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On the Fitzpatrick Theory

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This note is devoted to Professor Nobuyuki Kenmochi on the occasion of his 70th birthday

1. Introduction

In this note we illustrate a nonstandard variational technique that may be used to study variational and quasi-variational inequalities, and revisit a result of [36].

Let V be a real Banach space, $\beta : V \to \mathcal{P}(V')$ (the set of the parts of V') be a (possibly multi-valued) operator, and $z' \in V'$. A large class of nonlinear either stationary or evolutionary problems may be formulated in abstract form as

$$\text{find } u \in V \text{ such that} \qquad \beta(u) \ni z' \quad \text{in } V'. \tag{1.1}$$

In several cases β is maximal monotone in the sense of Minty and Browder; see e.g. [5,7,26]. In a more general set-up

$$\begin{aligned} &\beta(v) = \alpha_v(v) \qquad \forall v \in V, \text{ with} \\ &\alpha_v : V \to \mathcal{P}(V') \text{ maximal monotone for any } v \in V. \end{aligned} \tag{1.2}$$

In this case (1.1) reads

$$\text{find } u \in V \text{ such that} \qquad (\beta(u) =)\ \alpha_u(u) \ni z' \quad \text{in } V'. \tag{1.3}$$

Here are some examples:

(i) Let Ω be an open subset of $\mathbf{R}^N$ ($N \geq 1$), a measurable function $\phi = \phi(x, v, \xi)$ be continuous in $v \in \mathbf{R}$, and monotone in $\xi \in \mathbf{R}^N$. Setting $\alpha_v(w) := -\operatorname{div} \phi(x, v, \nabla w)$ in $\mathcal{D}'(\Omega)$ for any $v, w \in H^1_0(\Omega)$, (1.3) reads

$$\text{find } u \in H^1_0(\Omega) \text{ such that} \qquad -\operatorname{div} \phi(x, u, \nabla u) = z' \quad \text{in } H^{-1}(\Omega). \tag{1.4}$$

$\beta(u) = -\operatorname{div} \phi(x, u, \nabla u)$ is a typical *operator of the calculus of variations;* see e.g. [21,22].

(ii) It is well known that a multi-valued operator may also account for the presence of a constraint. For instance, let $\alpha_v : V \to V$ be single-valued and maximal monotone for any $v \in V$, K be a closed convex subset of V, and denote by ∂I_K the subdifferential of the indicator function of K (in the sense of convex analysis, see e.g. [12,14,31]). The inclusion

$$\alpha_u(u) + \partial I_K(u) \ni z' \tag{1.5}$$

is equivalent to the following variational inequality:

$$u \in K, \qquad \langle \alpha_u(u) - z', u - v \rangle \leq 0 \qquad \forall v \in K. \tag{1.6}$$

In particular, one may consider $\alpha_u(u) = -\operatorname{div}\phi(x,u,\nabla u)$ as in (1.4).

(iii) The inclusion (1.1) also encompasses a number of nonlinear evolutionary problems, e.g.,

$$D_t u + \alpha_u(u) + \partial I_K(u) \ni z' \qquad \text{in } W', \text{ in }]0,T[\ (D_t := \partial/\partial t). \tag{1.7}$$

In this case $V = L^p(0,T;W)$ for some real Banach space W, $p \in [2,+\infty[$, and K is a closed convex subset of W. Here also one may take $\alpha_u(u) = -\operatorname{div}\phi(x,u,\nabla u)$.

Note. The author is please to devote this little work to Professor Nobuyuki Kenmochi, a master and a friend.

2. Outline of the Fitzpatrick Theory

Next we briefly review a variational representation of maximal monotone operators, that was introduced by S. Fitzpatrick in the seminal paper [15].

