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On the Cattabriga problem appearing in the two phase problem of the viscous fluid flows

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Abstract

In this paper, we report results concerning the two phase problem for the viscous fluid flows without surface tension in a bounded region, which was announced in the RIMS Workshop on Mathematical Analysis in Fluid and Gas Dynamics organized by Professor Takayuki Kobayashi of Osaka University and Professor Tatsuo Iguchi of Keio University held at RIMS, Kyoto University, July 8–10, 2015. Especially, we prove the unique existence theorem for the Cattabriga problem which is obtained as a statinary problem for the linearized two phase problem system. Moreover, we proved the unique existence theorem for some weak Dirichlet problem with jump condition on the interface.

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1 Introduction

Let Ω be a bounded domain in N-dimensional Euclidean space $\mathbb{R}^N$ $(N \geq 2)$, and let Ω_+ be a subdomain of Ω. Let $\Omega_- = \Omega \setminus \overline{\Omega_+}$. The $\Omega_\pm$ are occupied by some viscous fluids. Let Γ_- and Γ be the boundary Ω and Ω_+, respectively. Note tht $\Gamma_- \cap \Gamma = \emptyset$. Assume that Γ_- and Γ are compact hypersurfaces of $W_r^{2-1/r}$ class $(N < r < \infty)$. Let $\Omega_{t,\pm}$, Γ_t and $\Gamma_{t,-}$ be the time evolution of $\Omega_\pm$, Γ and Γ_-, respectively. Set $\dot\Omega_t = \Omega_{t,+} \cup \Omega_{t,-}$ and $\dot\Omega = \Omega_+ \cup \Omega_-$. Then, the two phase problem for the viscous fluids without surface tension is formulated mathematically as follows:

$$
\begin{cases}
\partial_t\rho + \mathrm{div}\,(\rho\mathbf{v}) = 0 & \text{in } \dot\Omega_{t,\pm}, \\
\rho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) - \mathrm{Div}\,(\mathbf{S}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0 & \text{in } \dot\Omega_{t,\pm}, \\
[[(\mathbf{S}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t]] = 0, \quad [[\mathbf{v}]] = 0 & \text{on } \Gamma_t, \\
(\mathbf{S}_-(\mathbf{v}_-) - \mathfrak{p}_-\mathbf{I})\mathbf{n}_{t,-}|_{\Gamma_{t,-}} = 0 & \text{on } \Gamma_{t,-}, \\
(\rho, \mathbf{v})|_{t=0} = (\rho_* + \theta_-, \mathbf{v}_0) & \text{in } \dot\Omega
\end{cases}
\tag{1.1}
$$

for $t \in (0,T)$. Here, $\rho_* = \rho_*(x,t)$ is a piece-wise constant function defined by $\rho_*(x,t)|_{\Omega_{t,\pm}} = \rho_{*,\pm}$ with some positive constants $\rho_{*,\pm}$ describing the mass density of reference bodies $\Omega_\pm$; $\mathbf{v} = \mathbf{v}(x,t) = (v_1(x,t), \ldots, v_N(x,t))$ denotes a velocity field; $\mathfrak{p}$ a pressure field; and ρ a density field. In the case of the compressible fluids, the mass field $\rho_\pm = \rho|_{\Omega_{t,\pm}}$ are unknown functions; the pressure field $\mathfrak{p}_\pm = \mathfrak{p}|_{\Omega_{t,\pm}}$ are functions of mass densities $\rho_\pm$ as $\mathfrak{p}_\pm = P_\pm(\rho_\pm)$, that is, the barotropic fluids are considered, where $P_\pm(r)$ are C^∞ functions defined for $r > 0$ satisfying the conditions: $P_\pm'(r) > 0$ for $r > 0$ and $P_\pm(\rho_{*,\pm}) = 0$; and initial data θ_0 and $\mathbf{v}_0$ are prescribed functions. In the case of the incompressible fluids, the mass fields ρ is given by $\rho|_{\Omega_{t,\pm}} = \rho_{*,\pm}$, so that the balance of mass is read as $\mathrm{div}\,\mathbf{v}_\pm = 0$ in $\Omega_{t,\pm}$ with $\mathbf{v}_\pm = \mathbf{v}|_{\Omega_{t,\pm}}$, the pressure term $\mathfrak{p}_\pm = \mathfrak{p}|_{\Omega_\pm}$ are unknown functions, and for the initial data $\theta_0 = 0$ and $\mathbf{v}_0$ is a prescribed function.

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As for the remaining notation, $\mathbf{I}$ is the $N \times N$ unit matrix; $\mathbf{n}_t$ is the unit normal to Γ_t pointing from $\Omega_{t,+}$ to $\Omega_{t,-}$ while $\mathbf{n}_{t,-}$ is the unit outer normal of $\Gamma_{t,-}$; the $[[f]]$ denotes the jump quantity of f along Γ_t defined by

$$[[f]](x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_+}} f(x) - \lim_{\substack{x \to x_0 \\ x \in \Omega_-}} f(x) \quad \text{for } x_0 \in \Gamma_t;$$

and $\mathbf{S}$ is a stress tensor defined by

$$\mathbf{S}(\mathbf{u})|_{\Omega_\pm} = \mu_\pm \mathbf{D}(\mathbf{u}_\pm) + (\nu_\pm - \mu_\pm)\operatorname{div}\mathbf{u}_\pm \mathbf{I}, \quad \mathbf{D}(\mathbf{u}_\pm) = \nabla\mathbf{u}_\pm + (\nabla\mathbf{u}_\pm)^\top$$

with $\mathbf{u}_\pm = \mathbf{u}|_{\Omega_{t,\pm}}$, where $(\nabla\mathbf{u}_\pm)^\top$ denotes the transposed $\nabla\mathbf{u}$, and $\mu_\pm$ and $\nu_\pm$ are positive constants describing the first and second viscosity coefficinets. Furthermore, $\rho_t = \partial_t\rho = \partial\rho/\partial t$, for any matrix field K with (i,j) components K_{ij}, the quantity $\operatorname{Div} K$ is an N-vector with components $\sum_{j=1}^N \partial_j K_{ij}$, where $\partial_i = \partial/\partial x_j$, and for any vector of functions $\mathbf{w} = (w_1, \ldots, w_N)$, we set $\mathbf{w}_t = (\partial_t w_1, \ldots, \partial_t w_N)$, $\operatorname{div}\mathbf{w} = \sum_{j=1}^N \partial_j w_j$ and $\mathbf{w} \cdot \nabla\mathbf{w} = (\sum_{j=1}^N w_j\partial_j w_1, \ldots, \sum_{j=1}^N w_j\partial_j w_N)$.

Let $\mathbf{x} = \mathbf{x}(\xi, t)$ be a solution of the Cauchy problem

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) \quad \text{with } \mathbf{x}|_{t=0} = \xi.$$

The kinematic condition is:

$$\Gamma_t = \{x = \mathbf{x}(\xi, t) \mid \xi \in \Gamma\}, \quad \Gamma_{t,-} = \{x = \mathbf{x}(\xi, t) \mid \xi \in \Gamma_-\}.$$

Notation. Throughout the paper, for any domain D, $L_q(D)$, $W_q^n(D)$ and $B_{q,p}^s(D)$ denote the usual Lebesgue space, Sobolev space and Besov space, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^n(D)}$ and $\|\cdot\|_{B_{q,p}^s(D)}$ are their norms, where $1 \leq p, q \leq \infty$, n is any natural number and s is any non-negative real number. Let $W_{q,0}^1(D) = \{u \in W_q^1(D) \mid u|_{\partial D} = 0\}$, where ∂D is the boundary of D. Given function v defined on $\dot\Omega$ or $\dot\Omega_t$, we set $v_\pm = v|_{\Omega_\pm}$ or $v_\pm = v|_{\Omega_{t,\pm}}$. Given functions $v_\pm$ defined on $\Omega_\pm$ or on $\Omega_{t,\pm}$, v is defined by $v(x) = v_\pm(x)$ for $x \in \Omega_\pm$ or $v(x) = v_\pm(x)$ for $x \in \Omega_{t,\pm}$. Let

$$\begin{aligned}
&W_q^n(\dot\Omega) = \{v \in L_q(\dot\Omega) \mid v_\pm = v|_{\Omega_\pm} \in W_q^n(\Omega_\pm)\}, \quad B_{q,p}^s(\dot\Omega) = \{v \in L_q(\dot\Omega) \mid v_\pm = v|_{\Omega_\pm} \in B_{q,p}^s(\Omega_\pm)\},\\
&\|v\|_{L_q(\dot\Omega)} = \|v_+\|_{L_q(\Omega_+)} + \|v_-\|_{L_q(\Omega_-)}, \quad \|v\|_{W_q^n(\dot\Omega)} = \|v_+\|_{W_q^n(\Omega_+)} + \|v_-\|_{W_q^n(\Omega_-)},\\
&\|v\|_{B_{q,p}^s(\dot\Omega)} = \|v_+\|_{B_{q,p}^s(\Omega_+)} + \|v_-\|_{B_{q,p}^s(\Omega_-)}.
\end{aligned}$$

Let

$$\begin{aligned}
(u, v)_{\Omega_\pm} &= \int_{\Omega_\pm} u(x)\overline{v(x)}\,dx, \quad (u, v)_{\dot\Omega} = (u, v)_{\Omega_+} + (u, v)_{\Omega_-},\\
(u, v)_\Gamma &= \int_\Gamma u(x)\overline{v(x)}\,d\sigma_\Gamma, \quad (u, v)_{\Gamma_-} = \int_\Gamma u(x)\overline{v(x)}\,d\sigma_{\Gamma_-},
\end{aligned}$$

where $\overline{v(x)}$ denotes the complex conjugate of $v(x)$, and $d\sigma_\Gamma$ and $d\sigma_{\Gamma_-}$ denote the surface elements of Γ and Γ_-, respectively. For any two N vectors $\mathbf{a}^i = (a_1^i, \ldots, a_N^i)$ $(i = 1, 2)$, we set $\mathbf{a}^1 \cdot \mathbf{a}^2 = <\mathbf{a}^1, \mathbf{a}^2> = \sum_{j=1}^N a_j^1 a_j^2$. The $[[f]]$ denotes also the jump quantity of f along Γ defined by

$$[[f]](x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_+}} f(x) - \lim_{\substack{x \to x_0 \\ x \in \Omega_-}} f(x) \quad \text{for } x_0 \in \Gamma.$$

For two Banach spaces X, Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y, while $\|\cdot\|_{\mathcal{L}(X,Y)}$ denotes its norm. When $X = Y$, we use the abbreviation: $\mathcal{L}(X) = \mathcal{L}(X, X)$. The d-product space X^d is defined by of $X^d = \{\mathbf{u} = (u_1, \ldots, u_d) \mid u_i \in X \ (i = 1, \ldots, d)\}$, while its norm is written by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for short, where $\|\cdot\|_X$ is the norm of X. The boldface letter is used to represent vectors of functions. The letter C is used to represent generic constants and the value of C may change from line to line.

Statement of main results. Let $\mathbf{u}(\xi,t)$ be the Lagrangean description of the velocity field in $\dot\Omega$, and then the Euler coordinate x and the Lagrangean coordinate ξ are related by

$$x=\xi+\int_0^t \mathbf{u}(\xi,s)\,ds = X_{\mathbf{u}}(\xi,t) \quad \text{for } \xi\in\dot\Omega.$$

The Jacobi matrix of the transformation $x=X_{\mathbf{u}}(\xi,t)$ is invertible, if

$$\int_0^T \|\nabla\mathbf{u}(\cdot,t)\|_{L_\infty(\dot\Omega)}\,dt \le \sigma_0 \tag{1.2}$$

does hold with some small $\sigma_0>0$. By the Banach fixed point argument based on the maximal L_p-L_q maximal regularity theorem for the linearized equations, we can prove the local well-posedness which is stated in

Theorem 1.1. *Let $N<q,r<\infty$, $2<p<\infty$, and $R>0$. Assume that $\max(q,q')\le r$ and that Γ and Γ_- are both compact hyper-surfaces of $W_r^{2-1/r}$ class. Then, there exists a positive time $T>0$ depending on R such that for any initial data $\theta_{0,\pm}\in W_q^1(\Omega_\pm)$ and $\mathbf{v}_{0,\pm}\in B_{q,p}^{2(1-1/p)}(\Omega_\pm)^N$ with*

$$\|\theta_{0,\pm}\|_{W_q^1(\Omega_\pm)}+\|\mathbf{v}_{0,\pm}\|_{B_{q,p}^{2(1-1/p)}(\Omega_\pm)}\le R$$

satisfying the range condition:

$$\rho_{*,\pm}/2<\rho_{*,\pm}+\theta_{0,\pm}(x)\le 2\rho_{*,\pm}$$

and the compatibility conditions which is described as follows:

- *compressible-compressible case:*

$$[[(\mathbf{S}(\mathbf{v}_0)-P(\rho_0)\mathbf{I})\mathbf{n}]]=0,\quad [[\mathbf{v}_0]]=0$$
$$(\mathbf{S}_-(\mathbf{v}_{0,-})-P_-(\rho_{0,-})\mathbf{I})\mathbf{n}_-|_\Gamma=0,$$

  where $\rho_{0,\pm}=\rho_{*,\pm}+\theta_{0,\pm}$;

- *Ω_+ compressible and Ω_- incompressible case:*

$$[[\mathbf{S}(\mathbf{v}_0)\mathbf{n}-<\mathbf{S}(\mathbf{v}_0)\mathbf{n},\mathbf{n}>\mathbf{n}]]=0,\quad [[\mathbf{v}_0]]=0,\quad \operatorname{div}\mathbf{v}_{0,-}=0,$$
$$(\mathbf{S}_-(\mathbf{v}_{0,-})\mathbf{n}_- - <\mathbf{S}_-(\mathbf{v}_{0,-})\mathbf{n}_-,\mathbf{n}_->\mathbf{n}_-|_{\Gamma_-}=0,$$

- *Ω_+ incompressible and Ω_- compressible case:*

$$[[\mathbf{S}(\mathbf{v}_0)\mathbf{n}-<\mathbf{S}(\mathbf{v}_0)\mathbf{n},\mathbf{n}>\mathbf{n}]]=0,\quad [[\mathbf{v}_0]]=0,\quad \operatorname{div}\mathbf{v}_{0,+}=0,$$
$$(\mathbf{S}_-(\mathbf{v}_{0,-})-P_-(\rho_{0,-})\mathbf{I})\mathbf{n}_-|_{\Gamma_-}=0,$$

- *incompressible-incompressible case:*

$$[[\mathbf{S}(\mathbf{v}_0)\mathbf{n}-<\mathbf{S}(\mathbf{v}_0)\mathbf{n},\mathbf{n}>\mathbf{n}]]=0,\quad [[\mathbf{v}_0]]=0,\quad \operatorname{div}\mathbf{v}_{0,\pm}=0,$$
$$(\mathbf{S}_-(\mathbf{v}_{0,-})-<\mathbf{S}_-(\mathbf{v}_{0,-})\mathbf{n}_-,\mathbf{n}_->\mathbf{n}_-)|_{\Gamma_-}=0,$$

where $\mathbf{n}$ is the unit normal to Γ pointing from Ω_+ into Ω_-, while $\mathbf{n}_-$ is the unit outer normal to Γ_-, the equations (1.1) *described in the Lagrange coordinate admit unique solutions*

- *compressible-compressible case : $\theta_\pm$ and $\mathbf{u}_\pm$ with*

$$\theta_\pm\in W_p^1((0,T),W_q^1(\Omega_\pm)),\quad \mathbf{u}_\pm\in W_p^1((0,T),L_q(\Omega_\pm)^N)\cap L_p((0,T),W_q^2(\Omega_\pm)^N);$$

- *$\Omega_\pm$ comp.-$\Omega_\mp$ incomp. case : $\theta_\pm$, $\pi_\mp$ and $\mathbf{u}_\pm$ with*

$$\theta_\pm\in W_p^1((0,T),W_q^1(\Omega_\pm)),\quad \pi_\mp\in L_p((0,T),W_q^1(\Omega_\mp)),$$
$$\mathbf{u}_\pm\in W_p^1((0,T),L_q(\Omega_\pm)^N)\cap L_p((0,T),W_q^2(\Omega_\pm)^N);$$

- *incompressible-incompressible case:* $\pi_\pm$ *and* $\mathbf{u}_\pm$ *with*

$$\pi_\pm \in L_p((0,T), W^1_q(\Omega_\pm)), \quad \mathbf{u}_\pm \in W^1_p((0,T), L_q(\Omega_\pm)^N) \cap L_p((0,T), W^2_q(\Omega_\pm)^N);$$

where $\mathbf{u}$ *satisfies* (1.2). *Here,* $\theta_\pm$ *and* $\pi_\pm$ *denote the density fields and pressure fields in the Lagrange coordinate, that is,* $\rho_\pm(X_{\mathbf{u}}(\xi,t),t) = \theta_\pm(\xi,t)$ *and* $\mathfrak{p}_\pm(X_{\mathbf{u}}(\xi,t),t) = \pi_\pm(\xi,t)$ *for* $\xi \in \Omega_\pm$.

