,u_\varepsilon^\delta)||^2_{L^2(\mathbb{R}\times[0,T])} + ||\partial_x\beta(x,u_\varepsilon^\delta)||_{L^2(\mathbb{R}\times[0,T])}$$
$$+2||\partial_x\beta_\varepsilon(x,u_\varepsilon^\delta)||_{L^2(\mathbb{R}\times[0,T])}||\partial_x\beta(x,u_\varepsilon^\delta)||_{L^2(\mathbb{R}\times[0,T])})ds < C\tau,$$

by the assumptions $\{A1\}$, $\{A2\}$ and the first assertion.

□

Lemma 3.6. *A subsequence of $\{\beta(x,u_\varepsilon^\delta)\}_{\varepsilon,\delta>0}$ converges strongly to $\beta(x,u)$ in $L^2_{loc}(\Pi_T)$, where u is the $L^\infty(\Pi_T)$ weak $*$-limit of $\{u_\varepsilon^\delta\}_{\varepsilon,\delta>0}$. Furthermore,*

$$\beta(x,u) \in L^\infty(\Pi_T) \cap L^2(0,T;H^1(\mathbb{R})).$$

Moreover, we prove strong compactness of the total flux to (3). This result is the main idea of Karlsen-Risebro-Towers [9].

Lemma 3.7 (Compactness of the total flux). *We assume the condition $\{A5\}$. Let the total flux to (3):*

$$v_\varepsilon^\delta(x,t) = -A^\delta(x,u_\varepsilon^\delta) + \partial_x\beta_\varepsilon(x,u_\varepsilon^\delta). \tag{7}$$

Then, there exists a constant $C > 0$, independent of ε and δ, such that for all $t \in (0,T)$,

(i). $||v_\varepsilon^\delta(\cdot,t)||_{L^\infty(\mathbb{R})} \le C$,

(ii) $|v_\varepsilon^\delta(\cdot,t)|_{BV(\mathbb{R})} \le C$,

(iii) $||v_\varepsilon^\delta(\cdot,t+\tau) - v_\varepsilon^\delta(\cdot,t)||_{L^1(\mathbb{R})} \le C\sqrt{\tau}$ *for all* $\tau \ge 0$.

In particular, $\{v_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ is strongly compact in $L^1_{loc}(\Pi_T)$.

Proof. By the definition of v_ε^δ, it is clear that $\partial_x v_\varepsilon^\delta = \partial_t u_\varepsilon^\delta$. From this equality and (7), we have the following auxiliary problem:

$$\begin{cases} \partial_t v_\varepsilon^\delta = \partial_x([\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x v_\varepsilon^\delta) - [\partial_\xi A^\delta](x,u_\varepsilon^\delta)\partial_x v_\varepsilon^\delta + \gamma h(v_\varepsilon^\delta), \\ v_\varepsilon^\delta(x,0) = \partial_x\beta_\varepsilon(x,u_0^\delta(x)) - A^\delta(x,u_0^\delta(x)). \end{cases}$$

Here, we put

$$h(v_\varepsilon^\delta) = \overline{L_1} + \overline{L_2} - 2v_\varepsilon^\delta, \quad \overline{L_1} \equiv \operatorname*{ess\,inf}_{x\in\mathbb{R}}\{v_\varepsilon^\delta(x,0)\}, \quad \overline{L_2} \equiv \operatorname*{ess\,sup}_{x\in\mathbb{R}}\{v_\varepsilon^\delta(x,0)\}.$$

The proof of (i) is similar to the proof of Lemma 3.2.

We next prove (ii). By the equality $\partial_x v_\varepsilon^\delta = \partial_t u_\varepsilon^\delta$, it is inferred that

$$|v_\varepsilon^\delta|_{BV(\mathbb{R})} \equiv \int_{\mathbb{R}} |\partial_x v_\varepsilon^\delta| dx = \int_{\mathbb{R}} |\partial_t u_\varepsilon^\delta| dx.$$

By Lemma 3.3, we get the desired estimate (ii).

The proof of (iii) is similar to one of Karlsen-Risebro-Towers [9]. Therefore, we use the Frechét-Kolmogorov compactness theorem, then we obtain that $\{v_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ is strongly compact in $L^1_{loc}(\Pi_T)$. □

4 Proof of Theorem 2.1.

In this section, we prove the first main result. At first, we introduce a general form of the Panov compactness result to get strongly precompactness of $\{u_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ in $L^1_{loc}(\Pi_T)$.

Theorem 4.1 (Panov [13]). *Let $\Omega_T \equiv \Omega \times (0,T) \subset \mathbb{R}^{N+1}$ be an open set. Assume that the vector $\phi(x,\xi) \in (C(\mathbb{R}_\xi; BV(\Omega)))^{N+1}$ is non-degenerate with respect to ξ, i.e. for a.e. $x \in \Omega$ and for all $\lambda \in \mathbb{R}^{N+1}$, $\lambda \ne 0$, the map $\xi \mapsto (\lambda, \phi(x,\xi)) \ne$ constant on any nontrivial interval. Then, each bounded sequence $(u_k(x,t))_k \in L^\infty(\Omega_T)$, $L_1 \le u_k(x,t) \le L_2$ satisfying, for the Heviside function H and $k \in \mathbb{R}$,*

$$\nabla_{x,t} \cdot [H(u_k(x,t) - k)(\phi(x,u_k(x,t)) - \phi(x,k))] \text{ is precompact in } H^{-1}_{loc}(\Omega_T),$$

contains a subsequence which converges in $L^1_{loc}(\Omega_T)$.