Fitzpatrick associated to any operator $\alpha : V \to \mathcal{P}(V')$ the following function:

$$f_\alpha(v,v') := \sup\left\{\langle v',w\rangle - \langle w',w-v\rangle : w' \in \alpha(w)\right\} \qquad \forall (v,v') \in V \times V'. \tag{2.1}$$

(f_α was then named the *Fitzpatrick function* of α.) Being a pointwise supremum of a family of continuous and linear functions, f_α is convex and lower semicontinuous.

Theorem 2.1 *[15] If α is maximal monotone then*

$$f_\alpha(v,v') \geq \langle v',v\rangle \qquad \forall (v,v') \in V \times V', \tag{2.2}$$

$$f_\alpha(v,v') = \langle v',v\rangle \quad \Leftrightarrow \quad v' \in \alpha(v). \tag{2.3}$$

Defining the further function

$$J(v,v') := f_\alpha(v,v') - \langle v',v\rangle \qquad \forall (v,v') \in V \times V'. \tag{2.4}$$

(2.3) also reads

$$J(v,v') = \inf J = 0 \quad \Leftrightarrow \quad v' \in \alpha(v). \tag{2.5}$$

We may label "$J(v,v') = \inf J = 0$" a problem of *null-minimization.*

As it is expressed in (2.5), the maximal monotone relation is tantamount to minimizing J with respect to both variables. On the other hand, for a prescribed v', in order to determine v such that $v' \in \alpha(v)$ the functional $J(\cdot,v')$ is only minimized with respect to the first variable. As it is illustrated in Sect. 7 of [36], in this case it is necessary to prescribe the vanishing of the minimum value, in order to exclude the onset of spurious minimizers.

The prescription of the minimum value is a crucial issue of this theory, which thus differs from an ordinary variational principle.

Representative functions. The notion of Fitzpatrick function was extended as follows. One says that a convex and lower semicontinuous function $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$ (variationally) *represents* the operator $\alpha : V \to \mathcal{P}(V')$ whenever it fulfills the system (that we shall refer to as the *Fitzpatrick system)*

$$g(v,v') \geq \langle v', v\rangle \qquad \forall (v,v') \in V \times V', \tag{2.6}$$

$$g(v,v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in \alpha(v). \tag{2.7}$$

Accordingly, we shall say that g is a *representative* function of α, and that α is *representable.* Let us denote the class of these functions by $\mathcal{F}(V)$.

For instance, for any convex and lower semicontinuous function $\varphi : V \to \mathcal{P}(V')$, the classical Fenchel function [14]

$$g(v,v') := \varphi(v) + \varphi^*(v') \tag{2.8}$$

represents the operator $\partial\varphi$. In this case the Fitzpatrick system (2.6), (2.7) is reduced to the Fenchel system

$$\varphi(v) + \varphi^*(v') \geq \langle v', v\rangle \qquad \forall (v,v') \in V \times V', \tag{2.9}$$

$$\varphi(v) + \varphi^*(v') = \langle v', v\rangle \quad \Leftrightarrow \quad v' \in \partial\varphi(v). \tag{2.10}$$

This is a well-known result in convex analysis, see e.g. [12,14,31].

Representable operators are monotone; but, at variance with subdifferentials, they need not be either *cyclically* monotone or maximal monotone. Some results of this theory are briefly reviewed e.g. in [30,34,35].

Some results. Let us next assume that the Banach space V is reflexive, although this is not really needed for several of the results that follow. Besides the duality between V an V', let us consider the duality between the spaces $V \times V'$ and its dual $V' \times V$, and the corresponding convex conjugation. More specifically, for any function $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$, let us set

$$g^*(w',w) := \sup\left\{\langle w', v\rangle + \langle v', w\rangle - g(v,v') : (v,v') \in V \times V'\right\} \qquad \forall (w',w) \in V' \times V. \tag{2.11}$$

Here are some relevant results of this theory.