Next theorem is concerned with the global well-posedness theorem for small initial data.

Theorem 1.2. *Let* $N < q, r < \infty$, $2 < p < \infty$, *and* $R > 0$. *Assume that* $\max(q,q') \le r$, *that* Γ *and* Γ_- *are both compact hyper-surfaces of* $W^{2-1/r}_r$ *class, and that* $2/p + N/q < 1$. *Let* $\{\mathbf{p}_\ell\}_{\ell=1}^M$ *be the orthonormal basis of the rigid space* $\mathcal{R}_d = \{\mathbf{u} \mid \mathbf{D}(\mathbf{u}) = 0\}$ *with inner-product*

$$[\mathbf{u},\mathbf{v}] = (\rho_{*,+}\mathbf{u}_+, \mathbf{v}_+)_{\Omega_+} + (\rho_{*,-}\mathbf{u}_-, \mathbf{v}_-)_{\Omega_-}.$$

Then, there exists an $\epsilon > 0$ *such that if initial data* $\theta_{0,\pm}$ *(in the incompressible case, we interpret* $\theta_{0,\pm} = 0$ *) and* $\mathbf{v}_{0,\pm}$ *satisfies smallness condition:*

$$\|\theta_{0,\pm}\|_{W^1_q(\Omega_\pm)} + \|\mathbf{v}_0\|_{B^{2-1/p}_{q,p}(\Omega)} \le \epsilon,$$

and orthogonal condition:

$$((\rho_{*,+} + \theta_{0,+})\mathbf{v}_{0,+}), \mathbf{p}_\ell)_{\Omega_+} + ((\rho_{*,-} + \theta_{0,-})\mathbf{v}_{0,-}), \mathbf{p}_\ell)_{\Omega_-} = 0 \quad (\ell = 1, \ldots, M)$$

as well as regularity condition, range condition and compatibility condition, then the equations (1.1) *described in the Lagrange coordinate admit unique solutions defined on the whole time interval* $(0,\infty)$, *which decay exponentially.*

Remark 1.3. The rigid space $\mathcal{R}_d$ is the set of all N-vector of first order polynomials of the form: $Ax+\mathbf{b}$ with anti-symmetric $N \times N$ matrix A and constant N vector $\mathbf{b}$. Namely, $\mathcal{R}_d$ consists of all linear combinations of constant N vectors and polynomials of the form: $x_i\mathbf{e}_j - x_j\mathbf{e}_i$, where $\mathbf{e}_i = (0, \ldots, 0, \underset{ith}{1}, 0, \ldots, 0)$.

To prove Theorem 1.2, the main tool is the exponential stability of semi-group associated with the linearlized equations:

$$\begin{cases} \partial_t\theta + \gamma_0 \operatorname{div}\mathbf{v} = 0 & \text{in } \dot\Omega \times (0,\infty), \\ \partial_t\mathbf{v} - \gamma_1 \operatorname{Div}(\mathbf{S}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0 & \text{in } \dot\Omega \times (0,\infty), \\ [[(\mathbf{S}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}]] = 0, \quad [[\mathbf{v}]] = 0, & \\ (\mathbf{S}_-(\mathbf{v}_-) - \mathfrak{p}_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0, & \\ (\theta, \mathbf{v})|_{t=0} = (\theta_0, \mathbf{v}_0) & \text{in } \dot\Omega, \end{cases} \tag{1.3}$$

where γ_i $(i = 0,1)$ are piece-wise constant functions defined by $\gamma_i|_{\Omega_\pm} = \gamma_{i,\pm}$ with some positive constants $\gamma_{i,\pm}$. Moreover,

- the compressible-compressible case: $\mathfrak{p} = \gamma'\theta$ with some piece-wise constant function γ' defined by $\gamma'|_{\Omega_\pm} = \gamma'_\pm$ with some positive constants $\gamma'_\pm$;
- the $\Omega_\pm$ comp. - $\Omega_\mp$ incomp. case: $\mathfrak{p}_\pm = \gamma'_\pm\theta_\pm$, while $\theta_\mp = \theta_{0,\mp} = 0$ and $\mathfrak{p}_\mp$ is unknow function;
- the incompressible-incompressible case: $\theta = \theta_0 = 0$ and $\mathfrak{p}$ is unknown function.

In fact, to prove Theorem 1.2, the key step is to prove the existence of C^0 semigroup $\{T(t)\}_{t\ge0}$ associated with (1.3) on $\mathcal{H}_q(\Omega)$, which is analytic. Here,

$$\begin{aligned} &\mathcal{H}_q(\Omega) = \{(\theta,\mathbf{v}) \in W^1_q(\dot\Omega) \times L_q(\Omega)\} && \text{in the compressible-compressible case,} \\ &\mathcal{H}_q(\Omega) = \{(\theta_\pm,\mathbf{v}) \in W^1_q(\Omega_\pm) \times L_q(\Omega) \mid \operatorname{div}\mathbf{v}_\mp = 0\} && \text{in the } \Omega_\pm \text{ comp. - } \Omega_\mp \text{ incomp. case,} \\ &\mathcal{H}_q(\Omega) = \{\mathbf{v} \in L_q(\Omega) \mid \operatorname{div}\mathbf{v}_\mp = 0\} && \text{in the incompressible - incompressible case.} \end{aligned}$$

Moreover, if $\mathbf{v}$ satisfies the orthogonal condition:

$$(\gamma_1^{-1}\mathbf{v}, \mathbf{p}_\ell)_{\dot\Omega} = 0 \quad \text{for all } \ell = 1, \ldots, M, \tag{1.4}$$

then $\{T(t)\}_{t\ge0}$ is exponentially stable on $\mathcal{H}_q(\Omega)$, that is,

- compressible-compressible case:

$$\|T(t)(\theta,\mathbf{v})\|_{W^1_q(\dot\Omega)\times L_q(\Omega)} \le Ce^{-ct}\|(\theta,\mathbf{v})\|_{W^1_q(\dot\Omega)\times L_q(\Omega)};$$

- $\Omega_\pm$ compressible - $\Omega_\mp$ incompressible case:

$$\|T(t)(\theta_\pm,\mathbf{v})\|_{W^1_q(\Omega_\pm)\times L_q(\Omega)} \le Ce^{-ct}\|(\theta_\pm,\mathbf{v})\|_{W^1_q(\Omega_\pm)\times L_q(\Omega)};$$

- incompressible-incompressible case:

$$\|T(t)\mathbf{v}\|_{L_q(\Omega)} \le Ce^{-ct}\|\mathbf{v}\|_{L_q(\Omega)}$$

with some positive constants C and c for any $t > 0$.

To prove the exponential stability, one of key steps is to prove the unique existence theorem for the following problem:

$$\begin{cases} \gamma_0 \operatorname{div}\mathbf{v} = f & \text{in } \dot\Omega,\\ -\gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{v}) - \nabla\mathfrak{p}) = \mathbf{g} & \text{in } \dot\Omega,\\ [[(\mathbf{S}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}]] = [[\mathbf{h}]], \quad [[\mathbf{v}]] = 0 & \text{on } \Gamma,\\ (\mathbf{S}_-(\mathbf{v}_-) - \mathfrak{p}_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-} & \text{on } \Gamma_-. \end{cases} \tag{1.5}$$

Dividing the first equation in (1.5) by γ_0, we may assume that $\gamma_0 = 1$ in the following. This paper is concerned with problem (1.5) with $\gamma_0 = 1$, and we prove

Theorem 1.4. *Let $1 < q < \infty$ and $N < r < \infty$. Assume that $r \ge \max(q, q')$ with $q' = q/(q-1)$ and that Γ and Γ_- are both compact hyper-surfaces of $W^{2-1/r}_r$ class. Let $\{\mathbf{p}_\ell\}_{\ell=1}^M$ be the orthonormal basis of the rigid space $\mathcal{R}_d = \{\mathbf{u} \mid \mathbf{D}(\mathbf{u}) = 0\}$ with respect to the inner-product: $[\mathbf{u},\mathbf{v}] := (\gamma_1^{-1}\mathbf{u},\mathbf{v})_{\dot\Omega}$ on $L_q(\dot\Omega)$. Then, for any $f \in W^1_q(\dot\Omega)$, $\mathbf{g} \in L_q(\dot\Omega)$, $\mathbf{h} \in W^1_q(\dot\Omega)$ and $\mathbf{h}_- \in W^1_q(\Omega_-)$ satisfying the orthogonal condition:*

$$(\gamma_1^{-1}\mathbf{g},\mathbf{p}_\ell)_\Omega + (\mathbf{h},\mathbf{p}_\ell)_\Gamma + (\mathbf{h}_-,\mathbf{p}_\ell)_{\Gamma_-} = 0 \quad \textit{for all } \ell = 1,\ldots,M, \tag{1.6}$$

then problem (1.5) admits a unique solution $\mathbf{v} \in W^2_q(\dot\Omega)$ satisfying the orthogonal condition:

$$(\gamma_1^{-1}\mathbf{v},\mathbf{p}_\ell)_\Omega = 0 \quad \textit{for all } \ell = 1,\ldots,M \tag{1.7}$$

and the estimate:

$$\|\mathbf{v}\|_{W^2_q(\dot\Omega)} \le C(\|f\|_{W^1_q(\dot\Omega)} + \|\mathbf{g}\|_{L_q(\dot\Omega)} + \|\mathbf{h}\|_{W^1_q(\dot\Omega)} + \|\mathbf{h}_-\|_{W^1_q(\Omega_-)}). \tag{1.8}$$

Moreover, we discuss the unique solvability of the weak Dirichlet problem:

$$(\gamma_1\nabla u, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega). \tag{1.9}$$

For problem (1.9) we prove

Theorem 1.5. *Let $1 < q < \infty$ and $N < r < \infty$. Assume that $r \ge \max(q, q')$ and that both of Γ and Γ' are hyper-surfaces of $W^{2-1/r}_r$ class. Then, for any $\mathbf{f} \in L_q(\Omega)^N$, problem (1.9) admits a unique solution $u \in W^1_{q,0}(\Omega)$ satisfying the estimate: $\|u\|_{W^1_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}$.*

2 On the weak Dirichlet problem

In this section, we discuss the weak Dirichlet problem (1.9).

2.1 The weak Dirichle problem in $\mathbb{R}^N$

Let

$$\mathbb{R}^N_\pm = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \pm x_N > 0\}, \quad \mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

and set $\dot{\mathbb{R}}^N = \mathbb{R}^N_+ \cup \mathbb{R}^N_-$. First of all, we consider the variational equation:

$$\lambda(u, \varphi)_{\dot{\mathbb{R}}^N} + (\gamma_1 \nabla u, \nabla \varphi)_{\dot{\mathbb{R}}^N} = (\mathbf{f}, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in W^1_{q'}(\mathbb{R}^N), \tag{2.1}$$

where γ_1 is a piece-wise constant function defined by $\gamma|_{\mathbb{R}^N_\pm} = \gamma_{1,\pm}$ with some positive constants $\gamma_{1,\pm}$. To solve (2.1), we consider the strong form of (2.1):

$$\begin{cases} \lambda u_\pm - \gamma_{1,\pm}\Delta u_\pm = \operatorname{div}\mathbf{f}_\pm & \text{in } \mathbb{R}^N_\pm, \\ \gamma_{1,+}\partial_N u_+|_{x_N=0+} - \gamma_{1,-}\partial_N u_-|_{x_N=0-} = g, & \\ u_+|_{x_N=0+} = u_-|_{x_N=0-}. & \end{cases} \tag{2.2}$$

If $\mathbf{f} \in L_q(\mathbb{R}^N)^N$, then $\mathbf{f}_\pm = \mathbf{f}|_{\mathbb{R}^N_\pm} \in L_q(\Omega_\pm)^N$. Since $C^\infty_0(\mathbb{R}^N_\pm)$ is dense in $L_q(\mathbb{R}^N_\pm)$, we may assume that $\mathbf{f}_\pm \in C^\infty_0(\mathbb{R}^N_\pm)^N$. First of all, we construct solutions of (2.2). For any functions $h_\pm$ defined on $\pm x_N > 0$, let

$$h^o_\pm(x) = \begin{cases} h_\pm(x', x_N) & \pm x_N > 0, \\ -h_\pm(x', -x_N) & \pm x_N < 0, \end{cases} \quad h^e_\pm(x) = \begin{cases} h_\pm(x', x_N) & \pm x_N > 0, \\ h_\pm(x', -x_N) & \pm x_N < 0, \end{cases}$$

where $x' = (x_1, \ldots, x_{N-1})$. Let $\mathcal{F}$ and $\mathcal{F}^{-1}_\xi$ be Fourier transform and Fourier inverse transform defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\, dx, \quad \mathcal{F}^{-1}_\xi[f](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

Since $(\operatorname{div}\mathbf{f}_\pm)^o = \sum_{j=1}^{N-1}\partial_j(f^o_j) + \partial_N(f^e_N)$ with $\mathbf{f}_\pm = (f_{\pm 1}, \ldots, f_{\pm N})$, we have

$$\mathcal{F}[(\operatorname{div}\mathbf{f}_\pm)^o](\xi) = \sum_{j=1}^{N-1} i\xi_j \mathcal{F}[f^o_{\pm j}](\xi) + i\xi_N \mathcal{F}[f^e_{\pm N}](\xi).$$

Thus, if we set

$$u_{\pm 1} = \mathcal{F}^{-1}_\xi\Big[\frac{\mathcal{F}[(\operatorname{div}\mathbf{f}_\pm)^o](\xi)}{\lambda + \gamma_{1,\pm}|\xi|^2}\Big] = \mathcal{F}^{-1}_\xi\Big[\frac{\sum_{j=1}^{N-1} i\xi_j\mathcal{F}[f^o_{\pm j}](\xi) + i\xi_N\mathcal{F}[f^e_{\pm N}](\xi)}{\lambda + \gamma_{1,\pm}|\xi|^2}\Big] \tag{2.3}$$

we have

$$\lambda u_\pm - \gamma_{1,\pm}\Delta u_\pm = \operatorname{div}\mathbf{f}_\pm \quad \text{in } \mathbb{R}^N_\pm. \tag{2.4}$$