Using Theorem 4.1, we prove the following result:

Theorem 4.2. *We assume the conditions $\{A1\}$-$\{A5\}$. If $\varepsilon = c\delta$, for a constant $c > 0$, then a family of approximate solutions $\{u_\varepsilon\}_{\varepsilon>0} \equiv \{u_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ is strongly precompact in $L^1_{loc}(\Pi_T)$.*

Proof. Let $h(x,\xi) \in C^1(\mathbb{R}_\xi; L^\infty(\mathbb{R}))$. We rewrite the equation of (3) as follows:

$$\partial_t h(x,u_\varepsilon^\delta) + \partial_x A^\delta(x,u_\varepsilon^\delta) = \partial_t h(x,u_\varepsilon^\delta) - \partial_t u_\varepsilon^\delta + \partial_x^2 \beta_\varepsilon(x,u_\varepsilon^\delta). \tag{8}$$

Here, we define the corresponding entropy fluxes:

$$\begin{aligned}
\varphi_0(x,\xi) &\equiv H(\xi-k)(h(x,\xi)-h(x,k)),\\
\varphi_1(x,\xi) &\equiv H(\xi-k)(A(x,\xi)-A(x,k)),\\
\varphi_1^\delta(x,\xi) &\equiv H(\xi-k)(A^\delta(x,\xi)-A^\delta(x,k)),\\
\varphi_2(x,\xi) &\equiv -H(\xi-k)([\partial_x\beta](x,\xi)-[\partial_x\beta](x,k)),
\end{aligned}$$

where H stands for the Heaviside function and k is an arbitrarily fixed real number. We multiply (8) by $\eta'(u_\varepsilon^\delta) = H(u_\varepsilon^\delta - k)$ on both side of (8) to obtain the following equality:

$$\begin{aligned}
&(\partial_t,\partial_x)\cdot(\varphi_0(x,u_\varepsilon^\delta),\varphi_1(x,u_\varepsilon^\delta)+\varphi_2(x,u_\varepsilon^\delta))\\
&= \eta'(u_\varepsilon^\delta)(-\partial_x A^\delta(x,u_\varepsilon^\delta)+\partial_t h(x,u_\varepsilon^\delta)-\partial_t u_\varepsilon^\delta+\partial_x^2\beta_\varepsilon(x,u_\varepsilon^\delta))+\partial_x(\varphi_1(x,u_\varepsilon^\delta)+\varphi_2(x,u_\varepsilon^\delta))\\
&= -\eta'(u_\varepsilon^\delta)\partial_x A^\delta(x,u_\varepsilon^\delta)+\eta'(u_\varepsilon^\delta)\partial_t h(x,u_\varepsilon^\delta)-\eta'(u_\varepsilon^\delta)\partial_t u_\varepsilon^\delta\\
&\quad +\eta'(u_\varepsilon^\delta)\partial_x([\partial_x\beta](x,u_\varepsilon^\delta)+[\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta)+\partial_x\varphi_1(x,u_\varepsilon^\delta)+\partial_x\varphi_2(x,u_\varepsilon^\delta),
\end{aligned}$$

in the sense of distribution by the calculation (2). Here, it is deduced that

$$\partial_x(\eta'(u_\varepsilon^\delta)[\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta) \geq \eta'(u_\varepsilon^\delta)\partial_x([\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta),$$

in a way similar to the calculation of (5). Moreover, we see that

$$\eta'(u_\varepsilon^\delta)\partial_x[\partial_x\beta](x,u_\varepsilon^\delta)+\partial_x\varphi_2(x,u_\varepsilon^\delta) = \eta'(u_\varepsilon^\delta)[\partial_x^2\beta](x,k).$$

Thus, it is obtained that

$$\begin{aligned}
&\partial_t\varphi_0(x,u_\varepsilon^\delta)+\partial_x\varphi_1(x,u_\varepsilon^\delta)+\partial_x\varphi_2(x,u_\varepsilon^\delta)\\
&\leq \eta'(u_\varepsilon^\delta)(\partial_t h(x,u_\varepsilon^\delta)-[\partial_x A^\delta](x,k)+[\partial_x^2\beta](x,k)-\partial_t u_\varepsilon^\delta)\\
&\quad +\partial_x(\eta'(u_\varepsilon^\delta)[\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta)+\partial_x[\varphi_1-\varphi_1^\delta](x,u_\varepsilon^\delta).
\end{aligned}$$

By the Schwartz lemma on nonnegative distribution [15, Lemma 37.2], a nonnegative distribution is a nonnegative measure. Therefore, there exists $\mu_k^{\varepsilon,\delta}(x,t) \in \mathcal{M}(\Pi_T)$ such that

$$\begin{aligned}
&\partial_t\varphi_0(x,u_\varepsilon^\delta)+\partial_x\varphi_1(x,u_\varepsilon^\delta)+\partial_x\varphi_2(x,u_\varepsilon^\delta)\\
&= \eta'(u_\varepsilon^\delta)(\partial_t h(x,u_\varepsilon^\delta)-[\partial_x A^\delta](x,k)+[\partial_x^2\beta](x,k)-\partial_t u_\varepsilon^\delta)\\
&\quad +\partial_x(\eta'(u_\varepsilon^\delta)[\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta)+\partial_x[\varphi_1-\varphi_1^\delta](x,u_\varepsilon^\delta)+\mu_k^{\varepsilon,\delta}(x,t).
\end{aligned} \tag{9}$$

Here, $\mathcal{M}(\Pi_T)$ is a family of Radon measure on Π_T. We verify the right-hand side of (9). First, it holds that

$$\eta'(u_\varepsilon^\delta)(\partial_t h(x,u_\varepsilon^\delta) - \partial_t u_\varepsilon^\delta) \in \mathcal{M}_{b,loc}(\Pi_T),$$

by the Lipschitz continuity in time for u_ε^δ (Lemma 3.3). Here, $\mathcal{M}_{b,loc}(\Pi_T)$ is a family of locally bounded Radon measure. Moreover, it is observed that

$$\eta'(u_\varepsilon^\delta)(-[\partial_x A^\delta](x,k) + [\partial_x^2 \beta](x,k)) \in \mathcal{M}_{b,loc}(\Pi_T),$$

by the regularity assumptions {A1} and {A2}.

Next, we deal with the degenerate diffusion terms as follows:

$$\partial_x(\eta'(u_\varepsilon^\delta)[\partial_\xi \beta_\varepsilon](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta) = \partial_x(\eta'(u_\varepsilon^\delta)[\partial_\xi \beta](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta) + \varepsilon\partial_x(\eta'(u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta).$$

By the entropy dissipation bound (Lemma 3.4), we get the following convergence:

$$\int_{\Pi_T} |\varepsilon\eta'(u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta|^2 dxdt \le C\varepsilon \int_{\Pi_T} \varepsilon|\partial_x u_\varepsilon^\delta|^2 dxdt < C\varepsilon \to 0,$$

as $\varepsilon \downarrow 0$. On the other hand, we treat another part. To see this, we divide the domain Π_T as follows :

$$H := \{(x,t) \in \Pi_T \mid l(x,\beta(x,u(x,t))) < L(x,\beta(x,u(x,t)))\},$$

$$P := \{(x,t) \in \Pi_T \mid l(x,\beta(x,u(x,t))) = L(x,\beta(x,u(x,t)))\},$$

where $l(x,\xi) = \min\{\lambda \in [L_1,L_2] : \beta(x,\lambda) = \xi\}$, $L(x,\xi) = \max\{\lambda \in [L_1,L_2] : \beta(x,\lambda) = \xi\}$. We begin to consider the degenerate diffusion term on H. In fact, it follows that

$$[\partial_\xi \beta](x,u_\varepsilon^\delta) \to 0 \ \text{ a.e. on } H \quad \text{as } \varepsilon \downarrow 0.$$