Theorem 2.2 *[11,32] A function $g \in \mathcal{F}(V)$ represents a maximal monotone operator $\alpha : V \to \mathcal{P}(V')$ if and only if $g^* \in \mathcal{F}(V')$. In this case g^* represents the inverse operator $\alpha^{-1} : V' \to \mathcal{P}(V)$.*

The convex biconjugate function of f_α, denoted by $(f_\alpha)^{**}$, thus also represents α, whenever the operator α is maximal monotone.

Theorem 2.3 *[10,15,25,28] Let $\alpha : V \to \mathcal{P}(V')$ be a maximal monotone operator, f_α be its Fitzpatrick function, and $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function. Then*

$$g \in \mathcal{F}(V), \; g \text{ represents } \alpha \quad \Leftrightarrow \quad f_\alpha \leq g \leq (f_\alpha)^{**}. \tag{2.12}$$

Corollary 2.4 *If two functions $g_1, g_2 \in \mathcal{F}(V)$ represent a maximal monotone operator $V \to \mathcal{P}(V')$, then $\max\{g_1, g_2\} \in \mathcal{F}(V)$ represents the same operator.*

Theorem 2.5 *[3] Let $\alpha : V \to \mathcal{P}(V')$ be a maximal monotone operator, f_α be its Fitzpatrick function, and set*

$$\begin{aligned} F_\alpha(v,v',w,w') &:= f_\alpha(v+w, v'+w') + f_\alpha(v-w, v'-w') + \|w\|_V^2 + \|w'\|_{V'}^2 \\ &\qquad \forall (v,v'), (w,w') \in V \times V', \end{aligned} \tag{2.13}$$

$$\phi_\alpha(v,v') := \tfrac{1}{2} \inf\left\{F_\alpha(v,v',w,w') : (w,w') \in V \times V'\right\} \qquad \forall (v,v') \in V \times V'. \tag{2.14}$$

Then
$$\phi^*_\alpha(v',v) = \phi_\alpha(v,v') \qquad \forall (v,v') \in V. \tag{2.15}$$

Because of (2.15), the function ϕ_α is called a *self-dual representative* of α. Its use allows one to replace the null-minimization (2.5) by an ordinary minimization, since in this case it is granted that the minimum value vanishes.

Corollary 2.6 *Under the assumptions of Theorem 2.5, let us set*
$$\widetilde{J}(v,v') := \phi_\alpha(v,v') - \langle v',v\rangle \qquad \forall (v,v') \in V\times V'. \tag{2.16}$$
Then $\inf \widetilde{J} = 0$*, so that*
$$\widetilde{J}(v,v') = \inf \widetilde{J} \quad \Longleftrightarrow \quad v' \in \alpha(v). \tag{2.17}$$

Proof. By (2.15)
$$\widetilde{J}(v,v') = \tfrac{1}{2}[\phi_\alpha(v,v') + \phi^*_\alpha(v',v)] - \langle v',v\rangle \qquad \forall (v,v') \in V\times V'.$$
By the classical Fenchel system (2.9), (2.10) then $\inf \widetilde{J} = 0$. □

Existence methods. The above variational formulation may be used to prove existence of a solution for several problems of the form (1.1) for a representable operator β. We briefly illustrate some basic techniques.

(i) A subdifferential flow of the form $D_t u + \partial\varphi(u) \ni z'$, (with $\varphi : V \to \mathcal{P}(V')$ convex and lower semicontinuous), may be reformulated as a null-minimization problem along the lines of Brezis and Ekeland [6] and Nayroles [27]. (These two works predate [15], but already contain some elements of the Fitzpatrick theory). (1)(2)

(ii) An inclusion like (1.1) may be approximated by a sequence of inclusions (or equalities) for which existence of a solution is already known; uniform estimates may then be derived. This approximated problem may be represented as an equivalent null-minimization problem, and the limit may be taken in this formulation. If in this procedure the functional is also approximated, the Γ-convergence must also be proved — a nontrivial task for evolutionary problems; see e.g. [34,35].