In the following, we calculate $u_{\pm 1}(x', 0)$ and $(\partial_N u_{\pm 1})(x', 0)$. Recall that $\mathbf{f}_\pm \in C^\infty_0(\mathbb{R}^N_\pm)^N$. Especially, $f_{\pm N}(x', 0) = 0$ with $\mathbf{f}_\pm = (f_{\pm 1}, \ldots, f_{\pm N})$. Let

$$\hat{g}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} g(x', x_N)\, dx', \quad \mathcal{F}^{-1}_{\xi'}[g(\cdot, x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi', x_N)\, d\xi'.$$

The $\hat{g}$ and $\mathcal{F}^{-1}_{\xi'}[g]$ denote the partial Fourier transform with respect to x' and its inversion formula with respect to $\xi' = (\xi_1, \ldots, \xi_{N-1})$. Writing $\omega_\pm = \sqrt{\lambda/\gamma_{1,\pm} + |\xi'|^2}$, we have

$$\begin{aligned} &\hat{u}_{+1}(\xi', 0) = 0, \\ &(\partial_N \hat{u}_{+1})(\xi', 0) = -\sum_{j=1}^{N-1}\frac{\xi_j}{\gamma_{1,+}}\int_0^\infty e^{-y_N\omega_+}\hat{f}_{+j}(\xi', y_N)\, dy_N + \frac{\omega_+}{i\gamma_{1,+}}\int_0^\infty e^{-y_N\omega_+}\hat{f}_{+N}(\xi', y_N)\, dy_N. \end{aligned} \tag{2.5}$$

In fact, by the residue theorem

$$\hat{u}_{+1}(\xi', 0) = \sum_{j=1}^{N-1}\frac{1}{2\pi\gamma_{1,+}}\int_0^\infty\Big(\int_{-\infty}^\infty \frac{i\xi_j(e^{iy_N\xi_N} - e^{-iy_N\xi_N})}{\lambda/\gamma_{1,+} + |\xi'|^2 + \xi_N^2}\, d\xi_N\Big)\hat{f}_{+j}(\xi', y_N)\, dy_N$$

$$
\begin{aligned}
&+\frac{1}{2\pi\gamma_{1,+}}\int_0^\infty\Big(\int_{-\infty}^\infty \frac{i\xi_N(e^{iy_N\xi_N}+e^{-iy_N\xi_N})}{\lambda/\gamma_{1,+}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{+N}(\xi',y_N)\,dy_N\\
&=\sum_{j=1}^{N-1}\frac{1}{\gamma_{1,+}}\int_0^\infty\Big[\frac{i\xi_j e^{-y_N\omega_+}}{2i\omega_+}-\Big(-\frac{i\xi_j e^{-y_N\omega_+}}{-2i\omega_+}\Big)\Big]\hat f_{+j}(\xi',y_N)\,dy_N\\
&+\frac{1}{\gamma_{1,+}}\int_0^\infty\Big[\frac{i(i\omega_+)e^{-y_N\omega_+}}{2i\omega_+}+\Big(-\frac{i(-i\omega_+)e^{-y_N\omega_+}}{-2i\omega_+}\Big)\Big]\hat f_{+N}(\xi',y_N)\,dy_N\\
&=0.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
(\partial_N\hat u_{+1})(\xi',0)&=\sum_{j=1}^{N-1}\frac{-1}{2\pi\gamma_{1,+}}\int_0^\infty\Big(\int_{-\infty}^\infty\frac{\xi_j\xi_N(e^{iy_N\xi_N}-e^{-iy_N\xi_N})}{\lambda/\gamma_{1,+}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{+j}(\xi',y_N)\,dy_N\\
&-\frac{1}{2\pi\gamma_{1,+}}\int_0^\infty\Big(\int_{-\infty}^\infty\frac{\xi_N^2(e^{iy_N\xi_N}+e^{-iy_N\xi_N})}{\lambda/\gamma_{1,+}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{+N}(\xi',y_N)\,dy_N\\
&=\sum_{j=1}^{N-1}\frac{-1}{\gamma_{1,+}}\int_0^\infty\Big[\frac{\xi_j(i\omega_+)e^{-y_N\omega_+}}{2i\omega_+}+\frac{\xi_j(-i\omega_+)e^{-y_N\omega_+}}{-2i\omega_+}\Big)\Big]\hat f_{+j}(\xi',y_N)\,dy_N\\
&-\frac{1}{2\pi\gamma_{1,+}}\int_{-\infty}^\infty\int_0^\infty(e^{iy_N\xi_N}+e^{-iy_N\xi_N})\hat f_{+N}(\xi',y_N)\,dy_N\,d\xi_N\\
&+\frac{1}{\gamma_{1,+}}\int_0^\infty(\lambda/\gamma_{1,+}+|\xi'|^2)\Big(\frac{e^{-y_N\omega_+}}{2i\omega_+}-\frac{e^{-y_N\omega_+}}{-2i\omega_+}\Big)\hat f_{+N}(\xi',y_N)\,dy_N.
\end{aligned}
$$

Thus, using

$$
\frac{1}{2\pi}\int_{-\infty}^\infty\int_0^\infty(e^{iy_N\xi_N}+e^{-iy_N\xi_N})\hat f_{+N}(\xi',y_N)\,dy_N\,d\xi_N=\int_{-\infty}^\infty\mathcal F[f_{+N}^e](\xi)\,d\xi_N=\widehat{f_{+N}^e}(\xi',0)=0,
$$

we have (2.5). Similarly, we have

$$
\begin{aligned}
&\hat u_{-1}(\xi',0)=0,\\
&(\partial_N\hat u_{-1})(\xi',0)=\sum_{j=1}^{N-1}\frac{\xi_j}{\gamma_{1,-}}\int_{-\infty}^0 e^{y_N\omega_-}\hat f_{-j}(\xi',y_N)\,dy_N+\frac{\omega_-}{i\gamma_{1,-}}\int_{-\infty}^0 e^{y_N\omega_-}\hat f_{-N}(\xi',y_N)\,dy_N.
\end{aligned}
\tag{2.6}
$$

In fact,

$$
\begin{aligned}
\hat u_{-1}(\xi',0)&=\sum_{j=1}^{N-1}\frac{1}{2\pi\gamma_{1,-}}\int_{-\infty}^0\Big(\int_{-\infty}^\infty\frac{i\xi_j(-e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{\lambda/\gamma_{1,-}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{-j}(\xi',y_N)\,dy_N\\
&+\frac{1}{2\pi\gamma_{1,-}}\int_{-\infty}^0\Big(\int_{-\infty}^\infty\frac{i\xi_N(e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{\lambda/\gamma_{1,-}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{-N}(\xi',y_N)\,dy_N\\
&=\sum_{j=1}^{N-1}\frac{1}{\gamma_{1,-}}\int_{-\infty}^0\Big(-\frac{i\xi_j e^{y_N\omega_-}}{2i\omega_-}-\frac{i\xi_j e^{y_N\omega_-}}{-2i\omega_-}\Big)\hat f_{-j}(\xi',y_N)\,dy_N\\
&+\frac{1}{\gamma_{1,-}}\int_{-\infty}^0\Big(\frac{i(i\omega_-)e^{y_N\omega_-}}{2i\omega_-}-\frac{i(-i\omega_-)e^{y_N\omega_-}}{-2i\omega_-}\Big)\hat f_{-N}(\xi',y_N)\,dy_N\\
&=0;\\
(\partial_N\hat u_{-1})(\xi',0)&=\sum_{j=1}^{N-1}\frac{1}{2\pi\gamma_{1,-}}\int_{-\infty}^0\Big(\int_{-\infty}^\infty\frac{-\xi_j\xi_N(-e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{\lambda/\gamma_{1,-}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{-j}(\xi',y_N)\,dy_N\\
&+\frac{1}{2\pi\gamma_{1,-}}\int_{-\infty}^0\Big(\int_{-\infty}^\infty\frac{-\xi_N^2(e^{-iy_N\xi_N}+e^{iy_N\xi_N})}{\lambda/\gamma_{1,-}+|\xi'|^2+\xi_N^2}\,d\xi_N\Big)\hat f_{-N}(\xi',y_N)\,dy_N
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{N-1} \frac{1}{\gamma_{1,-}} \int_0^\infty \Big(\frac{\xi_j (i\omega_-) e^{y_N\omega_+}}{2i\omega_-} + \frac{\xi_j(-i\omega_-)e^{y_N\omega_-}}{-2i\omega_+} \Big) \hat f_{-j}(\xi', y_N)\, dy_N \\
&- \frac{1}{2\pi\gamma_{1,-}} \int_{-\infty}^\infty \int_0^\infty (e^{-iy_N\xi_N} + e^{iy_N\xi_N}) \hat f_{-N}(\xi', y_N)\, dy_N\, d\xi_N \\
&+ \frac{1}{\gamma_{1,-}} \int_{-\infty}^0 (\lambda/\gamma_{1,-} + |\xi'|^2) \Big(\frac{e^{y_N\omega_+}}{2i\omega_-} - \frac{e^{y_N\omega_+}}{-2i\omega_-} \Big) \hat f_{-N}(\xi', y_N)\, dy_N \\
&= \sum_{j=1}^{N-1} \frac{\xi_j}{\gamma_{1,-}} \int_{-\infty}^0 e^{y_N\omega_-} \hat f_{-j}(\xi', y_N)\, dy_N \\
&- \frac{1}{\gamma_{1,-}} \int_{-\infty}^\infty \mathcal{F}[f^e_{-N}](\xi)\, d\xi + \frac{\omega_-}{i\gamma_{1,-}} \int_{-\infty}^0 e^{y_N\omega_-} \hat f_{-N}(\xi', y_N)\, dy_N,
\end{aligned}
$$

and therefore we have (2.6).

Let

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \le \pi - \epsilon\} \quad \text{for } 0 < \epsilon < \pi/2.$$

Let $1 < q < \infty$, and then by the Fourier multiplier theorem,

$$|\lambda|^{1/2} \|u_{\pm 1}\|_{L_q(\mathbb{R}^N)} + \|\nabla u_{\pm 1}\|_{L_q(\mathbb{R}^N)} \le C_{q,\epsilon} \|f_\pm\|_{L_q(\mathbb{R}^N_\pm)} \tag{2.7}$$

for any $\lambda \in \Sigma_\epsilon$ with some constant $C_{q,\epsilon}$ depending solely on ϵ, q and $\gamma_{1,\pm}$.

Next, we construct the compensating function $u_{\pm 2}$. In view of (2.5) and (2.6), $u_{\pm 2}$ should satisfy the equations:

$$
\begin{cases}
(\lambda - \gamma_{1,\pm}\Delta) u_{\pm 2} = 0 & \text{in } \mathbb{R}^N_\pm, \\
\gamma_{1,+}\partial_N u_{+2}|_{x_N=0+} - \gamma_{1,-}\partial_N u_{-2}|_{x_N=0-} = h, & \\
u_{+2}|_{x_N=0+} = u_{-2}|_{x_N=0-}, &
\end{cases} \tag{2.8}
$$

where $h = g - (\partial_N u_+(\cdot, +0) - \partial_N u_-(\cdot, -0))$. We find $\hat u_{\pm 2}(\xi', x_N)$ of the forms: $\hat u_{\pm 2}(\xi', x_N) = \alpha_\pm e^{\mp\omega_\pm x_N}$. Obviously,

$$(\lambda + \gamma_{1,\pm}|\xi'|^2) \hat u_{\pm 2} - \gamma_{1,\pm} \partial_N^2 \hat u_{\pm 2} = 0.$$

Since $\gamma_{1,\pm}\partial_N \hat u_{\pm 2}(\xi', 0) = \mp\gamma_{1,\pm}\alpha_\pm\omega_\pm$ and $\hat u_{\pm 2}(\xi', 0) = \alpha_\pm$, from the interface condition of (2.8) and (2.5) and (2.6), we have

$$-\gamma_{1,+}\alpha_+\omega_+ - \gamma_{1,-}\alpha_-\omega_- = \hat h_+(\xi', 0) - \hat h_-(\xi', 0), \quad \alpha_+ = \alpha_-,$$

which, combined with (2.5) and (2.6), we have

$$
\begin{aligned}
\alpha_+ = \alpha_- &= -\frac{\hat h_+(\xi', 0) - \hat h_-(\xi', 0)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-} = -\frac{\hat g(\xi', 0)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-} \\
&+ \frac{1}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-} \Big\{ \sum_{j=1}^{N-1} \xi_j \int_0^\infty e^{-y_N\omega_+} \hat f_{+j}(\xi', y_N)\, dy_N + i\omega_+ \int_0^\infty e^{-y_N\omega_+} \hat f_{+N}(\xi', y_N)\, dy_N \\
&\qquad + \sum_{j=1}^{N-1} \xi_j \int_{-\infty}^0 e^{y_N\omega_-} \hat f_{-j}(\xi', y_N)\, dy_N - i\omega_- \int_{-\infty}^0 e^{y_N\omega_-} \hat f_{-N}(\xi', y_N)\, dy_N \Big\},
\end{aligned}
$$

so that

$$
\begin{aligned}
\hat u_{\pm 2}(\xi', x_N) &= -\frac{e^{\mp x_N} \hat g(\xi', 0)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-} \\
&+ \frac{e^{\mp\omega_\pm x_N}}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-} \Big\{ \sum_{j=1}^{N-1} \xi_j \int_0^\infty e^{-y_N\omega_+} \hat f_{+j}(\xi', y_N)\, dy_N + i\omega_+ \int_0^\infty e^{-y_N\omega_+} \hat f_{+N}(\xi', y_N)\, dy_N
\end{aligned}
$$

$$+\sum_{j=1}^{N-1}\xi_j\int_{-\infty}^0 e^{y_N\omega_+}\hat f_{-j}(\xi',y_N)\,dy_N - i\omega_-\int_{-\infty}^0 e^{y_N\omega_-}\hat f_{-N}(\xi',y_N)\,dy_N\Big\}.$$

Therefore, we have

$$\begin{aligned}
\hat u_{+2}(\xi',x_N) &= \int_0^\infty \frac{e^{-\omega_+(x_N+y_N)}\partial_N\hat g_+(\xi',y_N)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\,dy_N - \int_0^\infty \frac{e^{-\omega_+(x_N+y_N)}\omega_+\hat g_+(\xi',y_N)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\,dy_N\\
&+\sum_{j=1}^{N-1}\frac{\xi_j}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_0^\infty e^{-\omega_+(x_N+y_N)}\hat f_{+j}(\xi',y_N)\,dy_N\\
&+\frac{i\omega_+}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_0^\infty e^{-\omega_+(x_N+y_N)}\hat f_{+N}(\xi',y_N)\,dy_N\\
&+\sum_{j=1}^{N-1}\frac{\xi_j}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_0^\infty e^{-(\omega_+x_N+\omega_-y_N)}\hat f_{-j}(\xi',-y_N)\,dy_N\\
&-\frac{i\omega_-}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_0^\infty e^{-(\omega_+x_N+\omega_-y_N)}\hat f_{-N}(\xi',-y_N)\,dy_N;\\
\hat u_{-2}(\xi',x_N) &= -\int_{-\infty}^0 \frac{e^{\omega_-(x_N+y_N)}\partial_N\hat g_-(\xi',y_N)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\,dy_N - \int_{-\infty}^0 \frac{e^{\omega_-(x_N+y_N)}\omega_-\hat g_-(\xi',y_N)}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\,dy_N\\
&+\sum_{j=1}^{N-1}\frac{\xi_j}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_{-\infty}^0 e^{(\omega_-x_N+\omega_+y_N)}\hat f_{+j}(\xi',-y_N)\,dy_N\\
&+\frac{i\omega_-}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_{-\infty}^0 e^{(\omega_-x_N+\omega_+y_N)}\hat f_{+N}(\xi',-y_N)\,dy_N\\
&+\sum_{j=1}^{N-1}\frac{\xi_j}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_{-\infty}^0 e^{\omega_-(x_N+y_N)}\hat f_{-j}(\xi',y_N)\,dy_N\\
&-\frac{i\omega_-}{\gamma_{1,+}\omega_+ + \gamma_{1,-}\omega_-}\int_{-\infty}^0 e^{\omega_-(x_N+y_N)}\hat f_{-N}(\xi',y_N)\,dy_N. \qquad (2.9)
\end{aligned}$$

To estimate $u_{\pm 2}$, we introduce some symbol classes.