By the L^∞-bound (Lemma 3.2) and the entropy dissipation bound (Lemma 3.4) of u_ε^δ, we see that

$$\eta'(u_\varepsilon^\delta)[\partial_\xi \beta](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta \to 0 \ \text{ a.e. on } H \quad \text{as } \varepsilon \downarrow 0.$$

Secondly, we consider the degenerate diffusion term on P. By strong compactness of the total flux and the convergence of $\{u_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ a.e. on P (ref. [9, Lemma 3.3]), it is deduced that

$$\{\eta'(u_\varepsilon^\delta)[\partial_\xi \beta](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta\}_{\varepsilon,\delta>0} \text{ converges a.e. on } P. \tag{10}$$

On the other hand, by L^∞-bound and entropy dissipation bound, we have

$$\eta'(u_\varepsilon^\delta)[\partial_\xi \beta](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta \in L^2(\Pi_T). \tag{11}$$

By Lemma 3.7 (i), (10) and (11), the sequence $\{\eta'(u_\varepsilon^\delta)[\partial_\xi \beta](x,u_\varepsilon^\delta)\partial_x u_\varepsilon^\delta\}_{\varepsilon,\delta>0}$ converges strongly in $L^2(\Pi_T)$.

Finally, it holds that

$$\begin{aligned} |\varphi_1 - \varphi_1^\delta|(x,u_\varepsilon^\delta) &\le |A^\delta(x,u_\varepsilon^\delta) - A(x,u_\varepsilon^\delta)| + |A^\delta(x,k) - A(x,k)| \\ &\le 2 \max_{L_1 \le \xi \le L_2} |A^\delta(x,\xi) - A(x,\xi)| \to 0 \ \text{ in } L^2_{loc}(\mathbb{R}), \end{aligned}$$

as $\delta \downarrow 0$. Hence, we have $\partial_x[\varphi_1 - \varphi_1^\delta] \in H^{-1}_{c,loc}(\Pi_T)$ which is a family of functions that are precompact in $H^{-1}_{loc}(\Pi_T)$. Moreover, it follows that $\mu_k^{\varepsilon,\delta} \in \mathcal{M}_{b,loc}(\Pi_T)$. Therefore, we can use Theorem 4.1 using Lemma 4.3 below. Hence, we get the desired result. □

Lemma 4.3 (Murat). *Assume that a family (Q_ε) is bounded in $L^p(\Omega)$, $p > 2$, $\Omega \subset \mathbb{R}^N$ is an open set. Then,*

$$\nabla \cdot (Q_\varepsilon)_\varepsilon \in H^{-1}_{c,loc}(\Omega),$$

if $\nabla \cdot (Q_\varepsilon)_\varepsilon = p_\varepsilon + q_\varepsilon$ with $(q_\varepsilon)_\varepsilon \in H^{-1}_{c,loc}(\Omega)$ and $(p_\varepsilon)_\varepsilon \in \mathcal{M}_{b,loc}(\Omega)$.

Moreover, it should be checked that the limit function u constructed in Theorem 4.2 is a generalized solution to (P). In fact, u satisfies (P) in the sense of distribution and satisfies an entropy inequality in the sense of [4] and [10]. That is, it is inferred that there exists an entropy solution to (P).

Corollary 4.4. *Suppose that $\{A1\}$-$\{A5\}$ hold. The function u is the limit function constructed as the strong limit of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ in Theorem 4.2. Let v be another limit function as the strong limit of the sequence $\{v_\varepsilon\}_{\varepsilon>0}$, where v_ε solves the regularized problem (RP) corresponding to initial data v_0. Then, it holds the following properties:*

(i) the limit function u satisfy (P) in the sense of distribution.

(ii) the limit function u is an entropy solution to (P).

(iii) $||u(x,t) - v(x,t)||_{L^1(\mathbb{R})} \leq ||u_0(x) - v_0(x)||_{L^1(\mathbb{R})}$.

(iv) $|A(x, u(x,t)) - \partial_x\beta(x, u(x,t))|_{BV(\mathbb{R})} \leq C$, *for* $t \in (0,T)$.

(v) $||u(\cdot, t+\tau) - u(\cdot,t)||_{L^1(\mathbb{R})} \leq C\tau$, *for* $\tau \geq 0$.

Proof. By Theorem 4.2, we obtain the assertion (i) in a way similar to [9] and [21]. Using the result for (RP) in [4], it holds that

$$\int_{\mathbb{R}} |u_\varepsilon(x,t) - v_\varepsilon(x,t)|dx \leq \int_{\mathbb{R}} |u_0^\varepsilon(x) - v_0^\varepsilon(x)|dx. \tag{12}$$

As $\varepsilon \downarrow 0$, it is observed that the assertion (iii) holds for u_0, v_0 satisfying $\{$A5$\}$.

Moreover, the assertions (iv) and (v) are direct consequence of Lemma 3.7.

Finally, we prove the assertion (ii). Let u_ε^δ be the approximate solutions to (P). We set the following functions:

$$\eta(u_\varepsilon^\delta) = \operatorname{sgn}(u_\varepsilon^\delta - k)(u_\varepsilon^\delta - k),$$
$$q^1(x, u_\varepsilon^\delta) = \operatorname{sgn}(u_\varepsilon^\delta - k)(A^\delta(x, u_\varepsilon^\delta) - A^\delta(x,k)),$$
$$q^2(x, u_\varepsilon^\delta) = -\operatorname{sgn}(u_\varepsilon^\delta - k)([\partial_x\beta](x, u_\varepsilon^\delta) - [\partial_x\beta](x,k)),$$

for any $x \in \mathbb{R}$ and $k \in \mathbb{R}$. Then, we calculate that:

$$\begin{aligned}
&\partial_t\eta(u_\varepsilon^\delta) + \partial_x q^1(x, u_\varepsilon^\delta) + \partial_x q^2(x, u_\varepsilon^\delta) \\
&= \operatorname{sgn}(u_\varepsilon^\delta - k)(\partial_x^2\beta_\varepsilon(x, u_\varepsilon^\delta) - \partial_x A^\delta(x, u_\varepsilon^\delta)) + \operatorname{sgn}(u_\varepsilon^\delta - k)(A^\delta(x, u_\varepsilon^\delta) - A^\delta(x,k))_x \\
&\quad - \operatorname{sgn}(u_\varepsilon^\delta - k)([\partial_x\beta](x, u_\varepsilon^\delta) - [\partial_x\beta](x,k))_x \\
&= \operatorname{sgn}(u_\varepsilon^\delta - k)([\partial_\xi\beta_\varepsilon](x, u_\varepsilon^\delta)(u_\varepsilon^\delta)_x - A^\delta(x,k) + \partial_x\beta(x,k))_x \\
&= (\operatorname{sgn}(u_\varepsilon^\delta - k)[\partial_\xi\beta_\varepsilon](x, u_\varepsilon^\delta)(u_\varepsilon^\delta)_x)_x - \operatorname{sgn}'(u_\varepsilon^\delta - k)[\partial_\xi\beta_\varepsilon](x, u_\varepsilon^\delta)(u_\varepsilon^\delta)_x^2 \\
&\quad - \operatorname{sgn}(u_\varepsilon^\delta - k)(A^\delta(x,k) - \partial_x\beta(x,k))_x.
\end{aligned}$$

Therefore, it is deduced that

$$\operatorname{sgn}(u_\varepsilon^\delta - k)[(u_\varepsilon^\delta - k)_t + (A^\delta(x,u_\varepsilon^\delta) - A^\delta(x,k))_x - (\partial_x\beta(x,u_\varepsilon^\delta) - \partial_x\beta(x,k))_x]$$
$$+\operatorname{sgn}(u_\varepsilon^\delta - k)(A^\delta(x,k) - \partial_x\beta(x,k))_x = -\operatorname{sgn}'(u_\varepsilon^\delta - k)[\partial_\xi\beta_\varepsilon](x,u_\varepsilon^\delta)(u_\varepsilon^\delta)_x^2 \leq 0,$$

in the sense of distribution. That is, we get the following inequality:

$$\int_0^T\int_{\mathbb{R}} \operatorname{sgn}(u_\varepsilon^\delta - k)[(u_\varepsilon^\delta - k)\varphi_t + (A^\delta(x,u_\varepsilon^\delta) - A^\delta(x,k))\varphi_x - (\partial_x\beta(x,u_\varepsilon^\delta) - \partial_x\beta(x,k))\varphi_x$$
$$+(\partial_x A^\delta(x,k) - \partial_x^2\beta(x,k))\varphi]dxdt + \int_{\mathbb{R}} |u_0^\delta(x) - k|\varphi dx \geq 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times[0,T))^+$ and $k \in \mathbb{R}$. We take $\delta = c\varepsilon$, then we have the entropy inequality in Definition 2.2 as $\varepsilon \to 0$.

$\square$

Proof of Theorem 2.1. We remove the assumption {A5} by using the assertion (ii) in Corollary 4.4. If u_0 belongs to $BV(\mathbb{R})$, there exists a sequence $\{u_0^m\}_{m=1}^\infty$ such that each u_0^m satisfies {A5} and $u_0^m \to u_0$ in $L^1(\mathbb{R})$ as $m \to \infty$. Let u^m be a limit function of the sequence $\{u_\varepsilon\}$ with initial data u_0^m. Using the inequality (12), it holds that

$$\int_{\mathbb{R}} |u^m(x,t) - u^n(x,t)|dx \leq \int_{\mathbb{R}} |u_0^m(x) - u_0^n(x)|dx,$$

as $m,n \to \infty$. Therefore, $\{u^m\}_{m=1}^\infty$ is a Cauchy sequence in $L^1(\Pi_T)$. Hence, the limit function u is constructed under the assumptions {A1}-{A4}. In addition, it is also seen that the limit function u satisfies the assertions (i)-(v) in Corollary 4.4. $\square$

5 Proof of Theorem 2.3.

In this section, it may be confirmed that the limit function u is an unique entropy solution to (P). To see this, we prove the following assertion which is called Carrillo's lemma.

Lemma 5.1. *Let us assume {A1}-{A4} and {A6}. Let u be an entropy solution to (P). Then, it follows that*

$$\int_{\Pi_T} \operatorname{sgn}(u-k)[(u-k)\varphi_t + (A(x,u) - A(x,k))\varphi_x$$
$$-(\partial_x\beta(x,u) - \partial_x\beta(x,k))\varphi_x + (\partial_x^2\beta(x,k) - \partial_x A(x,k))\varphi]dxdt$$
$$= \lim_{\eta\to\infty}\int_{\Pi_T} \operatorname{sgn}_\eta'(\tilde\beta(u) - \tilde\beta(k))\gamma(x)(\partial_x\tilde\beta(u))^2\varphi dxdt,$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times(0,T))^+$ and $k \in \mathbb{R}\backslash E$. Here, $E \equiv \{\xi \in \mathbb{R} \mid \tilde\beta^{-1}(\xi)$ is discontinuous at $\xi\}$.

Proof. By the assertion (i) in Corollary 4.4, we have the following equality:

$$\int_{\Pi_T} (u\varphi_t + A(x,u)\varphi_x - \partial_x\beta(x,u)\varphi_x)dxdt = 0,$$

for $\varphi \in C_0^\infty(\mathbb{R}\times(0,T))$. Here, we set $\varphi = \operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi$ for $\eta > 0$, $k \in \mathbb{R}\backslash E$ and $\phi \in C_0^\infty(\mathbb{R}\times(0,T))$. Then, the first term of the above equality is calculated

$$\begin{aligned}
&\int_{\Pi_T} u(\operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi)_t dxdt = -\int_{\Pi_T} u_t \operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi dxdt \\
&= \int_{\Pi_T} [\int_k^u \operatorname{sgn}_\eta(\tilde\beta(\xi) - \tilde\beta(k))d\xi]\phi_t dxdt \to \int_{\Pi_T} |u-k|\phi_t dxdt,
\end{aligned}$$