(iii) (1.1) may be reformulated via a *self-dual* representative function, see e.g. [3]. This approach was investigated by Ghoussoub and coworkers; see e.g. [16,17,18] and references therein.

(iv) Along the lines of [35], here in Sect. 3 existence of a solution of an inclusion like (1.1) is proved, first by reformulating the problem via a representative function, and then applying an extension of the classical minimax theorem of Ky Fan; see Theorem 3.3 ahead.

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(1) They pointed out that the gradient flow $D_t u + \partial\psi(u) = z'$ is tantamount to the null-minimization of the functional
$$\Phi(v,z') = \int_0^T [\psi(v) + \psi(z' - D_t v)]\, dt + \tfrac{1}{2}(\|v(T)\|^2_H - \|u(0)\|^2_H) - \langle z',v\rangle,$$
as v ranges in $H^1(0,T;V') \cap L^2(0,T;V)$ $(\subset C^0([0,T];H))$. (Here $V \subset H = H' \subset V'$ with dense inclusions).

(2) A different saddle-point approach is also at the basis of the results of Sect. 3 of the present note.

A look at the literature. Aftyer the pioneering work of Fitzpatrick [15] and its rediscovery by Martinez-Legaz and Théra [23] and also by Burachik and Svaiter [10], a recent but rapidly expanding literature has been devoted to this theory in the last fifteen years; see e.g. [3,11,18,24,25,28,29]. This may be compared with the approach that is developed in the monograph [16], and with that based on the notion of *bipotential* of Buliga, de Saxcé and Vallée, see e.g. [9].

The analysis of inclusions of the form (1.3) classically lead to the introduction of the class of pseudo-monotone operators in the sense of Brezis, and successive extensions; see e.g. [4,8] and the surveys [20,38]. This extended the classical theory of maximal monotone operators, see e.g. [2,5,7]. Apparently, the corresponding pseudo-monotone flow $D_t u + \alpha_u(u) \ni z'$ has been less studied in that abstract set-up.

As this author dealt with a variational approach for equations of the form (1.3) and (1.4) also in other works (with special reference to quasilinear evolutionary problems), a comparison seems in order. In [34] the method (iii) was used, and in particular quasilinear maximal monotone equations and first-order flows were formulated as null-minimization problems. The *structural stability,* namely, the dependence of the solution on data and operators, was then studied via De Giorgi's notion of Γ-convergence. In [35] this method was applied to the homogenization of monotone quasilinear PDEs with a single nonlinearity. In [37] the structural stability of pseudo-monotone equations and the corresponding doubly-nonlinear first-order flow were studied without using the Fitzpatrick theory.

Representation of nonmonotone operators. The present analysis may be extended in several directions. For instance, in [36] this author suggested to generalize the notion of representative function (see the system (2.6) and (2.7)) as follows.

Let us still assume that V is a real reflexive Banach space. An (in general nonconvex) function $g : V \times V' \to \mathbf{R} \cup \{+\infty\}$ is said to represent an (in general nonmonotone) operator $\beta : V \to \mathcal{P}(V')$, whenever g is weakly lower semicontinuous and fulfills a *generalized Fitzpatrick system*

$$g(v, v') \geq \langle v', v \rangle \qquad \forall (v, v') \in V \times V', \tag{2.18}$$
$$g(v, v') = \langle v', v \rangle \quad \Leftrightarrow \quad v' \in \beta(v). \tag{2.19}$$

(We assumed V to be reflexive. If V were not so, we would require g to be lower semicontinuous with respect to the product of the weak topology of V by the weak star topology of V'.)

For instance, for any $v \in V$ let a monotone operator $\alpha_v : w \mapsto \alpha_v(w)$ be represented by a function $g_{\alpha_v} \in \mathcal{F}(V)$, and set

$$\beta(v) := \alpha_v(v), \qquad g_\beta(v, v') := g_{\alpha_v}(v, v') \qquad \forall (v, v') \in V \times V'. \tag{2.20}$$

It is promptly seen that the function g_β then fulfills (2.18) and (2.19).