Definition 2.1. Let Ξ be a domain in $\mathbb{C}$ and let $m(\xi',\lambda)$ ($\lambda=\gamma+i\tau\in\Xi$) be a function defined for $(\xi',\lambda)\in(\mathbb{R}^{N-1}\setminus\{0\})\times\Xi$. Assume that $m(\xi,\lambda)$ is an infinitely many differentiable function with respect to $\xi\in\mathbb{R}^{N-1}\setminus\{0\}$ for each $\lambda\in\Xi$.

(1) $m(\xi',\lambda)$ is called a multiplier of order s with type 1 on Ξ if the estimates:

$$|\partial_{\xi'}^{\kappa'}m(\xi',\lambda)|\le C_{\alpha'}(|\lambda|^{1/2}+|\xi'|)^{s-|\kappa'|} \qquad (2.10)$$

hold for any multi-index $\kappa'\in\mathbb{N}_0^{N-1}$ and $(\xi',\lambda)\in\Xi$ and $(\xi',\lambda)\in\Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and Ξ.

(2) $m(\xi',\lambda)$ is called a multiplier of order s with type 2 on Ξ if the estimates:

$$|\partial_{\xi'}^{\kappa'}m(\xi',\lambda)|\le C_{\kappa'}(|\lambda|^{1/2}+|\xi'|)^{s}|\xi'|^{-|\kappa'|} \qquad (2.11)$$

hold for any multi-index $\kappa'\in\mathbb{N}_0^{N-1}$ and $(\xi',\lambda)\in\Xi$ with some constants $C_{\kappa'}$ depending solely on κ' and Ξ.

Let $\mathbf{M}_{s,i}(\Xi)$ be the set of all multipliers of order s with type i on Ξ ($i=1,2$).

Obviously, $\mathbf{M}_{s,i}(\Xi)$ are vector spaces on $\mathbb{C}$. Moreover, the following lemma follows from the fact: $(|\lambda|^{1/2}+|\xi'|)^{-|\alpha'|}\le|\xi'|^{-|\alpha'|}$ and the Leibniz rule immediately.

Lemma 2.2. *Let s_1, s_2 be two real numbers. Then, the following three assertions hold.*

(1) *Given $m_i \in \mathbf{M}_{s_i,1}(\Xi)$ $(i = 1, 2)$, we have $m_1 m_2 \in \mathbf{M}_{s_1+s_2,1}(\Xi)$.*

(2) *Given $\ell_i \in \mathbf{M}_{s_i,i}(\Xi)$ $(i = 1, 2)$, we have $\ell_1\ell_2 \in \mathbf{M}_{s_1+s_2,2}(\Xi)$.*

(3) *Given $n_i \in \mathbf{M}_{s_i,2}(\Xi)$ $(i = 1, 2)$, we have $m_1 m_2 \in \mathbf{M}_{s_1+s_2,2}(\Xi)$.*

In what follows, we use the following lemma due to Shibata and Shimizu [3, Lemma 5.4].

Lemma 2.3. *Let $0 < \vartheta < \pi/2$ and $1 < q < \infty$. Given $\ell_0(\xi,', \lambda) \in \mathbb{M}_{0,1}(\Sigma_\vartheta)$ and $\ell_1(\xi', \lambda) \in \mathbb{M}_{0,2}(\Sigma_\vartheta)$, we define the operators $L_j(\lambda)$ $(j = 1, 2, 3, 4)$ by*

$$
\begin{aligned}
[L_1(\lambda)h](x) &= \int_0^\infty \mathcal{F}'^{-1}[\ell_0(\xi', \lambda)\lambda^{1/2}e^{-A_k(\xi',\lambda)(x_N+y_N)}\mathcal{F}[h](\xi', y_N)](x')\, dy_N,\\
[L_2(\lambda)h](x) &= \int_0^\infty \mathcal{F}'^{-1}[\ell_1(\xi, \lambda)Ae^{-A_k(\xi',\lambda)(x_N+y_N)}\mathcal{F}'[h](\xi', y_N)](x')\, dy_N,\\
[L_3(\lambda)h](x) &= \int_0^\infty \mathcal{F}'^{-1}[\ell_1(\xi', \lambda)Ae^{-A(x_N+y_N)}\mathcal{F}[h](\xi', y_N))](x')\, dy_N,\\
[L_4(\lambda)h](x) &= \int_0^\infty \mathcal{F}_{\xi'}'^{-1}[\ell_1(\xi', \lambda)A^2\mathcal{M}_k(\xi', x_N + y_N, \lambda)\mathcal{F}'[h](\xi', y_N)](x')\, dy_N.
\end{aligned}
$$

Then, L_i is a bounded linear operator on $L_q(\mathbb{R}^N_+)$ and

$$\|L_i(\lambda)h\|_{L_q(\mathbb{R}^N_+)} \le C\|h\|_{L_q(\mathbb{R}^N_+)}.$$

Using the identities:

$$\omega_\pm = \frac{\lambda\gamma_{1,\pm}^{-1} + |\xi'|^2}{\omega_\pm}, \quad 1 = \frac{\lambda\gamma_{1,\pm}^{-1} + |\xi'|^2}{\omega_\pm^2},$$

and applying Lemma 2.3, we have

$$\|\lambda^{1/2}u_{\pm2}, \nabla u_{\pm2})\|_{L_q(\mathbb{R}^N_\pm)} \le C\{\|\mathbf{f}_+|_{L_q(\mathbb{R}^N_+)} + \|\mathbf{f}_-|_{L_q(\mathbb{R}^N_-)} + \|(\lambda^{1/2}g_\pm, \nabla g_\pm)\|_{L_q(\mathbb{R}^N_\pm)}\}. \tag{2.12}$$

Setting $u_\pm = u_{\pm1} + u_{\pm2}$ and combining (2.7) and (2.12) yield that $u_\pm$ satisfy the estimate:

$$\|(\lambda^{1/2}u_\pm, \nabla u_\pm)\|_{L_q(\mathbb{R}^N_\pm)} \le C\{\|\mathbf{f}_+\|_{L_q(\mathbb{R}^N_+)} + \|\mathbf{f}_-\|_{L_q(\mathbb{R}^N_-)} + \|(\lambda^{1/2}g_\pm, \nabla g_\pm)\|_{L_q(\mathbb{R}^N_\pm)}\}. \tag{2.13}$$

Moreover, by the Fourier multiplier theorem and Lemma 2.3, we see that $u_\pm \in W_q^2(\mathbb{R}^N_\pm)$ and $u_\pm$ satisfies (2.2). Since $\mathbf{f}_\pm|_{x_N=\pm0} = 0$, assuming that $g = 0$ in (2.2), using the integration by parts and defining u by $u(x) = u_\pm(x)$ for $x \in \mathbb{R}^N_\pm$, we have

Theorem 2.4. *Let $1 < q < \infty$ and $0 < \vartheta < \pi/2$. Set*

$$\Sigma_\vartheta = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \vartheta\}.$$

Then, for any $\mathbf{f} \in L_q(\dot{\mathbb{R}}^N)$ and $\lambda \in \Sigma_\vartheta$, the variational problem (2.1) admits a unique solution $u \in W_q^1(\mathbb{R}^N)$ satisfying the estimate:

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(\mathbb{R}^N)} \le C\|\mathbf{f}\|_{L_q(\dot{\mathbb{R}}^N)}. \tag{2.14}$$

with some constant $C > 0$.

2.2 Bent half-space problem

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. Writing $\nabla\Phi(x) = \mathcal{A} + B(x)$ and $\nabla\Phi^{-1}(y) = \mathcal{A}_{-1} + B_{-1}(y)$, we assume that $\mathcal{A}$ and $\mathcal{A}_{-1}$ are orthonormal matrices with constant coefficients and $B(x)$ and $B_{-1}(y)$ are matrices of functions in $W^1_r(\mathbb{R}^N)$ with $N < r < \infty$ such that

$$\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \leq M_1, \quad \|\nabla(B, B_{-1})\|_{L_r(\mathbb{R}^N)} \leq M_2. \tag{2.15}$$

We will choose M_1 small enough eventually, so that we may assume that $0 < M_1 \leq 1 \leq M_2$ in the following. Set $D_\pm = \Phi(\mathbb{R}^N_\pm)$, $\Gamma_0 = \Phi(\mathbb{R}^N_0)$ and let $\mathbf{n}_0$ be the unit outer normal to Γ_0. Setting $\Phi^{-1} = (\Phi_{-1,1}, \ldots, \Phi_{-1,N})$, we see that Γ_0 is represented by $x_N = \Phi_{-1,N}(y) = 0$, which furnishes that

$$\mathbf{n}_0 = \frac{\nabla\Phi_{-1,N}}{|\nabla\Phi_{-1,N}|} = \frac{(\mathcal{A}_{N1} + B_{N1}, \ldots, \mathcal{A}_{NN} + B_{NN})}{(\sum_{i=1}^N(\mathcal{A}_{Ni} + B_{Ni})^2)^{1/2}}, \tag{2.16}$$

where we have set $\mathcal{A}_{-1} = (\mathcal{A}_{ij})$ and $B_{-1} = (B_{ij})$. In particular, $\mathbf{n}_0$ is defined on the whole $\mathbb{R}^N$. Since $\sum_{i=1}^N(\mathcal{A}_{Ni} + B_{Ni})^2 = 1 + \sum_{i=1}^N(2\mathcal{A}_{Ni}B_{Ni} + B_{Ni}^2)$, by (2.15) $\|\nabla\mathbf{n}_0\|_{L_r(\mathbb{R}^N)} \leq C_N M_2$.

Moreover, we have

$$\frac{\partial}{\partial y_j} = \sum_{k=1}^N \frac{\partial x_k}{\partial y_j}\frac{\partial}{\partial x_k} = \sum_{k=1}^N(\mathcal{A}_{ij} + B_{ij}(\Phi(x)))\frac{\partial}{\partial x_k}. \tag{2.17}$$

By (2.15),

$$\|B_{jk} \circ \Phi\|_{L_\infty(\mathbb{R}^N)} \leq CM_1, \quad \|\nabla(B_{jk} \circ \Phi)\|_{L_r(\mathbb{R}^N)} \leq CM_2. \tag{2.18}$$

Let $\ddot{\mathbb{R}}^N = D_+ \cup D_-$, and

$$(u, v)_{\ddot{\mathbb{R}}^N} = \int_{D_+} u(x)\overline{v(x)}\,dx + \int_{D_-} u(x)\overline{v(x)}\,dx..$$

In this subsection, we consider the variational equations:

$$\lambda(u, \varphi)_{\ddot{\mathbb{R}}^N} + (\gamma_1\nabla\mathbf{u}, \nabla\varphi)_{\ddot{\mathbb{R}}^N} = (\mathbf{f}, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in W^1_q(\mathbb{R}^N) \tag{2.19}$$

with $\mathbf{f} = (f_1, \ldots, f_N) \in L_q(\mathbb{R}^N)$, and γ_1 is a piecewise smooth function defined by $\gamma_1|_{D_\pm} = \gamma_{1,\pm}$ with some positive constants $\gamma_\pm$. By the transformation: $y = \Phi(x)$, the equation (2.19) is transformed to the equation:

$$\lambda(vJ, \varphi)_{\dot{\mathbb{R}}^N} + ((\gamma_1 \circ \Phi)J\sum_{j,k,\ell=1}^N(\mathcal{A}_{jk} + B_{jk} \circ \Phi)(\mathcal{A}_{j\ell} + B_{j\ell} \circ \Phi)\partial_k v, \partial_\ell\varphi)_{\dot{\mathbb{R}}^N} = (\mathbf{F}, \nabla\varphi)_{\dot{\mathbb{R}}^N} \tag{2.20}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$, where $J = \det\Phi$ and $\mathbf{F} = (F_1, \ldots, F_N)$ with $F_k = \sum_{j=1}^N(\mathcal{A}_{jk} + B_{jk} \circ \Phi)f_j$. By (2.15),

$$\|J - 1\|_{L_\infty(\mathbb{R}^N)} \leq CM_1, \quad \|\nabla J\|_{L_r(\mathbb{R}^N)} \leq CM_2. \tag{2.21}$$

Let $Jv = w$, and then

$$\begin{aligned} J&\sum_{j,k,\ell=1}^N(\mathcal{A}_{jk} + B_{jk} \circ \Phi)(\mathcal{A}_{j\ell} + B_{j\ell} \circ \Phi)\partial_k v \\ &= \sum_{j,k,\ell=1}^N(\mathcal{A}_{jk} + B_{jk} \circ \Phi)(\mathcal{A}_{j\ell} + B_{j\ell} \circ \Phi)\partial_k w - \{\sum_{j,k,\ell=1}^N(\mathcal{A}_{jk} + B_{jk} \circ \Phi)(\mathcal{A}_{j\ell} + B_{j\ell} \circ \Phi)\partial_k J\}J^{-2}w. \end{aligned}$$