as $\eta \to 0$ by Lemma 3.3. Moreover, it is observed that

$$\begin{aligned}
&\int_{\Pi_T} (A(x,u) - \partial_x\beta(x,u))(\operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi)_x dxdt \\
&= \int_{\Pi_T} (A(x,u) - A(x,k) - \partial_x\beta(x,u) + \partial_x\beta(x,k))(\operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi)_x dxdt \\
&\quad + \int_{\Pi_T} (A(x,k) - \partial_x\beta(x,k))(\operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi)_x dxdt \\
&= \int_{\Pi_T} (A(x,u) - A(x,k)) \operatorname{sgn}'_\eta(\tilde\beta(u) - \tilde\beta(k))\partial_x\tilde\beta(u)\phi dxdt \\
&\quad - \int_{\Pi_T} \operatorname{sgn}'_\eta(\tilde\beta(u) - \tilde\beta(k))\partial_x\gamma(x)(\tilde\beta(u) - \tilde\beta(k))\partial_x\tilde\beta(u)\phi dxdt \\
&\quad - \int_{\Pi_T} \operatorname{sgn}'_\eta(\tilde\beta(u) - \tilde\beta(k))\gamma(x)(\partial_x\tilde\beta(u))^2\phi dxdt \\
&\quad + \int_{\Pi_T} (A(x,u) - A(x,k) - \partial_x\beta(x,u) + \partial_x\beta(x,k)) \operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi_x dxdt \\
&\quad - \int_{\Pi_T} (\partial_x A(x,k) - \partial_x^2\beta(x,k)) \operatorname{sgn}_\eta(\tilde\beta(u) - \tilde\beta(k))\phi dxdt \\
&\to -\lim_{\eta\to 0}\int_{\Pi_T} \operatorname{sgn}'_\eta(\tilde\beta(u) - \tilde\beta(k))\gamma(x)(\partial_x\tilde\beta(u))^2\phi dxdt \\
&\quad + \int_{\Pi_T} \operatorname{sgn}(u-k)(A(x,u) - A(x,k) - \partial_x\beta(x,u) + \partial_x\beta(x,k))\phi_x dxdt \\
&\quad - \int_{\Pi_T} \operatorname{sgn}(u-k)(\partial_x A(x,k) - \partial_x^2\beta(x,k))\phi dxdt,
\end{aligned}$$

as $\eta \to 0$. Hence, we get the desired result. □

Next, we prove a Kato's type inequality. To see this, we introduce test functions. Let a non-negative function $\delta(\sigma) \in C_0^\infty(\mathbb{R})$ satisfying

$$\delta(\sigma) = \delta(-\sigma), \quad \delta(\sigma) = 0, \quad \text{for } |\sigma| \geq 1, \quad \text{and} \quad \int_{\mathbb{R}} \delta(\sigma)d\sigma = 1.$$

For $\rho > 0$, we set

$$\delta_\rho(t) = \frac{1}{\rho}\delta\left(\frac{t}{\rho}\right), \quad \text{and} \quad \omega_\rho(x) = \frac{1}{2\rho^N}\delta\left(\frac{|x|^2}{\rho^2}\right).$$

For the above functions, we can see that

$$\partial_t\delta_\rho(t-s) = \frac{1}{\rho^2}\delta'\left(\frac{t-s}{\rho}\right) = -\partial_s\delta_\rho(t-s),$$

$$\partial_x\omega_\rho(x-y) = \frac{1}{\rho^{N+2}}(x-y)\delta'\left(\frac{|x-y|^2}{\rho^2}\right) = -\partial_y\omega_\rho(x-y).$$

Here, we define the function $\varphi = \varphi(x,t,y,s) \in C_0^\infty(\Pi_T \times \Pi_T)$ by

$$\varphi(x,t,y,s) = \psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\omega_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right),$$

where $\psi = \psi(x,t) \in C_0^\infty(\Pi_T)$ is another non-negative test function. Having in mind the above test function, we deal with the following assertion:

Proposition 5.2. *Let us assume $\{A1\}$-$\{A4\}$ and $\{A6\}$. Let u and v be entropy solutions to (P). Moreover, it additionally assume that $A(\cdot,\xi) \in W^{1,1}(\mathbb{R})$ for $\xi \in [L_1, L_2]$. Then, there exists a positive constant C such that*

$$\begin{aligned}\int_{\Pi_T} &\operatorname{sgn}(u-v)[(u-v)\varphi_t + (A(x,u)-A(x,v))\varphi_x \\ &-(\partial_x\beta(x,u)-\partial_x\beta(x,v))\varphi_x]dxdt + C\int_{\Pi_T}|u-v|\varphi dxdt \geq 0,\end{aligned} \tag{13}$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times(0,T))^+$.

Proof. By $A(\cdot,\xi) \in W^{1,1}(\mathbb{R})$ for $\xi \in [L_1, L_2]$, the entropy inequality for u in Definition 2.2 can be written below:

$$\begin{aligned}\int_{\Pi_T} &\operatorname{sgn}(u-k)[(u-k)\varphi_t + (A(x,u)-A(x,k))\varphi_x - (\partial_x\beta(x,u)-\partial_x\beta(x,k))\varphi_x \\ &+(\partial_x A(x,k) - \partial_x^2\beta(x,k))\varphi]dxdt \geq 0,\end{aligned} \tag{14}$$

for all $\varphi \in C_0^\infty(\mathbb{R}\times(0,T))^+$ and $k \in \mathbb{R}$. Let $v(y,s)$ be another entropy solution to (P) in $(y,s) \in \mathbb{R}\times(0,T)$. We set $k = v(y,s)$ in (14) and integrate both side with respect to $(y,s) \in \mathbb{R}\times(0,T)$, then we get the following inequality:

$$\begin{aligned}\int_{\Pi_T\times\Pi_T} &\operatorname{sgn}(u-v)[(u-v)\varphi_t + (A(x,u)-A(x,v))\varphi_x - (\partial_x\beta(x,u)-\partial_x\beta(x,v))\varphi_x \\ &+(\partial_x A(x,v) - \partial_x^2\beta(x,v))\varphi]dxdtdyds \geq 0.\end{aligned}$$

Here, we write the right hand-side in the above inequality by $I(\Pi_T\times\Pi_T)$. By the entropy inequality (14), it follows that

$$I(\Pi_T\times\Pi_T) = I(\Pi_T\times(\Pi_T\backslash\mathscr{E}_v)) + I(\Pi_T\times\mathscr{E}_v) \geq I(\Pi_T\times(\Pi_T\backslash\mathscr{E}_v)).$$

Here, we set

$$\mathscr{E}_u = \{(x,t) \in \Pi_T \mid \tilde{\beta}(u(x,t)) \in E\}, \quad \mathscr{E}_v = \{(y,s) \in \Pi_T \mid \tilde{\beta}(v(y,s)) \in E\}.$$