Existence of a solution of (1.1) might be proved by extending the methods (i)–(iii) that we mentioned above. Although the lack of convexity precludes the use of duality, examples may be constructed starting from the convex case; one may thus exploit the standard theory, as we do in this note. This also calls for an extension of the stability results that were studied for maximal monotone operators in [34].

3. Existence via Minimax

In this section we first assume that $\alpha : V \to \mathcal{P}(V')$ is maximal monotone, and deal with the inclusion $\alpha(u) \ni z'$ for a prescribed $z' \in V'$. If V is reflexive and α is coercive, existence of a solution is well known. Here we reformulate that inclusion in terms of a representative function of α, and prove existence of a solution of the associated null-minimization problem via a minimax method. Afterwards we consider the nonmonotone inclusion $\alpha_u(u) \ni z'$, and restate the theorem of [36] of existence of a solution.

In order to perform this program, we need a simple extension of the classical minimax theorem of Ky Fan, that here we recall.

Lemma 3.1 *(Ky Fan) [13] Let C be a convex subset of a real Hausdorff topological vector space X, and $\Phi : C \times C \to \mathbf{R}$ be such that*

$$\Phi(\cdot, y) \text{ is lower semicontinuous, } \forall y \in C, \tag{3.1}$$

$$\Phi(x, \cdot) \text{ is quasi-concave, } \forall x \in C, \tag{3.2}$$

$$\Phi(x, x) \leq 0, \ \forall x \in C, \tag{3.3}$$

$$\begin{aligned} &\exists \text{ compact convex set } K \subset X, \exists y_0 \in C \cap K : \\ &\Phi(x, y_0) > 0 \quad \forall x \in C \setminus K. \end{aligned} \tag{3.4}$$

Then

$$\exists \widetilde{x} \in C \cap K : \ \sup_{y \in C} \Phi(\widetilde{x}, y) = \inf_{x \in C} \sup_{y \in C} \Phi(x, y) \leq 0. \tag{3.5}$$

Corollary 3.2 *Let X be the dual of a real Banach space equipped with the weak star topology, C be a convex subset of X, and Φ be as above. Lemma 3.1 then holds under the assumption*

$$\exists M > 0 \text{ such that } \quad \sup_{\|y\| \leq M} \ \inf_{\|x\| > M} \Phi(x, y) > 0, \tag{3.6}$$

in place of the condition (3.4).

Proof. As the set $K = \{x \in X : \|x\| \leq M\}$ is weakly star compact, (3.6) yields (3.4) for this topology. □

The (maximal) monotone problem. Let us assume that

$$\begin{aligned} &V \text{ is real reflexive Banach space, } \quad z' \in V', \\ &\alpha : V \to \mathcal{P}(V') \text{ is maximal monotone,} \end{aligned} \tag{3.7}$$

and consider the inclusion

$$\text{find } u \in V \text{ such that } \quad \alpha(u) \ni z' \quad \text{in } V'. \tag{3.8}$$

Next we prove existence of a solution via an associated representative function.

Theorem 3.3 *Let a mapping $\psi \in \mathcal{F}(V)$ represent α, and be such that*

$$\inf_{v' \in V'} \frac{\psi(v, v')}{\|v\|_V} \to +\infty \quad \text{as } \|v\|_V \to +\infty. \tag{3.9}$$

Then there exists $u \in V$ such that

$$\psi(u, z') = \langle z', u \rangle. \tag{3.10}$$

As $\psi \in \mathcal{F}(V)$ represents α, it is promptly checked that the function $g = \psi$ fulfills the system (2.6), (2.7). The condition (3.9) entails the coerciveness of the operator α, and by (2.7) the equality (3.10) is equivalent to the inclusion $\alpha(u) \ni z'$. We thus retrieve a classical result, namely, the surjectivity of coercive maximal monotone operators acting on a reflexive Banach space; see e.g. [2,5,7].