Noting

$$\sum_{j=1}^N \mathcal{A}_{jk}\mathcal{A}_{j\ell} = \delta_{k\ell} = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}$$

and letting $P = (P_{k\ell}(x))$ and $Q(x) = (Q_1(x), \ldots, Q_N(x))$ with

$$P_{k\ell} = \sum_{j=1}^{N}\{\mathcal{A}_{jk}(B_{j\ell}\circ\Phi) + \mathcal{A}_{j\ell}(B_{jk}\circ\Phi) + (B_{jk}\circ\Phi)(B_{j\ell}\circ\Phi)\},$$

$$Q_\ell = -\sum_{j,k=1}^{N}(\mathcal{A}_{jk} + B_{jk}\circ)(\mathcal{A}_{j\ell} + B_{j\ell}\circ)(\partial_k J)J^{-2},$$

we have

$$(\lambda w, \varphi)_{\dot{\mathbb{R}}^N} + ((\gamma_1\circ\Phi)\nabla w, \nabla\varphi)_{\dot{\mathbb{R}}^N} + ((\gamma_1\circ\Phi)(P\nabla w + Qw), \nabla\varphi)_{\dot{\mathbb{R}}^N} = (\mathbf{F}, \nabla\varphi)_{\mathbb{R}^N} \tag{2.22}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$. By Sobolev's imbedding theorem,

$$\|ab\|_{L_q(\dot{\mathbb{R}}^N)} \le C\|a\|_{L_r(\dot{\mathbb{R}}^N)}\|b\|^{1-\frac{N}{r}}_{L_q(\dot{\mathbb{R}}^N)}\|\nabla b\|^{\frac{N}{r}}_{L_q(\dot{\mathbb{R}}^N)} \tag{2.23}$$

for any $a \in L_r(\dot{\mathbb{R}}^N)$ and $b \in W^1_q(\dot{\mathbb{R}}^N)$ provided $N < r < \infty$ (cf. [2, Lemma 2.4]). So, applying (2.23) and using (2.18) and (2.21), we have

$$\|P\nabla w + Qw\|_{L_q(\dot{\mathbb{R}}^N)} \le C(M_1 + \sigma)\|\nabla w\|_{L_q(\dot{\mathbb{R}}^N)} + C_\sigma M_2\|w\|_{L_q(\dot{\mathbb{R}}^N)} \tag{2.24}$$

for any small $\sigma > 0$ with some constants C and C_σ, where C_σ is a constant such that $C_\sigma \to \infty$ as $\sigma \to 0$.
Given $z \in W^1_q(\mathbb{R}^N)$, let $w \in W^1_q(\mathbb{R}^N)$ be a solution to the variational equation:

$$(\lambda w, \varphi)_{\dot{\mathbb{R}}^N} + ((\gamma_1\circ\Phi)\nabla w, \nabla\varphi)_{\dot{\mathbb{R}}^N} = (\mathbf{F} - (\gamma_1\circ\Phi)(P\nabla z + Qz), \nabla\varphi)_{\mathbb{R}^N} \tag{2.25}$$

for any $\varphi \in W^1_{q'}(\mathbb{R}^N)$. By Theorem 2.4 and (2.24), such w uniquely exists, which satisfies the estimate:

$$\|(\lambda^{1/2}w, \nabla w)\|_{L_q(\mathbb{R}^N)} \le C(M_1 + \sigma)\|\nabla z\|_{L_q(\mathbb{R}^N)} + C_\sigma M_2\|z\|_{L_q(\mathbb{R}^N)} + C\|\mathbf{f}\|_{L_q(\mathbb{R}^N)}.$$

Choosing $\sigma > 0$ and $M_1 > 0$ small enough and $|\lambda|$ large enough, by the Banach fixed point theorem we have

Theorem 2.5. *Let $1 < q < \infty$ and $0 < \vartheta < \pi/2$. For $\lambda_0 > 0$, we set*

$$\Sigma_{\vartheta,\lambda_0} = \{\lambda \in \Sigma_\vartheta \mid |\lambda| \ge \lambda_0\}.$$

Then, there exists a $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\vartheta,\lambda_0}$ and $\mathbf{f} \in L_q(\mathbb{R}^N)$ problem (2.19) admits a unique solution $u \in W^1_q(\mathbb{R}^N)$ satisfying the estimate:

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(\mathbb{R}^N)} \le C\|\mathbf{f}\|_{L_q(\mathbb{R}^N)}.$$

Next, for the later use we consider two more variational problems. The first one is the variational problem in $\mathbb{R}^N$:

$$\lambda(u, \varphi)_{\mathbb{R}^N} + (\gamma\nabla u, \nabla\varphi)_{\mathbb{R}^N} = (\mathbf{f}, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in W^1_{q'}(\mathbb{R}^N), \tag{2.26}$$

where γ is a positive constant. Then, we have

Theorem 2.6. *Let $1 < q < \infty$ and $0 < \vartheta < \pi/2$. Then, for any $\lambda \in \Sigma_\vartheta$ and $\mathbf{f} \in L_q(\mathbb{R}^N)$ problem (2.27) admits a unique solution $u \in W^1_q(\mathbb{R}^N)$ satisfying the estimate:*

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(\mathbb{R}^N)} \le C\|\mathbf{f}\|_{L_q(\mathbb{R}^N)}.$$

The second one is the variational problem in D_+:

$$\lambda(u, \varphi)_{D_+} + (\gamma\nabla u, \nabla\varphi)_{D_+} = (\mathbf{f}, \nabla\varphi)_{D_+} \quad \text{for any } \varphi \in W^1_{q',0}(D_+). \tag{2.27}$$

Employing the similar argumentation to the proof of Theorem 2.5, we have

Theorem 2.7. *Let $1 < q < \infty$ and $0 < \vartheta < \pi/2$. Then, there exists a $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\vartheta,\lambda_0}$ and $\mathbf{f} \in L_q(D_+)$ problem (2.19) admits a unique solution $u \in W^1_{q,0}(D_+)$ satisfying the estimate:*

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(D_+)} \le C\|\mathbf{f}\|_{L_q(D_+)}.$$

2.3 A proof of Theorem 1.5

To prove Theorem 1.5, first we consider the variational problem:

$$\lambda(u,\varphi)_{\dot\Omega} + (\gamma^1\nabla u, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega). \tag{2.28}$$

And then, we have

Theorem 2.8. *Leet $1 < q < \infty$, $N < r < \infty$ and $0 < \vartheta < \pi/2$. Assume that $\max(q,q') \le r$ and that Γ and Γ_- are compact hypersurfaces of class $W^{2-1/r}_r$. Then, there exists a $\lambda_1 > 0$ such that for any $\mathbf{f} \in L_q(\Omega)^N$ and $\lambda \in \Sigma_{\vartheta,\lambda_1}$, problem (2.28) admits a unique solution $u \in W^1_{q,0}(\Omega)$ satisfying the estimate:*

$$\|(\lambda^{1/2}u, \nabla u)\|_{L_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}.$$

To prove Theorem 2.1, we start with

Proposition 2.9. *Let $N < r < \infty$ and let Γ and Γ_- be compact hyper-surfaces of $W^{2-1/r}_r$. Set $\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma_-$. Let M_1 be any positive number $\in (0,1)$. Then, there exist constants $M_2 > 0$, $0 < d < 1$, open sets $U_\pm \subset \Omega_\pm$, finitely many N-vector of functions $\Phi^i_j \in W^2_r(\mathbb{R}^N)^N$ $(i = 0,1, j = 1,\dots,K_i)$, and points $x^i_j \in \Gamma_i$ $(i = 0,1, j = 1,\dots,K_i)$ such that the following assertions hold:*

(i) *The maps: $\mathbb{R}^N \ni x \mapsto \Phi^i_j(x) \in \mathbb{R}^N$ $(i = 0,1)$ are bijective.*

(ii) $\Omega = \left(\bigcup_{j=1}^{K_0} \Phi^0_j(\mathbb{R}^N) \cap B_d(x^0_j)\right) \cup \left(\bigcup_{j=1}^{K_1} \Phi^1_j(\mathbb{R}^N_+) \cap B_d(x^1_j)\right) \cup U_+ \cup U_-$,
$\Phi^0_j(\mathbb{R}^N_0) \cap B_d(x^0_j) = \Gamma \cap B_d(x^0_j)$, $\quad \Phi^0_j(\mathbb{R}^N) \cap B_d(x^0_j) = \Omega \cap B_d(x^0_j)$,
$\Phi^1_j(\mathbb{R}^N_0) \cap B_d(x^1_j) = \Gamma_- \cap B_d(x^1_j)$, $\quad \Phi^1_j(\mathbb{R}^N_+) \cap B_d(x^1_j) = \Omega_- \cap B_d(x^1_j)$.

(iii) *There exist C^∞ functions ζ^i_j, $\tilde\zeta^i_j$ $(i = 0,1,\ j = 1,\dots,K_i)$, $\zeta^2_\pm$, and $\tilde\zeta^2_\pm$ such that*

$$0 \le \zeta^i_j,\ \tilde\zeta^i_j \le 1,\ 0 \le \zeta^2_\pm,\ \tilde\zeta^2_\pm \le 1 \quad \operatorname{supp}\zeta^i_j, \operatorname{supp}\tilde\zeta^i_j \subset B_{d^i}(x^i_j), \quad \operatorname{supp}\zeta^2_\pm, \operatorname{supp}\tilde\zeta^2_\pm \subset U_\pm,$$
$$\|(\zeta^i_j, \tilde\zeta^i_j)\|_{W^2_\infty(\mathbb{R}^N)},\ \|(\zeta^2_\pm, \tilde\zeta^2_\pm)\|_{W^2_\infty(\mathbb{R}^N)} \le c_0, \quad \tilde\zeta^i_j = 1 \text{ on } \operatorname{supp}\zeta^i_j, \quad \tilde\zeta^2_\pm = 1 \text{ on } \operatorname{supp}\zeta^2_\pm,$$
$$\sum_{i=0}^{1}\sum_{j=1}^{K_i} \zeta^i_j + \zeta^2_+ + \zeta^2_- = 1 \text{ on } \overline\Omega, \quad \sum_{j=1}^{\infty}\zeta^i_j = 1 \text{ on } \Gamma^i \ (i = 0,1).$$

Here, c_0 is a constant which depends on M_2, N, q and r.

(iv) *$\nabla\Phi^i_j = \mathcal{A}^i_j + B^i_j$, $\nabla(\Phi^i_j)^{-1} = \mathcal{A}^i_{j,-} + B^i_{j,-}$, where $\mathcal{A}^i_j$ and $\mathcal{A}^i_{j,-}$ are $N \times N$ constant orthonormal matrices, and B^i_j and $B^i_{j,-}$ are $N \times N$ matrices of $W^{1+i}_r(\mathbb{R}^N)$ functions defined on $\mathbb{R}^N$ which satisfy the conditions: $\|B^i_j\|_{L_\infty(\mathbb{R}^N)} \le M_1$, $\|B^i_{j,-}\|_{L_\infty(\mathbb{R}^N)} \le M_1$, $\|\nabla B^i_j\|_{W^i_r(\mathbb{R}^N)} \le M_2$ and $\|\nabla B^i_{j,-}\|_{W^i_r(\mathbb{R}^N)} \le M_2$ for $i = 0$, 1 and $j = 1,\dots K_i$. Here, $W^0_r(\mathbb{R}^N) = L_r(\mathbb{R}^N)$.*

Since Γ and Γ_- are compact hyper-surfaces of $W^{2-1/r}_r$ class, employing the argumentations due to Enomoto and Shibata [1, Proposition 6.1], we can prove Proposition 2.9, so that we may omit its proof.
Let $\ddot{\mathbb{R}}^N_j = \Phi^0_j(\mathbb{R}^N_+) \cup \Phi^0_j(\mathbb{R}^N_-)$, $D^1_j = \Phi^1_j(\mathbb{R}^N_+)$, and $\Gamma^1_j = \partial D^1_j = \Phi^1_j(\mathbb{R}^N_0)$. Given $\mathbf{f} \in L_q(\Omega)^N$, let u^0_j, u^1_j and $u^2_\pm$ be solutions to the following variational problems:

$$\lambda(u^0_j,\varphi)_{\ddot{\mathbb{R}}^N_j} + (\gamma^0_j\nabla u^0_j, \nabla\varphi)_{\ddot{\mathbb{R}}^N_j} = (\tilde\zeta^0_j\mathbf{f}, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in W^1_{q'}(\mathbb{R}^N), \tag{2.29}$$
$$\lambda(u^1_j,\varphi)_{D^1_j} + (\gamma_{1,-}\nabla u^1_j, \nabla\varphi)_{D^1_j} = (\tilde\zeta^1_j\mathbf{f}, \nabla\varphi)_{D^1_j} \quad \text{for any } \varphi \in W^1_{q',0}(D^1_j), \tag{2.30}$$
$$\lambda(u^2_\pm,\varphi)_{\mathbb{R}^N} + (\gamma_{1,\pm}\nabla u^2_\pm, \nabla\varphi)_{\mathbb{R}^N} = (\tilde\zeta^2_\pm\mathbf{f}, \nabla\varphi)_{\mathbb{R}^N} \quad \text{for any } \varphi \in W^1_{q'}(\mathbb{R}^N). \tag{2.31}$$

Here, γ^0_j are piece-wise constant functions defined by $\gamma^0_j|_{\Phi^0_j(\mathbb{R}^N_\pm)} = \gamma_{1,\pm}$. By Theorem 2.5, Theorem 2.6 and Theorem 2.7, there exists a constant $\lambda_1 \ge 1$ such that for any $\lambda \in \Sigma_{\vartheta,\lambda_1}$ problems (2.29), (2.30) and

(2.31) admit unique solutions $u_j^0 \in W_q^1(\mathbb{R}^N)$, $u_j^1 \in W_q^1(D_j^1)$ and $u_\pm^2 \in W_q^1(\mathbb{R}^N)$ satisfying the estimates:

$$\begin{aligned}
\|(\lambda^{1/2}u_j^0, \nabla u_j^0)\|_{L_q(\mathbb{R}^N)} &\leq C\|\tilde\zeta_j^0 \mathbf{f}\|_{L_q(\mathbb{R}^N)}, \\
\|(\lambda^{1/2}u_j^1, \nabla u_j^1)\|_{L_q(D_j^1)} &\leq C\|\tilde\zeta_j^1 \mathbf{f}\|_{L_q(D_j^1)}, \\
\|(\lambda^{1/2}u_\pm^2, \nabla u_j^0)\|_{L_q(\mathbb{R}^N)} &\leq C\|\tilde\zeta_\pm^2 \mathbf{f}\|_{L_q(\mathbb{R}^N)}.
\end{aligned} \tag{2.32}$$

Let $\mathcal{A}(\lambda)$ be an operator defined by

$$\mathcal{A}(\lambda)\mathbf{f} = \sum_{i=0}^{1}\sum_{j=1}^{K_i} \zeta_j^i u_j^i + \zeta_+^2 u_+^2 + \zeta_-^2 u_-^2,$$

and then noting that $(\sum_{i=0}^1 \sum_{j=1}^{K_i} \nabla\zeta_j^i + \nabla\zeta_+^2 + \nabla\zeta_-^2 = 0$, by (2.29), (2.30) and (2.31) we have

$$\lambda(\mathcal{A}(\lambda)\mathbf{f}, \varphi)_{\dot\Omega} + (\gamma^1\nabla\mathcal{A}(\lambda)\mathbf{f}, \nabla\varphi)_{\dot\Omega} = (\mathbf{f} + \mathcal{R}_1(\lambda)\mathbf{f}, \nabla\varphi)_{\dot\Omega} + (\mathcal{R}_2(\lambda)\mathbf{f}, \varphi)_{\dot\Omega} + (\mathcal{R}_3(\lambda)\mathbf{f}, \varphi)_\Gamma \tag{2.33}$$

for any $\varphi \in W_{q',0}^1(\Omega)$ with

$$\begin{aligned}
\mathcal{R}_1(\lambda)\mathbf{f} &= 2\sum_{i=0}^{1}\sum_{j=1}^{K_i} \gamma_j^i(\nabla\zeta_j^i)u_j^i + \gamma_{1,+}(\nabla\zeta_+^2)u_+^2 + \gamma_{1,-}(\nabla\zeta_-^2)u_-^2 \\
\mathcal{R}_2(\lambda)\mathbf{f} &= -\{\sum_{i=0}^{1}\sum_{j=1}^{K_i} \gamma_j^i(\Delta\zeta_j^i)u_j^i + \gamma_{1,+}(\Delta\zeta_+^2)u_+^2 + \gamma_{1,-}(\Delta\zeta_-^2)u_-^2\}, \\
\mathcal{R}_3(\lambda)\mathbf{f} &= -\sum_{j=1}^{K_0}(\gamma_{1,+} - \gamma_{1,-})(u_j^0(\nabla\zeta_j^0)\cdot\mathbf{n})|_\Gamma.
\end{aligned}$$