Taking into account Lemma 5.1, we see that

$$I(\Pi_T \times (\Pi_T \backslash \mathscr{E}_v)) = \lim_{\eta \to 0} \int_{\Pi_T \times (\Pi_T \backslash \mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde{\beta}(u) - \tilde{\beta}(v))\gamma(x)(\partial_x \tilde{\beta}(u))^2 \varphi dxdtdyds.$$

In view of this, the following inequality is valid:

$$\begin{aligned}
&\int_{\Pi_T \times \Pi_T} \mathrm{sgn}(u-v)[(u-v)\varphi_t + (A(x,u) - A(x,v))\varphi_x - (\partial_x \beta(x,u) - \partial_x \beta(x,v))\varphi_x \\
&\qquad\qquad + (\partial_x A(x,v) - \partial_x^2 \beta(x,v))\varphi] dxdtdyds \\
&\geq \lim_{\eta \to 0} \int_{\Pi_T \times (\Pi_T \backslash \mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde{\beta}(u) - \tilde{\beta}(v))\gamma(x)(\partial_x \tilde{\beta}(u))^2 \varphi dxdtdyds \\
&= \lim_{\eta \to 0} \int_{(\Pi_T \backslash \mathscr{E}_u) \times (\Pi_T \backslash \mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde{\beta}(u) - \tilde{\beta}(v))\gamma(x)(\partial_x \tilde{\beta}(u))^2 \varphi dxdtdyds.
\end{aligned}$$

Similarly, we also get another inequality:

$$\begin{aligned}
&\int_{\Pi_T \times \Pi_T} \mathrm{sgn}(v-u)[(v-u)\varphi_s + (A(y,v) - A(y,u))\varphi_y - (\partial_y \beta(y,v) - \partial_y \beta(y,u))\varphi_y \\
&\qquad\qquad + (\partial_y A(y,u) - \partial_y^2 \beta(y,u))\varphi] dxdtdyds \\
&\geq \lim_{\eta \to 0} \int_{(\Pi_T \backslash \mathscr{E}_u) \times \Pi_T} \mathrm{sgn}'_\eta(\tilde{\beta}(u) - \tilde{\beta}(v))\gamma(y)(\partial_y \tilde{\beta}(v))^2 \varphi dxdtdyds \\
&= \lim_{\eta \to 0} \int_{(\Pi_T \backslash \mathscr{E}_u) \times (\Pi_T \backslash \mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde{\beta}(u) - \tilde{\beta}(v))\gamma(y)(\partial_y \tilde{\beta}(v))^2 \varphi dxdtdyds.
\end{aligned}$$

Summing up the above two inequalities, we see that

$$\begin{aligned}
&\int_{\Pi_T \times \Pi_T} \mathrm{sgn}(u-v)[(u-v)(\varphi_t + \varphi_s) + (A(x,u) - A(x,v))\varphi_x + (A(y,v) - A(y,u))\varphi_y \\
&\qquad\qquad - (\partial_x \beta(x,u) - \partial_x \beta(x,v))\varphi_x - (\partial_y \beta(y,v) - \partial_y \beta(y,u))\varphi_y \\
&\qquad\qquad + (\partial_x A(x,v) - \partial_x^2 \beta(x,v))\varphi + (\partial_y A(y,u) - \partial_y^2 \beta(y,u))\varphi] dxdtdyds \\
&\geq \lim_{\eta \to 0} \int_{(\Pi_T \backslash \mathscr{E}_u) \times (\Pi_T \backslash \mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde{\beta}(u) - \tilde{\beta}(v))[\gamma(x)(\partial_x \tilde{\beta}(u))^2 + \gamma(y)(\partial_y \tilde{\beta}(v))^2]\varphi dxdtdyds \\
&\equiv I_{RHS}.
\end{aligned}$$

We calculate the left-hand side in the above inequality, respectively. To see this, we use the test function $\varphi = \psi(\frac{x+y}{2}, \frac{t+s}{2})\omega_\rho(\frac{x-y}{2})\delta_\rho(\frac{t-s}{2})$, for $\rho > 0$. Then, the first term is computed that

$$\int_{\Pi_T \times \Pi_T} |u-v|(\varphi_t + \varphi_s) dxdtdyds = \int_{\Pi_T \times \Pi_T} |u-v|(\psi_t + \psi_s)\omega_\rho \delta_\rho dxdtdyds.$$

Secondly, the convection terms are considered below:

$$\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)[(A(x,u)-A(x,v))\varphi_x+(A(y,v)-A(y,u))\varphi_y$$
$$+(\partial_x A(x,v)+\partial_y A(y,u))\varphi]dxdtdyds$$
$$=\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)[(A(x,u)-A(y,v))\varphi_x+[(A(y,v)-A(x,v))\varphi]_x$$
$$-(A(y,v)-A(x,v))\varphi_y-[(A(x,u)-A(y,u))\varphi]_y]dxdtdyds$$
$$=\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)[(A(x,u)-A(y,v))(\varphi_x+\varphi_y)$$
$$+[(A(y,v)-A(x,v))\varphi]_x-[(A(x,u)-A(y,u))\varphi]_y]dxdtdyds\equiv\sum_{i=1}^{3} I_A^i.$$

Let us put $\varphi=\psi(\frac{x+y}{2},\frac{t+s}{2})\omega_\rho(\frac{x-y}{2})\delta_\rho(\frac{t-s}{2})$, for $\rho>0$, then $I_A^2+I_A^3$ is equal to

$$\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)\{[(A(y,v)-A(x,v))_x-(A(x,u)-A(y,u))_y]\psi\omega_\rho\delta_\rho$$
$$+[(A(y,v)-A(x,v))\psi_x-(A(x,u)-A(y,u))\psi_y]\omega_\rho\delta_\rho$$
$$+[(A(y,v)-A(x,v))(\omega_\rho)_x-(A(x,u)-A(y,u))(\omega_\rho)_y]\psi\delta_\rho\}dxdtdyds\equiv\sum_{i=4}^{6} I_A^i.$$

Letting $\rho\to 0$, the convergence $I_A^5\to 0$ hold. Moreover, it follows that

$$I_A^4\to\int_{\Pi_T} \operatorname{sgn}(u-v)[(\partial_x A)(x,u)-(\partial_x A)(x,v)]\psi dxdt.$$