Proof. This argument is based on reformulating the equation (3.10) as a minimax problem, and then applying the classical Fan theorem. This proof follows the lines of the more general argument of Sect. 5 of [36]. We split it into three steps.

(i) First we set

$$K(v,t) := \sup_{t' \in V'} \{\langle v, t' \rangle - \psi^*(t', t)\} \qquad \forall v, t \in V. \tag{3.11}$$

By a standard procedure,

$$\begin{aligned}
K(v,t) &= \sup_{t' \in V'} \Big\{\langle v, t' \rangle - \sup_{(w,w') \in V \times V'} \{\langle w, t' \rangle + \langle w', t \rangle - \psi(w, w')\}\Big\} \\
&= \sup_{t' \in V'} \inf_{(w,w') \in V \times V'} \{\langle v - w, t' \rangle - \langle w', t \rangle + \psi(w, w')\} \\
&= \sup_{t' \in V'} \inf_{w \in V} \Big\{\langle v - w, t' \rangle + \inf_{w' \in V'} \{ - \langle w', t \rangle + \psi(w, w')\}\Big\} \\
&= \inf_{w' \in V'} \{ - \langle w', t \rangle + \psi(v, w')\} \qquad \forall v, t \in V.
\end{aligned} \tag{3.12}$$

By (3.11) and (3.12) we infer that

$$\begin{aligned}
&K(\cdot, t) \text{ is convex and lower semicontinuous } \forall t \in V, \\
&K(v, \cdot) \text{ is concave and upper semicontinuous } \forall v \in V.
\end{aligned} \tag{3.13}$$

By (3.7) and Theorem 2.2, ψ and ψ^* are both representative functions; therefore they fulfill the Fitzpatrick system (2.6), (2.7). Thus

$$K(t,t) = \inf_{w' \in V'} \{ - \langle w', t \rangle + \psi(t, w')\} \geq 0 \qquad \forall t \in V, \tag{3.14}$$

$$K(t,t) = \sup_{t' \in V'} \{\langle t, t' \rangle - \psi^*(t', t)\} \leq 0 \qquad \forall t \in V, \tag{3.15}$$

whence

$$K(t,t) = 0 \qquad \forall t \in V. \tag{3.16}$$

Thus (t, t) is a saddle point of K for any $t \in V$.

(ii) Next we set

$$\Phi(v,t) := K(v,t) + \langle z', t - v \rangle \qquad \forall v, t \in V, \tag{3.17}$$

whence

$$\Phi(v,v) = K(v,v) \overset{(3.16)}{=} 0 \qquad \forall v \in V. \tag{3.18}$$

By (3.11),

$$\begin{aligned}\sup_{t\in V} \Phi(v,t) &= \sup_{(t,t')\in V\times V'} \big(\langle z',t\rangle + \langle v,t'\rangle - \psi^*(t',t)\big) - \langle z',v\rangle \\ &= \psi^{**}(v,z') - \langle z',v\rangle = \psi(v,z') - \langle z',v\rangle \qquad \forall v\in V.\end{aligned} \tag{3.19}$$

Because of (3.19)

$$v \mapsto \Phi(v,t) \text{ is concave and weakly lower semicontinuous, } \forall t\in V. \tag{3.20}$$

Moreover,

$$\Phi(v,0) \overset{(3.17)}{=} K(v,0) - \langle z',v\rangle \overset{(3.14)}{\geq} \inf_{w'\in V'} \psi(v,w') - \|z'\|_{V'}\|v\|_V \qquad \forall v\in V; \tag{3.21}$$

by (3.9) then

$$\exists M>0: \qquad \|v\|>M \ \Rightarrow\ \Phi(v,0)>0. \tag{3.22}$$

(iii) By (3.18), (3.20) and (3.22), we may apply Fan's Theorem via Corollary 3.2, selecting $X = V$ equipped with the weak topology and $C = V$. Therefore there exists $u \in V$ (with $\|u\| \leq M$) such that

$$\sup_{t\in V} \Phi(u,t) = \inf_{v\in V}\sup_{t\in V} \Phi(v,t) \leq 0. \tag{3.23}$$