By Poincarés' inequality,

$$\|\varphi\|_{L_q(\Omega)} \leq C\|\nabla\varphi\|_{L_q(\Omega)} \tag{2.34}$$

for any $\varphi \in W_{q',0}^1(\Omega)$, so that by (2.32)

$$|(\mathcal{R}_2(\lambda)\mathbf{f}, \varphi)_{\dot\Omega}| \leq C|\lambda|^{-1/2}\|\mathbf{f}\|_{L_q(\Omega)}\|\nabla\varphi\|_{L_{q'}(\Omega)}. \tag{2.35}$$

for any $\varphi \in W_{q',0}^1(\Omega)$. By the interpolation inequality for the trace operator and (2.32) we have

$$\|\mathcal{R}_3(\lambda)\mathbf{f}\|_{L_q(\Gamma)} \leq (\sum_{j=1}^{K_0}\|(\nabla\zeta_j^0)u_j^0\|_{L_q(\Omega)}^{1-1/q}\|\nabla((\nabla\zeta_j^0)u_j^0)\|_{L_q(\Omega)}^{1/q}) \leq C|\lambda|^{-\frac{1}{2q'}}\|\mathbf{f}\|_{L_q(\Omega)},$$

which, combined with (2.34), furnishes that

$$|(\mathcal{R}_3(\lambda)\mathbf{f}, \varphi)_\Gamma| \leq C|\lambda|^{-\frac{1}{2q'}}\|\mathbf{f}\|_{L_q(\Omega)}\|\nabla\varphi\|_{L_{q'}(\Omega)}. \tag{2.36}$$

for any $\varphi \in W_{q',0}^1(\Omega)$. By the Hahn-Banach theorem, there exists an operator $\mathcal{R}_4(\lambda) \in \mathcal{L}(L_q(\Omega)^N)$ such that

$$(\mathcal{R}_2(\lambda)\mathbf{f}, \varphi)_{\dot\Omega} + (\mathcal{R}_3(\lambda)\mathbf{f}, \varphi)_\Gamma = (\mathcal{R}_4(\lambda)\mathbf{f}, \nabla\varphi)_{\dot\Omega}$$

for any $\varphi \in W_{q',0}^1(\Omega)$, and moreover by (2.35) and (2.36)

$$\|\mathcal{R}_4(\lambda)\mathbf{f}\|_{L_q(\Omega)} \leq C|\lambda|^{-\frac{1}{2q'}}\|\mathbf{f}\|_{L_q(\Omega)} \tag{2.37}$$

for any $\lambda \in \Sigma_{\vartheta,\lambda_1}$.

By (2.33) we have

$$\lambda(\mathcal{A}(\lambda)\mathbf{f}, \varphi)_{\dot\Omega} + (\gamma^1\nabla\mathcal{A}(\lambda)\mathbf{f}, \nabla\varphi)_{\dot\Omega} = ((I + (\mathcal{R}_1(\lambda) + \mathcal{R}_4(\lambda))\mathbf{f}, \nabla\varphi)_\Omega \quad \text{for any } \varphi \in W_{q',0}^1(\Omega),$$

where I denotes the identity operator from $L_q(\Omega)$ onto itself. By (2.32), we have

$$\|\mathcal{R}_1(\lambda)\mathbf{f}\|_{L_q(\Omega)} \leq C|\lambda|^{1/2}\|\mathbf{f}\|_{L_q(\Omega)},$$

which, combined with (2.37), furnishes that

$$\|\mathcal{R}_1(\lambda) + \mathcal{R}_4(\lambda)\|_{\mathcal{L}(L_q(\Omega)^N)} \leq 1/2$$

for any $\lambda \in \Sigma_{\vartheta,\lambda_2}$ with some large constant $\lambda_2 \geq \lambda_1$. Thus, $v = \mathcal{A}(\lambda)(I + \mathcal{R}_1(\lambda) + \mathcal{R}_4(\lambda))^{-1}\mathbf{f}$ solves problem (2.28) uniquely, which satisfies the estimate:

$$\|(\lambda^{1/2}v, \nabla v)\|_{L_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}.$$

This completes the proof of Theorem 2.8. □

Finally, we give a

Proof of Theorem 1.5. For any $\mathbf{f}$ and $\mathbf{g} \in L_q(\Omega)^N$, we have $(\mathbf{f}, \nabla\varphi)_\Omega = (\mathbf{g}, \nabla\varphi)_\Omega$ for any $\varphi \in W^1_{q',0}(\Omega)$ provided that $\operatorname{div}(\mathbf{f} - \mathbf{g}) = 0$ in Ω, so that we consider the quotient space $\dot{L}_q(\Omega) = L_q(\Omega)^N \setminus PL_q(\Omega)$, where $PL_q(\Omega) = \{\mathbf{f} \in L_q(\Omega)^N \mid \operatorname{div}\mathbf{f} = 0 \text{ in } \Omega\ \}$. By the Helmholtz decomposition, for any $\mathbf{f} \in L_q(\Omega)^N$, there exist $\mathbf{g} \in L_q(\Omega)^N$ and $\psi \in W^1_q(\Omega)$ uniquely such that $\mathbf{f} = \mathbf{g} + \nabla\psi$ and $\operatorname{div}\mathbf{g} = 0$ in Ω. Here, $\psi \in W^1_q(\Omega)$ is a unique solution to the weak Neumann problem:

$$(\nabla\psi, \nabla\varphi)_\Omega = (\mathbf{f}, \nabla\varphi)_\Omega \quad \text{for any } \varphi \in W^1_{q'}(\Omega).$$

In other words, ψ is a weak solution to the Neumann problem:

$$\Delta\psi = \operatorname{div}\mathbf{f} \quad \text{in } \Omega, \quad \mathbf{n}\cdot\nabla\psi = \mathbf{n}_-\cdot\mathbf{f} \quad \text{on } \Gamma_-.$$

For $\mathbf{f} \in L_q(\Omega)^N$, let $[\mathbf{f}]$ be the representation of $\mathbf{f}$ in $\dot{L}_q(\Omega)$, and then $[\mathbf{f}] = \nabla\psi$. If $\operatorname{div}\mathbf{f} = 0$, then $[\mathbf{f}] = 0$. Moreover, by the regularity theorem of the solutions to the Neumann problem, $[\mathbf{f}] \in W^1_q(\Omega)^N$ provided that $\mathbf{f} \in W^1_q(\Omega)^N$.

Under these preparation, we prove Theorem 1.5. Let λ be a large positive number such that $\lambda > \lambda_1$, where λ_1 is the number given in Theorem 2.8, and then by Theorem 2.8 for any $\mathbf{f} \in \dot{L}_q(\Omega)^N$, problem (2.28) admits a unique solution $u \in W^1_{q,0}(\Omega)$ satisfying the estimate: $\|u\|_{W^1_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$. Let $\mathcal{R}$ be an operator $\in \mathcal{L}(\dot{L}_q(\Omega), W^1_{q,0}(\Omega))$ defined by $\mathcal{R}\mathbf{f} = u$. We look for a solution (1.9) of the form: $u = \mathcal{R}\mathbf{g}$ with $\mathbf{g} \in \dot{L}_q(\Omega)$, and then

$$(\gamma_1\nabla u, \nabla\varphi)_{\dot{\Omega}} = (\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} + (\lambda u, \varphi)_{\dot{\Omega}} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega). \tag{2.38}$$

Since Ω is a bounded domain whose boundary is a hyper-surface of $W^{2-1/r}_r$ class, there exists a $h \in W^2_q(\Omega)$ solving the Dirichlet problem:

$$\Delta h = -\lambda u \quad \text{in } \Omega, \quad h|_{\Gamma_-} = 0$$

uniquely and satisfying the estimate:

$$\|h\|_{W^2_q(\Omega)} \leq C\|\lambda u\|_{L_q(\Omega)} \leq C\|\mathbf{g}\|_{L_q(\Omega)}.$$

Let $\mathcal{S}$ be an operator defined by $\mathcal{S}\mathbf{g} = [\nabla h]$, and then

$$(\lambda u, \varphi)_{\dot{\Omega}} = -(\Delta h, \varphi)_{\dot{\Omega}} = (\nabla\mathcal{S}\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega),$$

and therefore the equation (2.38) is transformed to

$$(\gamma_1\nabla\mathcal{R}\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} = ((I + \mathcal{S})\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega), \tag{2.39}$$

where I is the identity operator in $\mathcal{L}(\dot{L}_q(\Omega))$. Since $\nabla h \in W^1_q(\Omega)^N$, $\mathcal{S}\mathbf{g} \in W^1_q(\Omega)^N$. Moreover, Ω is bounded, so that the Rellich compactness theorem yields that $\mathcal{S}$ is a compact operator from $\dot{L}_q(\Omega)$ into itself. Thus, in view of the Riesz-Schauder theorem, especially the Fredholm alternative principle, in order to prove that the $I+\mathcal{S}$ is invertible, it suffices to prove that the kernel of $I+\mathcal{S}$ is trivial. Thus, let $\mathbf{g} \in \dot{L}_q(\Omega)$ satisfy $(I+\mathcal{S})\mathbf{g} = 0$, and then by (2.39)

$$(\gamma_1 \nabla \mathcal{R}\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} = 0 \quad \text{for any } \varphi \in W^1_{q',0}(\Omega). \tag{2.40}$$

The task is to prove $\mathbf{g} = 0$ in $\dot{L}_q(\Omega)$, that is, $\operatorname{div}\mathbf{g} = 0$. First, we consider the case $2 \leq q < \infty$. Since $\mathcal{R}\mathbf{g} \in W^1_{q,0}(\Omega) \subset W^1_{2,0}(\Omega) \subset W^1_{q',0}(\Omega)$, setting $\varphi = \mathcal{R}\mathbf{g}$ in (2.40), we have $\|\gamma_1\nabla\mathcal{R}\mathbf{g}\|_{L_2(\dot{\Omega})} = 0$, that is, $\gamma_1\nabla\mathcal{R}\mathbf{g} = 0$ in $\dot{\Omega}$, which yields that $\mathcal{R}\mathbf{g}$ is constant in $\Omega_\pm$. Thus, $\mathcal{R}\mathbf{g}(x) = c_\pm$ for $x \in \Omega_\pm$ with some constants $c_\pm$. But, $\mathcal{R}\mathbf{g} \in W^1_{q,0}(\Omega)$, which furnishes that $[[\mathcal{R}\mathbf{g}]] = 0$ and $\mathcal{R}\mathbf{g}|_{\Gamma_-} = 0$. Thus, $c_+ = c_- = 0$, which implies that $\mathcal{R}\mathbf{g} = 0$. So, we have $\mathbf{g} = 0$ in $\dot{L}_q(\Omega)$. In fact, we have

$$0 = \lambda(\mathcal{R}\mathbf{g}, \varphi)_{\dot{\Omega}} + (\gamma_1\nabla\mathcal{R}\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} = (\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega),$$

which furnishes that $\operatorname{div}\mathbf{g} = 0$. Thus, the inverse operator $(I+\mathcal{S})^{-1}$ of $I+\mathcal{S}$ exists in $\mathcal{L}(\dot{L}_q(\Omega))$. In view of (2.39), we see that for any $\mathbf{f} \in \dot{L}_q(\Omega)$, $u = \mathcal{R}(I+\mathcal{S})^{-1}\mathbf{f} \in W^1_{q,0}(\Omega)$ is a solution of (1.9) satisfying the estimate:

$$\|u\|_{W^1_q(\Omega)} \leq C\|\mathbf{f}\|_{\dot{L}_q(\Omega)}.$$

Next, we consider the case where $1 < q < 2$. In this case, $2 < q' < \infty$, so that for any $\mathbf{h} \in L_{q'}(\Omega)^N$ there exists a $\varphi \in W^1_{q',0}(\Omega)$ which solves the variational equation:

$$(\gamma_1\nabla\varphi, \nabla\psi)_{\dot{\Omega}} = (\mathbf{h}, \nabla\psi)_{\dot{\Omega}} \quad \text{for any } \psi \in W^1_{q,0}(\Omega).$$

Setting $\psi = \mathcal{R}\mathbf{g}$, by (2.40) we have

$$0 = (\gamma_1\nabla\mathcal{R}\mathbf{g}, \nabla\varphi)_{\dot{\Omega}} = (\mathbf{h}, \nabla\mathcal{R}\mathbf{g})_{\dot{\Omega}}.$$

The arbitrary choice of $\mathbf{f}$ implies that $\nabla\mathcal{R}\mathbf{g} = 0$ in $\dot{\Omega}$, so that $\mathcal{R}\mathbf{g} = c_\pm$ in $\Omega_\pm$ with some constants $c_\pm$. But, $\mathcal{R}\mathbf{g} \in W^1_{q,0}(\Omega)$, so that $\mathcal{R}\mathbf{g} = 0$. Therefore, employing the same argumentation as above, we see that $(I+\mathcal{S})^{-1}$ exists in $\mathcal{L}(\dot{L}_q(\Omega))$, which furnishes the existence of a solution $u \in W^1_{q,0}(\Omega)$ of problem (1.9) for any $\mathbf{f} \in L_q(\Omega)^N$ satisfying the estimate: $\|u\|_{W^1_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$.

The uniqueness of solutions follows from the existence of solutions of the dual problem. In fact, let $u \in W^1_{q,0}(\Omega)$ satisfy the homogeneous equation:

$$(\gamma_1\nabla u, \nabla\varphi)_{\dot{\Omega}} = 0 \quad \text{for any } \varphi \in W^1_{q',0}(\Omega).$$

For any $\mathbf{f} \in L_{q'}(\Omega)$, let $\varphi \in W^1_{q',0}(\Omega)$ be a solution of the variational problem:

$$(\gamma_1\nabla\varphi, \nabla\psi)_{\dot{\Omega}} = (\mathbf{f}, \nabla\psi)_{\dot{\Omega}} \quad \text{for any } \psi \in W^1_{q',0}(\Omega).$$

Setting $\psi = u$, we have

$$0 = (\gamma_1\nabla\varphi, \nabla u)_{\dot{\Omega}} = (\mathbf{f}, \nabla u)_{\dot{\Omega}}.$$

The arbitrary choice of $\mathbf{f}$ implies that $\nabla u = 0$ in Ω, so that u is a constant in Ω. But, $u \in W^1_{q,0}(\Omega)$, so that $u = 0$. This proves the uniqueness of solutions, so that we have completed the proof of Theorem 1.5.