In addition, we see that

$$I_A^6=\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)[(A(y,v)-A(x,v))+(A(x,u)-A(y,u))](\omega_\rho)_x\psi\delta_\rho\}dxdtdyds$$
$$=\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)[(A(x,u)-A(x,v))-(A(y,u)-A(y,v))](\omega_\rho)_x\psi\delta_\rho\}dxdtdyds,$$

by the property of ω_ρ. Thirdly, we investigate the diffusion terms as follows:

$$\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)[-(\partial_x\beta(x,u)-\partial_x\beta(x,v))\varphi_x-(\partial_y\beta(y,v)-\partial_y\beta(y,u))\varphi_y$$
$$-\partial_x^2\beta(x,v))\varphi-\partial_y^2\beta(y,u)\varphi]dxdtdyds$$
$$=\int_{\Pi_T\times\Pi_T} \operatorname{sgn}(u-v)\{[(-\partial_x\gamma(x)\tilde\beta(u)+\partial_x\gamma(x)\tilde\beta(v))\varphi_x+(\partial_y\gamma(y)\tilde\beta(v)-\partial_y\gamma(y)\tilde\beta(u))\varphi_y]$$
$$+[-\gamma(x)\partial_x\tilde\beta(u)\varphi_x+\gamma(y)\partial_y\tilde\beta(v)\varphi_y]+[\partial_x^2\gamma(x)\tilde\beta(v)-\partial_y^2\gamma(y)\tilde\beta(u)]\varphi\}dxdtdyds\equiv\sum_{i}^{3} I_\beta^i.$$

We also consider each term in the above equality, respectively. We start by checking I_β^1 below:

$$I_\beta^1=\int_{\Pi_T\times\Pi_T}\operatorname{sgn}(u-v)[-(\partial_x\gamma(x)\tilde{\beta}(u)-\partial_y\gamma(y)\tilde{\beta}(v))(\varphi_x+\varphi_y)$$

$$+(\partial_x\gamma(x)\tilde{\beta}(u)\varphi_y-\partial_y\gamma(y)\tilde{\beta}(v)\varphi_x+\partial_x\gamma(x)\tilde{\beta}(v)\varphi_x-\partial_y\gamma(y)\tilde{\beta}(u)\varphi_y)]dxdtdyds\equiv\sum_{i=1}^{2}I_\beta^{1,i}.$$

Especially, $I_\beta^{1,2}$ is computed that

$$\begin{aligned}I_\beta^{1,2}&=-\int_{\Pi_T\times\Pi_T}\operatorname{sgn}(u-v)(\partial_x\gamma(x)-\partial_y\gamma(y))(\tilde{\beta}(u)-\tilde{\beta}(v))\varphi_x dxdtdyds\\&=\int_{\Pi_T\times\Pi_T}\{\partial_x^2\gamma(x)[\operatorname{sgn}(u-v)(\tilde{\beta}(u)-\tilde{\beta}(v))]\varphi\\&\qquad+(\partial_x\gamma(x)-\partial_y\gamma(y))[\operatorname{sgn}(u-v)(\tilde{\beta}(u)-\tilde{\beta}(v))]_x\varphi\}dxdtdyds.\end{aligned}$$

Let us also put $\varphi=\psi(\frac{x+y}{2},\frac{t+s}{2})\omega_\rho(\frac{x-y}{2})\delta_\rho(\frac{t-s}{2})$, for $\rho>0$, then we obtain

$$\lim_{\rho\to 0}I_\beta^{1,2}=\int_{\Pi_T}\partial_x^2\gamma(x)\operatorname{sgn}(u-v)(\tilde{\beta}(u)-\tilde{\beta}(v))\psi dxdt,$$

by $\gamma\in C^2(\mathbb{R})$ and $\tilde{\beta}(u)\in H^1(\mathbb{R})$ for a.e. $t\in(0,T)$. Meanwhile, we see that

$$\lim_{\rho\to 0}I_\beta^{3}=-\int_{\Pi_T}\partial_x^2\gamma(x)\operatorname{sgn}(u-v)(\tilde{\beta}(u)-\tilde{\beta}(v))\psi dxdt.$$

On the other hand, we deal with I_β^2. Taking into account the definition of $\mathscr{E}_u$ and $\mathscr{E}_v$, the following calculation is valid:

$$\begin{aligned}I_\beta^2&=\int_{\Pi_T\times\Pi_T}\operatorname{sgn}(u-v)[-\gamma(x)\partial_x\tilde{\beta}(u)\varphi_x+\gamma(y)\partial_y\tilde{\beta}(v)\varphi_y]dxdtdyds\\&=\int_{(\Pi_T\setminus\mathscr{E}_u)\times\Pi_T}\operatorname{sgn}(\tilde{\beta}(u)-\tilde{\beta}(v))\gamma(x)\partial_x\tilde{\beta}(u)\varphi_y dxdtdyds\\&\quad-\int_{\Pi_T\times(\Pi_T\setminus\mathscr{E}_v)}\operatorname{sgn}(\tilde{\beta}(u)-\tilde{\beta}(v))\gamma(y)\partial_y\tilde{\beta}(v)\varphi_x dxdtdyds\\&=\int_{(\Pi_T\setminus\mathscr{E}_u)\times(\Pi_T\setminus\mathscr{E}_v)}\operatorname{sgn}_\eta'(\tilde{\beta}(u)-\tilde{\beta}(v))(\gamma(x)+\gamma(y))\partial_x\tilde{\beta}(u)\partial_y\tilde{\beta}(v)\varphi dxdtdyds.\end{aligned}$$

Therefore, it is observed that

$$
\begin{aligned}
& I_{RHS} - I_\beta^2 \\
&= \lim_{\eta\to 0}\int_{(\Pi_T\setminus\mathscr{E}_u)\times(\Pi_T\setminus\mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde\beta(u)-\tilde\beta(v))[\gamma(x)(\partial_x\tilde\beta(u))^2+\gamma(y)(\partial_y\tilde\beta(v))^2 \\
&\qquad\qquad -(\gamma(x)+\gamma(y))\partial_x\tilde\beta(u)\partial_y\tilde\beta(v)]\varphi dxdtdyds \\
&= \lim_{\eta\to 0}\int_{(\Pi_T\setminus\mathscr{E}_u)\times(\Pi_T\setminus\mathscr{E}_v)} \mathrm{sgn}'_\eta(\tilde\beta(u)-\tilde\beta(v))([\sqrt{\partial_x\gamma(x)}\partial_x\tilde\beta(u)-\sqrt{\partial_y\gamma(y)}\partial_y\tilde\beta(v)]^2 \\
&\qquad\qquad -[\sqrt{\partial_x\gamma(x)}-\sqrt{\partial_y\gamma(y)}]^2\partial_x\tilde\beta(u)\partial_y\tilde\beta(v))\varphi dxdtdyds.
\end{aligned}
$$