Hence, recalling that $\psi \in \mathcal{F}(V)$,

$$0 \overset{(2.6)}{\leq} \psi(u,z') - \langle z',u\rangle \overset{(3.19)}{=} \sup_{t\in V} \Phi(u,t) \leq 0. \tag{3.24}$$

Thus $\psi(u,z') = \langle z',u\rangle$. □

Remark. The proof of existence is trivialized whenever the representative function ψ is self-dual, in the sense of (2.15). Setting $J_{z'}(v) = \psi(v,z') - \langle z',v\rangle$ for any $v\in V$, in this case by Corollary 2.6

$$J_{z'}(u) = \inf_{v\in V} J_{z'}(v) \qquad \Leftrightarrow \qquad \alpha(u) \ni z' \text{ in } V', \tag{3.25}$$

and existence of a minimizer directly follows from the coerciveness and lower semicontinuity of $J_{z'}$.

A nonmonotone problem. Next we deal with the nonmonotone inclusion

$$\alpha_u(u) \ni z' \qquad \text{in } V'. \tag{3.26}$$

More specifically, we assume that a maximal monotone operator $\alpha_v : V \to \mathcal{P}(V')$ is represented (in the sense of Fitzpatrick) by a function $\psi_v \in \mathcal{F}(V)$ for any $v \in V$. We formulate the inclusion $\alpha_u(u) \ni z'$ variationally, and state a theorem of existence of a solution.

Theorem 3.4 *[36] Let $\alpha_z : V \to \mathcal{P}(V')$ be a maximal monotone operator for any $z\in V$, and let α_z be represented by a function $V\times V\times V' \to \mathbf{R}\cup\{+\infty\} : (z,v,v') \mapsto \psi_z(v,v')$ such that*

$$\psi_z \in \mathcal{F}(V), \quad \psi_z^* \in \mathcal{F}(V') \qquad \forall z\in V, \tag{3.27}$$

$$\inf_{v'\in V'} \frac{\psi_v(v,v')}{\|v\|_V} \to +\infty \quad \text{as } \|v\|_V \to +\infty, \tag{3.28}$$

$$\inf_{v\in V} \frac{\psi_v(v,v')}{\|v'\|_{V'}} \to +\infty \quad \text{as } \|v'\|_{V'} \to +\infty, \tag{3.29}$$

$$V\times V' \to \mathbf{R}\cup\{+\infty\} : (v,v') \mapsto \psi_v(v,v') \text{ is weakly lower semicontinuous.} \tag{3.30}$$

For any $z' \in V'$, then there exists $u \in V$ such that

$$\psi_u(u, z') = \langle z', u \rangle. \tag{3.31}$$

This equation is equivalent to the inclusion $\alpha_u(u) \ni z'$ in V'.

We refer the reader to [36] for the argument and for application of this result to problems like those that we outlined in Sect. 1.

Remark. Similarly to what we pointed out in the previous remark, the proof of existence of (3.31) is also trivialized whenever for any $z \in V$ the (assumed maximal monotone) operator $v \mapsto \alpha_z(v)$ is represented by a self-dual function ψ_z, in the sense of (2.15). Theorem 2.5 above (see [3]) provides a way to construct a large class of examples. Setting $\widetilde{J}_{z'}(v) = \psi_v(v, z') - \langle z', v \rangle$, in this case

$$\widetilde{J}_{z'}(u) = \inf_{v \in V} \widetilde{J}_{z'}(v) \qquad \Leftrightarrow \qquad \alpha_u(u) \ni z' \text{ in } V', \tag{3.32}$$

and existence of a minimizer directly follows from the coerciveness and weak lower semicontinuity of $\widetilde{J}_{z'}$.

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