3 Stokes equations and reduced Stokes equations

Let λ be a complex number. In this section, we consider the following generalized resolvent problem of the Stokes equations:

$$\begin{cases} \operatorname{div}\mathbf{v} = f & \text{in } \dot{\Omega}, \\ \lambda\mathbf{v} - \gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{v}) - \nabla\mathfrak{p}) = \mathbf{g} & \text{in } \dot{\Omega}, \\ [[(\mathbf{S}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}]] = [[\mathbf{h}]], \quad [[\mathbf{v}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{v}_-) - \mathfrak{p}_-\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-} & \text{on } \Gamma_-. \end{cases} \tag{3.1}$$

Let $K(\mathbf{u})$ be a solution to the variational problem:

$$(\gamma_1\nabla K(\mathbf{u}),\nabla\varphi)_{\dot\Omega}=(\gamma_1\mathrm{Div}\,\mathbf{S}(\mathbf{u})-\nabla\mathrm{div}\,\mathbf{u},\nabla\varphi)_{\dot\Omega}\quad\text{for any }\varphi\in W^1_{q',0}(\Omega),\tag{3.2}$$

subject to

$$[[K(\mathbf{u})]]=[[<\mathbf{S}(\mathbf{u})\mathbf{n},\mathbf{n}>+\mathrm{div}\,\mathbf{u}]],\quad K(\mathbf{u})|_{\Gamma_-}=(<\mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_-,\mathbf{n}_->+\mathrm{div}\,\mathbf{u}_-)|_{\Gamma_-}.$$

In view of Theorem 1.5, for any $\mathbf{u}\in W^2_q(\dot\Omega)^N\cap W^1_q(\Omega)^N$ problem (3.2) admits a unique solution $K(\mathbf{u})\in W^1_q(\dot\Omega)$ possessing the estimate:

$$\|K(\mathbf{u})\|_{W^1_q(\dot\Omega)}\le C\|\mathbf{u}\|_{W^2_q(\dot\Omega)}.\tag{3.3}$$

We also consider the following equations:

$$\begin{cases}\lambda\mathbf{u}-\gamma_1(\mathrm{Div}\,\mathbf{S}(\mathbf{u})-\nabla K(\mathbf{u}))=\mathbf{g} & \text{in }\dot\Omega,\\ [[(\mathbf{S}(\mathbf{u})-K(\mathbf{u})\mathbf{I})\mathbf{n}]]=[[\mathbf{h}]],\quad [[\mathbf{u}]]=0 & \text{on }\Gamma,\\ (\mathbf{S}_-(\mathbf{u}_-)-K(\mathbf{u})\mathbf{I})\mathbf{n}_-|_{\Gamma_-}=\mathbf{h}_-|_{\Gamma_-} & \text{on }\Gamma_-.\end{cases}\tag{3.4}$$

Problem (3.1) is called the Stokes equations, while problem (3.4) is called the reduced Stokes equations. We discuss the equivalence of both problems (3.1) and (3.4).

First, we assume that problem (3.1) is solvable. Let $J_q(\Omega)$ be the set of all N-vector of functions $\mathbf{g}$ satisfying the condition:

$$(\mathbf{g},\nabla\varphi)_{\dot\Omega}=0\quad\text{for any }\varphi\in W^1_{q',0}(\Omega).\tag{3.5}$$

Let $\mathbf{g}\in J_q(\Omega)$. Let $\mathbf{v}$ and $\mathfrak{p}$ be solutions of problem (3.1) with $f=0$. Then, noting that $\mathrm{div}\,\mathbf{v}=0$ in $\dot\Omega$ and $[[\mathbf{v}]]=0$, by (3.1), (3.2) and (3.5) we have

$$\begin{aligned}0&=(\lambda\mathbf{v}-\gamma_1(\mathrm{Div}\,\mathbf{S}(\mathbf{u})-\nabla\mathfrak{p}),\nabla\varphi)_{\dot\Omega}\\&=-\lambda(\mathrm{div}\,\mathbf{v},\varphi)_{\dot\Omega}-(\gamma_1\mathrm{Div}\,\mathbf{S}(\mathbf{u})-\nabla\mathrm{div}\,\mathbf{v},\nabla\varphi)_{\dot\Omega}+(\gamma_1\nabla\mathfrak{p},\nabla\varphi)_{\dot\Omega}\\&=(\gamma_1\nabla(\mathfrak{p}-K(\mathbf{u})),\nabla\varphi)_{\dot\Omega}\quad\text{for any }\varphi\in W^1_{q',0}(\Omega).\end{aligned}$$

Moreover,

$$[[\mathfrak{p}]]=[[<\mathbf{S}(\mathbf{v})\mathbf{n},\mathbf{n}>]]=[[K(\mathbf{v})]],\quad \mathfrak{p}|_{\Gamma_-}=<\mu_-\mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_-,\mathbf{n}_->|_{\Gamma_-}=K(\mathbf{v})|_{\Gamma_-},$$

because $\mathrm{div}\,\mathbf{v}=0$ in $\dot\Omega$. Thus, the uniqueness implies that $\mathfrak{p}=K(\mathbf{v})$, which shows that $\mathbf{v}$ is a solution of problem (3.4).

Conversely, we assume that problem (3.4) is uniquely solvable. Moreover, we assume that it does hold the uniqueness of the variational problem:

$$(\lambda w,\varphi)_{\dot\Omega}+(\nabla w,\nabla\varphi)_{\dot\Omega}=(\mathbf{f},\nabla\varphi)_{\dot\Omega}\quad\text{for any }\varphi\in W^1_{q',0}(\Omega),\tag{3.6}$$

subject to $[[w]]=g$ and $w|_{\Gamma_-}=g|_{\Gamma_-}$. Let $f\in W^1_q(\dot\Omega)$, $\mathbf{g}\in L_q(\Omega)^N$, $\mathbf{h}\in W^1_q(\dot\Omega)^N$ and $\mathbf{h}_-\in L_q(\Omega_-)^N$ be given. Let $\theta\in W^1_q(\dot\Omega)$ be a solution to the variational problem:

$$(\gamma_1\nabla\theta,\nabla\varphi)_{\dot\Omega}=-(\mathbf{g},\nabla\varphi)_{\dot\Omega}\quad\text{for any }\varphi\in W^1_{q',0}(\Omega),\tag{3.7}$$

subject to $[[\theta]]=<[[\mathbf{h}]],\mathbf{n}>$ and $\theta|_{\Gamma_-}=<\mathbf{h}_-|_{\Gamma_-},\mathbf{n}_->$. Using this θ, we rewrite (3.1) of the form:

$$\begin{cases}\mathrm{div}\,\mathbf{v}=f_\pm & \text{in }\dot\Omega,\\ \lambda\mathbf{v}-\gamma_1(\mathrm{Div}\,\mathbf{S}(\mathbf{v})-\nabla(\mathfrak{p}+\theta))=\mathbf{g}+\gamma_1\nabla\theta & \text{in }\dot\Omega,\\ [[(\mathbf{S}(\mathbf{v})-(\mathfrak{p}+\theta)\mathbf{I})\mathbf{n}]]=[[\mathbf{h}]]-[[\theta]]\mathbf{n},\quad [[\mathbf{v}]]=0 & \text{on }\Gamma,\\ (\mathbf{S}_-(\mathbf{v}_-)-(\mathfrak{p}+\theta)\mathbf{I})\mathbf{n}_-|_{\Gamma_-}=\mathbf{h}_-|_{\Gamma_-}-\theta|_{\Gamma_-}\mathbf{n}_- & \text{on }\Gamma_-.\end{cases}$$

By (3.7), $\mathbf{g}+\gamma_1\nabla\theta\in J_q(\Omega)$, $<[[\mathbf{h}]],\mathbf{n}>-[[\theta]]=0$ on Γ, and $<\mathbf{h}_-|_{\Gamma_-},\mathbf{n}_->-\theta|_{\Gamma_-}=0$ on Γ_-. Thus, in (3.4) we may assume that

$$\mathbf{g}\in J_q(\Omega),\quad <[[\mathbf{h}]],\mathbf{n}>=0,\quad <\mathbf{h}_-|_{\Gamma_-},\mathbf{n}_->=0.\tag{3.8}$$

Let $\psi \in W_q^1(\dot\Omega)$ be a solution to the variational equation:

$$(\gamma_1\nabla\psi, \nabla\varphi)_{\dot\Omega} = -\{\lambda(f,\varphi)_{\dot\Omega} + (\nabla f, \nabla\varphi)_{\dot\Omega}\} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega), \tag{3.9}$$

subject to $[[\psi]] = [[f]]$ and $\psi|_{\Gamma_-} = f|_{\Gamma_-}$. In fact, since $|\lambda(f,\varphi)_{\dot\Omega}| \leq |\lambda|\|f\|_{L_q(\Omega)}\|\nabla\varphi\|_{L_{q'}(\Omega)}$ for any $\varphi \in W^1_{q',0}(\Omega)$ as follows from Poincarés' inequality, by the Harn-Banach theorem there exists a $\mathbf{G} \in L_q(\Omega)^N$ such that

$$\lambda(f,\varphi)_{\dot\Omega} = (\mathbf{G}, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega)$$

with $\|\mathbf{G}\|_{L_q(\Omega)} \leq C|\lambda|\|f\|_{L_q(\Omega)}$. Let $\mathbf{u} \in W_q^2(\dot\Omega) \cap W_q^1(\Omega)$ be a solution to the reduced Stokes equations:

$$\begin{cases} \lambda\mathbf{u} - \gamma_1(\mathrm{Div}\,\mathbf{S}(\mathbf{u}) - \nabla K(\mathbf{u})) = \mathbf{g} + \gamma_1\nabla\psi & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{u}) - K(\mathbf{u})\mathbf{I})\mathbf{n}]] = [[\mathbf{h}]] - [[f]]\mathbf{n}, \quad [[\mathbf{u}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{u}_-) - K(\mathbf{u})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-} - f|_{\Gamma_-}\mathbf{n}_- & \text{on } \Gamma_-. \end{cases} \tag{3.10}$$

Note that $[[\mathbf{u}]] = 0$ in (3.10). For any $\varphi \in W^1_{q',0}(\Omega)$, by (3.8) and (3.10) we have

$$\begin{aligned}(\gamma_1\nabla\psi, \nabla\varphi)_{\dot\Omega} &= (\lambda\mathbf{u} - \gamma_1(\mathrm{Div}\,\mathbf{S}(\mathbf{u}) - \nabla K(\mathbf{u})), \nabla\varphi)_{\dot\Omega} \\ &= -\{\lambda(\mathrm{div}\,\mathbf{u}, \varphi)_{\dot\Omega} + (\nabla\mathrm{div}\,\mathbf{u}, \nabla\varphi)_{\dot\Omega}\},\end{aligned}$$

which, combined with (3.9), furnishes that

$$\lambda(f - \mathrm{div}\,\mathbf{u}, \varphi)_{\dot\Omega} + (\nabla(f - \mathrm{div}\,\mathbf{u}), \nabla\varphi)_{\dot\Omega} = 0 \quad \text{for any } \varphi \in W^1_{q',0}(\Omega).$$

Moreover, by (3.2), (3.8) and (3.10),

$$[[\mathrm{div}\,\mathbf{u}]] = -[[< \mathbf{S}(\mathbf{u})\mathbf{n}, \mathbf{n} > -K(\mathbf{u})]] = [[f]], \quad \mathrm{div}\,\mathbf{u}|_{\Gamma_-} = -(< \mathbf{S}(\mathbf{u})\mathbf{n}_-, \mathbf{n}_- > -K(\mathbf{u}))|_{\Gamma_-} = f|_{\Gamma_-},$$

and therefore by the uniqueness we have $\mathrm{div}\,\mathbf{u} = f$ in $\dot\Omega$. Since $[[\psi]] = [[f]]$ and $\psi|_{\Gamma_1} = f|_{\Gamma_1}$, $\mathbf{u}$ and $\mathfrak{p} = K(\mathbf{u}) - \psi$ solve problem (3.1).

4 A proof of Theorem 1.4

Concerning the reduced Stokes equations (3.4), we have

Theorem 4.1. *Let $1 < q < \infty$, $N < r < \infty$ and $0 < \epsilon < \pi/2$. Let*

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \mid |\arg\lambda| \leq \pi - \epsilon, \ \ |\lambda| \geq \lambda_0\}$$

for $\lambda_0 > 0$. Assume that $r \geq \max(q, q')$ with $q' = q/(q-1)$ and that Γ and Γ_- are both compact hyper-surfaces of $W_r^{2-1/r}$ class. Then, there exists a constant $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $\mathbf{g} \in L_q(\Omega)^N$, $\mathbf{h} \in W_q^1(\dot\Omega)^N$, and $\mathbf{h}_- \in W_q^1(\Omega_-)$, problem (3.4) admits a unique solution $\mathbf{u} \in W_q^2(\dot\Omega)^N$ possessing the estimate:

$$\|(\lambda\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u})\|_{L_q(\dot\Omega)} \leq C\{\|\mathbf{g}\|_{L_q(\dot\Omega)} + \|(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})\|_{L_q(\dot\Omega)} + \|(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}_-)\|_{L_q(\Omega_-}\}. \tag{4.1}$$

Theorem 4.1 was proved by Saito and Sri [5]. In fact, the interface problem in the whole space with interface $x_N = 0$ and the one phase problem in the half-space were treated in Shibata and Shimizu [3, 4]. And then, according the the method due to Shibata [2], we can construct a parametrix by using the partition of unity, so that arguing similarly to Shibata [2] we can prove Theorem 4.1.

Moreover, if $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{h}_-$ in Theorem 4.1 satisfy the orthogonal condition (1.6), then $\mathbf{u}$ satisfies the orthogonal condition (1.7), that is

$$(\gamma_1^{-1}\mathbf{u}, \mathbf{p}_\ell)_{\dot\Omega} = 0 \quad \text{for all } \ell = 1, \ldots, M. \tag{4.2}$$

In fact, multiplyingthe first equation in (3.4) by $\gamma_1^{-1}\mathbf{u}$ and using divergence theorem of Gauß, we have

$$(\gamma_1^{-1}\mathbf{g}, \mathbf{p}_\ell)_{\dot\Omega} = \lambda(\gamma_1^{-1}, \mathbf{u}, \mathbf{p}_\ell)_{\dot\Omega} - (\mathrm{Div}\,(\mathbf{S}(\mathbf{u}) - K(\mathbf{u})\mathbf{I}), \mathbf{p}_\ell)_{\dot\Omega}$$

$$= \lambda(\gamma_1^{-1}, \mathbf{u}, \mathbf{p}_\ell)_{\dot\Omega} - ([[\mathbf{h}]], \mathbf{p}_\ell)_\Gamma - (\mathbf{h}_-, \mathbf{p}_\ell)_\Gamma + \frac{1}{2}(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{p}_\ell))_{\dot\Omega} - (K(\mathbf{u}), \operatorname{div}\mathbf{p}_\ell)_{\dot\Omega}.$$

Since $\mathbf{D}(\mathbf{p}_\ell) = 0$ and $\operatorname{div}\mathbf{p}_\ell = 0$, the condition (1.6) yields that $\lambda(\gamma_1^{-1}, \mathbf{u}, \mathbf{p}_\ell)_{\dot\Omega} = 0$. Since $\lambda \neq 0$, $\mathbf{u}$ satisfies the orthogonal condition (4.2).