Consequently, we see that

$$
\begin{aligned}
&\int_{\Pi_T}\mathrm{sgn}(u-v)[(u-v)\partial_t\psi+(A(x,u)-A(y,v))\partial_x\psi \\
&\qquad -(\partial_x\gamma(x)\tilde\beta(u)-\partial_y\gamma(y)\tilde\beta(v))\partial_x\psi]dxdt \\
&\quad+\int_{\Pi_T}\mathrm{sgn}(u-v)([\partial_xA](x,u)-[\partial_xA](x,v))\psi dxdt+\lim_{\rho\to 0}I_A^6 \\
&\geq -\lim_{\rho\to 0}\lim_{\eta\to 0}\int_{(\Pi_T\setminus\mathscr{E}_u)\times(\Pi_T\setminus\mathscr{E}_v)}\mathrm{sgn}'_\eta(\tilde\beta(u)-\tilde\beta(v))[\sqrt{\partial_x\gamma(x)}-\sqrt{\partial_y\gamma(y)}]^2 \\
&\qquad\qquad \partial_x\tilde\beta(u)\partial_y\tilde\beta(v)\varphi dxdtdyds,
\end{aligned}
$$

as $\rho\to 0$. The right-hand side of the above inequality equal to zero using the method of Kalrsen-Ohlberger [6, Proof of Theorem 2.1]. Furthermore, we compute that

$$
\begin{aligned}
\lim_{\rho\to 0}I_A^6 &\leq \frac{C||\delta'||_{L^\infty(\mathbb{R})}}{4}\lim_{\rho\to 0}\int_{\Pi_T\times\Pi_T}\frac{|x-y|^2}{\rho^2}\frac{\chi_{|x-y|<2\rho}}{\rho^N}|u-v|\psi\delta_\rho dxdtdyds \\
&= \frac{C||\delta'||_{L^\infty(\mathbb{R})}}{4}\int_{\Pi_T}|u-v|\psi dxdt.
\end{aligned}
$$

Meanwhile, we obtain

$$
\begin{aligned}
&\int_{\Pi_T}\mathrm{sgn}(u-v)([\partial_xA](x,u)-[\partial_xA](x,v))\psi dxdt \\
&\leq ||\partial_\xi\partial_xA(x,\xi)||_{L^\infty(\mathbb{R}^2)}\int_{\Pi_T}|u-v|\psi dxdt,
\end{aligned}
$$

by $\{A7\}$. Hence, we conclude the desired inequality. □

Theorem 5.3. *Let us assume $\{A1\}$-$\{A4\}$ and $\{A6\}$-$\{A9\}$. Let u and v be entropy solutions to (P) associated with initial functions u_0 and v_0. Then, there exists a positive constant C such that*

$$
||u(\cdot,t)-v(\cdot,t)||_{L^1(\mathbb{R})}\leq e^{Ct}||u_0-v_0||_{L^1(\Omega)},
$$

for a.e. $t \in (0,T)$. *In particular, for each initial value* u_0, *an entropy solution is uniquely determined.*

Proof. By the assumption {A8} and Kato's type inequality (13), it is seen that

$$\tag{15} \begin{aligned} &\int_{\Pi_T} \operatorname{sgn}(u-v)[(u-v)\varphi_t + (A(x,u)-A(x,v))\varphi_x \\ &\qquad -(\partial_x\beta(x,u)-\partial_x\beta(x,v))\varphi_x]dxdt + C\int_{\Pi_T} |u-v|\varphi dxdt \geq 0 \end{aligned}$$

for all $\varphi \in C_0^\infty(\Pi_T \backslash \{x_m\}_{m=1}^M)^+$. Here, $\{x_m\}_{m=1}^M$ is a family of jump points for $A(\cdot,\xi)$ with respect to x for $\xi \in [L_1, L_2]$. For near the jump points, the second and third terms in the above inequality make the following form:

$$J \equiv \sum_{m=1}^{M} \int_0^T [\operatorname{sgn}(u-v)\{(A(x,u)-A(x,v)) - (\partial_x\beta(x,u)-\partial_x\beta(x,v))\}]_{x=\xi_m-}^{x=\xi_m+} \phi(\xi_m,t)dt,$$

for $\phi \in C_0^\infty(\Pi_T)$. Applying the crossing condition {A9} and the method of Karlsen-Risebro-Towers [10], it is observed that $J \leq 0$. Therefore, we have the inequality (15) for all $\psi \in C_0^\infty(\Pi_T)$.

In the inequality (15), we substitute the following test function:

$$\varphi_r(x) = \int_{\mathbb{R}} \delta(|x-y|)\chi_{|y|<r} dy \quad \text{and} \quad \lambda_\rho(t) = \int_{-\infty}^{t} (\delta_\rho(\tau-t_1) - \delta_\rho(\tau-t_2))d\tau,$$

for $0 < t_1 < t_2 < T$ and $r > 1$. Then, it follows that

$$\partial_x\varphi_r(x) = 0, \quad \text{for } |x| < r-1 \text{ or } |x| > r+1.$$

Let us put $\psi(x,t) = \varphi_r(x)\lambda_\rho(t)$, then it is deduced that

$$\begin{aligned} &\lim_{r\to\infty} \int_{\Pi_T} \operatorname{sgn}(u-v)[(A(x,u)-A(x,v))\psi_x + (\beta(x,u)-\beta(x,v))\psi]dxdt \\ &\leq C \lim_{r\to\infty} \int_0^T \int_{||x|-r|\leq 1} (|u|+|v|)dxdt = 0, \end{aligned}$$

by $u, v \in L^1(\mathbb{R})$ for a.e. $t \in (0,T)$. Hence we have

$$\int_{\Pi_T} |u-v|(\lambda_\rho)_t dxdt + C\int_{\Pi_T} |u-v|\lambda_\rho dxdt \geq 0.$$

Letting $\rho \to 0$, it is deduced that

$$\int_{\mathbb{R}} |u(x,t_1)-v(x,t_1)|dxdt - \int_R |u(x,t_2)-v(x,t_2)|dxdt + C\int_{t_1}^{t_2} |u-v|dxdt \geq 0.$$

Using Gronwall's inequality, we can get

$$||u(\cdot,t_2)-v(\cdot,t_2)||_{L^1(\mathbb{R})} \leq e^{C(t_2-t_1)}||u(\cdot,t_1)-v(\cdot,t_1)||_{L^1(\mathbb{R})}.$$

Letting $t_1 \to 0$ and setting $t_2 = T$, we obtain the desired result. □

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