¿From now on, we prove Theorem 1.4. For this purpose, first we consider the reduced Soktes equations:

$$\begin{cases} -\gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{v}) - \nabla K(\mathbf{v})) = \mathbf{g} & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{v}) - K(\mathbf{v})\mathbf{I})\mathbf{n}]] = [[\mathbf{h}]], \quad [[\mathbf{v}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{v}_-) - K(\mathbf{v})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = \mathbf{h}_-|_{\Gamma_-} & \text{on } \Gamma_-. \end{cases} \tag{4.3}$$

By Theorem 4.1, there exists a large $\lambda_1 > 0$ such that problem (3.4) with $\lambda = \lambda_1$ admits a unique solution $\mathbf{u} \in W_q^2(\dot\Omega)$ possessing the estimate:

$$\|\mathbf{u}\|_{W_q^2(\dot\Omega)} \leq C\{\|\mathbf{g}\|_{L_q(\dot\Omega)} + \|\mathbf{h}\|_{W_q^1(\dot\Omega)} + \|\mathbf{h}_-\|_{W_q^1(\Omega_-)}\}. \tag{4.4}$$

Here and in the following $\lambda_1 > 0$ is a fixed positive constant. We look for a solution $\mathbf{v}$ of the equations (4.3) of the form $\mathbf{v} = \mathbf{u} + \mathbf{w}$, and then $\mathbf{w}$ satisfies the equations:

$$\begin{cases} -\gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{w}) - \nabla K(\mathbf{w})) = \mathbf{f} & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{w}) - K(\mathbf{w})\mathbf{I})\mathbf{n}]] = 0, \quad [[\mathbf{w}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{w}_-) - K(\mathbf{w})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0 & \text{on } \Gamma_-. \end{cases} \tag{4.5}$$

witn $\mathbf{f} = -\lambda_1\mathbf{u}$. Moreover, by (4.2), $\mathbf{f}$ satisfies the orthogonal condition:

$$(\gamma_1^{-1}\mathbf{f}, \mathbf{p}_\ell)_{\dot\Omega} = 0 \quad \text{for all } \ell = 1, \ldots, M. \tag{4.6}$$

In view of Theorem 4.1 again, there exists a unique $\mathbf{z} \in W_q^2(\dot\Omega)$ which solves the equations:

$$\begin{cases} \lambda_1\mathbf{z} - \gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{z}) - \nabla K(\mathbf{z})) = \mathbf{f} & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{z}) - K(\mathbf{z})\mathbf{I})\mathbf{n}]] = 0, \quad [[\mathbf{z}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{z}_-) - K(\mathbf{z})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0 & \text{on } \Gamma_-. \end{cases} \tag{4.7}$$

Moreover, $\mathbf{f}$ satisfies the orthogonal condition (4.6), so that $\mathbf{z}$ also satisfies the orthgonal condition (4.6). Let $\mathcal{R}$ be an operator defined by $\mathcal{R}\mathbf{f} = \mathbf{z}$. Then, we have

$$\begin{aligned} &\|\mathcal{R}\mathbf{f}\|_{W_q^2(\dot\Omega)} \leq C\|\mathbf{f}\|_{L_q(\dot\Omega)}, \\ &(\gamma_1^{-1}\mathcal{R}\mathbf{f}, \mathbf{p}_\ell)_{\dot\Omega} = 0 \quad \text{for all } \ell = 1, \ldots, M. \end{aligned} \tag{4.8}$$

We look for a solution of (4.5) of the form $\mathbf{z} = \mathcal{R}\mathbf{g}$. Inserting this formula into (4.5) and using (4.7), we have

$$\begin{cases} -\gamma_1(\operatorname{Div}\mathbf{S}(\mathcal{R}\mathbf{g}) - \nabla K(\mathcal{R}\mathbf{g})) = \mathbf{g} - \lambda_1\mathcal{R}\mathbf{g} & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathcal{R}\mathbf{g}) - K(\mathcal{R}\mathbf{g})\mathbf{I})\mathbf{n}]] = 0, \quad [[\mathcal{R}\mathbf{g}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-((\mathcal{R}\mathbf{g})_-) - K(\mathcal{R}\mathbf{g})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0 & \text{on } \Gamma_-. \end{cases} \tag{4.9}$$

Since $\mathcal{R}\mathbf{g} \in W_q^2(\dot\Omega)^N$ and since $[[\mathcal{R}\mathbf{g}]] = 0$, $\mathcal{R}\mathbf{g} \in W_q^1(\Omega)$. Moreover, Ω is bounded, so that the Rellich compactness theorem yields that $\mathcal{R}$ is a compact operator from $L_q(\dot\Omega)^N$ into itself. Thus, to prove that $(I - \lambda_1\mathcal{R})$ is invertible, it is sufficient to prove that the kernel of $I - \lambda_1\mathcal{R}$ is trivial, where I denotes the identity operator of $\mathcal{L}(L_q(\dot\Omega)^N)$. Let $\mathbf{g}$ be an element in $L_q(\dot\Omega)^N$ such that $(I - \lambda_1\mathcal{R})\mathbf{g} = 0$. The task is to prove that $\mathbf{g} = 0$. Let $\mathbf{u} = \mathcal{R}\mathbf{g}$, and then by (4.9), $\mathbf{u} \in W_q^2(\dot\Omega)$ satisfies the equations:

$$\begin{cases} -\gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{u}) - \nabla K(\mathbf{u})) = 0 & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{u}) - K(\mathbf{u})\mathbf{I})\mathbf{n}]] = 0, \quad [[\mathbf{u}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-((\mathbf{u}_-) - K(\mathbf{u})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0 & \text{on } \Gamma_-. \end{cases} \tag{4.10}$$

As was observed in Sect. 3, $\mathbf{u}$ satisifies the divergence free condition:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \dot\Omega. \tag{4.11}$$

In fact, for any $\varphi \in W^1_{q',0}(\Omega)$ by (4.10) with $\gamma_1 = 1$ and (3.2),

$$0 = (\operatorname{Div} \mathbf{S}(\mathbf{u}) - \nabla K(\mathbf{u})), \nabla\varphi)_{\dot\Omega} = (\nabla \operatorname{div} \mathbf{u}, \nabla\varphi)_{\dot\Omega},$$

subject to $[[\operatorname{div} \mathbf{u}]] = [[< \mathbf{S}(\mathbf{u})\mathbf{n}, \mathbf{n} > -K(\mathbf{u})]] = 0$ and $\operatorname{div} \mathbf{u}|_{\Gamma_-} = (< \mathbf{S}_-(\mathbf{u}_-)\mathbf{n}_-, \mathbf{n}_- > -K(\mathbf{u}))|_{\Gamma_-} = 0$. Thus, the uniqueness implies (4.11).

First, we consider the case where $2 \le q < \infty$. In this case, $\mathbf{u} \in W^2_q(\dot\Omega)^N \subset W^2_2(\dot\Omega)^N$, so that multiplying the first equation in (4.10) by $\mathbf{u}$ and using the divergence theorem of Gauß, the interface condition and the boundary condition in (4.10) and (4.11), we have

$$\frac{\mu_+}{2}\|\mathbf{D}(\mathbf{u})\|^2_{L_2(\Omega_+)} + \frac{\mu_-}{2}\|\mathbf{D}(\mathbf{u})\|^2_{L_2(\Omega_-)} = 0. \tag{4.12}$$

But, $\mathbf{u}$ satisfies (2.40) and $\{\mathbf{p}_\ell\}_{\ell=,1}^M$ is the orthonormal basis of $\mathcal{R}_d$ with respect to the inner-product $[\mathbf{u}, \mathbf{v}] := (\gamma_1^{-1}\mathbf{u}, \mathbf{v})_{\dot\Omega}$, so that $\mathbf{u} = \mathcal{R}\mathbf{g} = 0$. Since $\mathcal{R}\mathbf{g}$ satisfies (4.9), we have $\mathbf{g} = 0$, which furnishes that $(I - \lambda_1\mathcal{R})^{-1} \in \mathcal{L}(L_q(\dot\Omega)^N)$. Thus, $\mathbf{w} = \mathcal{R}(I - \lambda_1\mathcal{R})^{-1}\mathbf{f}$ solves the equations (4.5). Moreover, by (4.8) $\mathbf{w}$ satisifes

$$\begin{aligned} &\|\mathbf{w}\|_{W^2_q(\dot\Omega)} \le C\|\mathbf{f}\|_{L_q(\dot\Omega)}, \\ &(\gamma_1^{-1}\mathbf{w}, \mathbf{p}_\ell)_{\dot\Omega} = 0 \quad \text{for all } \ell = 1, \ldots, M. \end{aligned} \tag{4.13}$$

Next, we consider the case where $1 < q < 2$. We consider the problem

$$\begin{cases} -(\operatorname{Div} \mathbf{S}(\mathbf{z}) - \nabla K'(\mathbf{z})) = \mathbf{g} & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{z}) - K'(\mathbf{z})\mathbf{I})\mathbf{n}]] = 0, \quad [[\mathbf{z}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{z}_-) - K'(\mathbf{z})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = 0 & \text{on } \Gamma_-, \end{cases} \tag{4.14}$$

where $K'(\mathbf{z})$ is a solution to the variational equation:

$$(\nabla K'(\mathbf{z}), \nabla\varphi)_{\dot\Omega} = (\operatorname{Div} \mathbf{S}(\mathbf{z}) - \nabla \operatorname{div} \mathbf{z}, \nabla\varphi)_\Omega \quad \text{for any } \varphi \in W^1_{q,0}(\Omega),$$

subject to

$$[[K'(\mathbf{z})]] = [[< \mathbf{S}(\mathbf{z})\mathbf{n}, \mathbf{n} > + \operatorname{div} \mathbf{z}]], \quad K'(\mathbf{z})|_{\Gamma_-} = (< \mathbf{S}_-(\mathbf{z}_-)\mathbf{n}_-, \mathbf{n}_- > + \operatorname{div} \mathbf{z}_-)|_{\Gamma_-}.$$

Employing the same argument as above, we may assume that problem (4.14) is uniquely solvable for $2 \le q' < \infty$. Thus, for any $\mathbf{g} \in J_{q'}(\Omega)$, let $\mathbf{z} \in W^2_{q'}(\dot\Omega)^N$ be a solution of (4.14). As was seen in Sect. 3, $\mathbf{z} \in J_{q'}(\Omega)$. Noting $\mathbf{u}$ satisfies the first equation of (4.10) with $\gamma_1 = 1$ and using (4.14) and the divergence theorem of Gauß, we have

$$\begin{aligned} (\mathbf{u}, \mathbf{g})_{\dot\Omega} &= -(\mathbf{u}, \operatorname{Div} \mathbf{S}(\mathbf{z}) - \nabla K'(\mathbf{z}))_{\dot\Omega} = \frac{\mu_+}{2}(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{z}))_{\Omega_+} + \frac{\mu_-}{2}(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{z}))_{\Omega_-}, \\ 0 &= (\operatorname{Div} \mathbf{S}(\mathbf{u}) - \nabla K(\mathbf{u}), \mathbf{z})_{\dot\Omega} = \frac{\mu_+}{2}(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{z}))_{\Omega_+} + \frac{\mu_-}{2}(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{z}))_{\Omega_-}, \end{aligned}$$

and therefore, $(\mathbf{u}, \mathbf{g})_{\dot\Omega} = 0$. For any $\mathbf{f} \in L_{q'}(\dot\Omega)^N$, let $\psi \in W^1_{q',0}(\Omega)$ be a solution to the variational problem:

$$(\nabla\psi, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q,0}(\Omega).$$

Since $\mathbf{f} - \nabla\psi \in J_q(\Omega)$, we have

$$(\mathbf{u}, \mathbf{f})_{\dot\Omega} = (\mathbf{u}, \mathbf{f} - \nabla\psi)_{\dot\Omega} + (\mathbf{u}, \nabla\psi)_{\dot\Omega} = (\mathbf{u}, \nabla\psi)_{\dot\Omega}.$$

Since $(\mathbf{u}, \nabla\psi)_{\dot\Omega} = 0$ as follows from (4.11) and $[[\mathbf{u}]] = 0$, we have $(\mathbf{u}, \mathbf{f})_{\dot\Omega} = 0$, which, combined with the arbitrary choice of $\mathbf{f}$, furnishes that $\mathbf{u} = 0$. Thus, employing the same argument as above, $(I - \lambda_1\mathcal{R})^{-1}$ exists in $\mathcal{L}(\mathrm{L}_q(\dot\Omega)^N)$, so that $\mathbf{w} = \mathcal{R}(I - \lambda_1\mathcal{R})^{-1}\mathbf{f}$ solves the equations (4.5) and satisfies (4.13). The uniqueness of solutions follows from the existence of solutions of the dual problem, so that we have proved

Theorem 4.2. *Let $1 < q < \infty$, $N < r < \infty$ and $0 < \epsilon < \pi/2$. Assume that $r \geq \max(q, q')$ with $q' = q/(q-1)$ and that Γ and Γ_- are both compact hyper-surfaces of $W_r^{2-1/r}$ class. Then, for any $\mathbf{g} \in L_q(\dot\Omega)$, $\mathbf{h} \in W_q^1(\dot\Omega)$ and $\mathbf{h}_- \in W_q^1(\Omega_-)$ satisfying the orthogonal condition (1.6), problem (4.3) admits a unique solution $\mathbf{v} \in W_q^2(\dot\Omega)$ satisfying the orthogonal condition (1.4) and the estimate:*

$$\|\mathbf{v}\|_{W_q^2(\dot\Omega)} \leq C\{\|\mathbf{g}\|_{L_q(\dot\Omega)} + \|(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})\|_{L_q(\dot\Omega)} + \|(\lambda^{1/2}\mathbf{h}_-, \nabla\mathbf{h}_-)\|_{L_q(\Omega_-}\}. \tag{4.15}$$

As was discussed in Sect. 3, Theorem 1.4 can be proved by using Theorem 4.2. In fact, let $f \in W_q^1(\dot\Omega)$, $\mathbf{g} \in L_q(\dot\Omega)^N$, $\mathbf{h} \in W_q^1(\dot\Omega)$ and $\mathbf{h}_- \in W_q^1(\Omega_-)^N$ be given and satisfy the orthogonal condition (1.6). Let $\theta \in W_q^1(\dot\Omega)$ be a solution to the variational equations:

$$(\gamma_1\nabla\theta, \nabla\varphi)_{\dot\Omega} = -(\mathbf{g}, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega),$$

subject to $[[\theta]] =< [[\mathbf{h}]], \mathbf{n} >$ on Γ and $\theta|_{\Gamma_-} =< \mathbf{h}_-, \mathbf{n}_- > |_{\Gamma_-}$ on Γ_-. By Theorem 1.5, such θ exists and satisfies the estimate:

$$\|\theta\|_{W_q^1(\dot\Omega)} \leq C(\|\mathbf{g}\|_{L_q(\dot\Omega)} + \|\mathbf{h}\|_{W_q^1(\dot\Omega)} + \|\mathbf{h}_-\|_{W_q^1(\Omega_-)}).$$

Using this θ, we rewrite problem (1.5) with $\gamma_0 = 1$ of the form:

$$\begin{cases} \operatorname{div}\mathbf{v} = f & \text{in } \dot\Omega, \\ -\gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{v}) - \nabla(\mathfrak{p}+\theta)) = \mathbf{g} + \gamma_1\nabla\theta & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{v}) - (\mathfrak{p}+\theta)\mathbf{I})\mathbf{n}]] = [[\mathbf{h}]] - < [[\mathbf{h}]], \mathbf{n} > \mathbf{n}, \quad [[\mathbf{v}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{v}_-) - K(\mathbf{v})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = (\mathbf{h}_- - < \mathbf{h}_-, \mathbf{n}_- > \mathbf{n}_-)|_{\Gamma_-} & \text{on } \Gamma_-. \end{cases} \tag{4.16}$$

Note that $\mathbf{g} + \gamma_1\nabla\theta \in J_q(\Omega)$, $< [[\mathbf{h}]] - < [[\mathbf{h}]], \mathbf{n} > \mathbf{n}, \mathbf{n} >= 0$ and $< \mathbf{h}_- - < \mathbf{h}_-, \mathbf{n}_- > \mathbf{n}_-, \mathbf{n}_- >= 0$.

Next, let $\psi \in W_q^1(\dot\Omega)$ be a solution to the variational equation:

$$(\gamma_1\nabla\psi, \nabla\varphi)_{\dot\Omega} = -(\nabla f, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q',0}(\Omega), \tag{4.17}$$

subject to $[[\psi]] = [[f]]$ on Γ and $\psi|_{\Gamma_-} = f|_{\Gamma_-}$ on Γ_-. Let $\mathbf{v} \in W_q^2(\dot\Omega)^N$ be a solution to the reduced Stokes equations:

$$\begin{cases} -\gamma_1(\operatorname{Div}\mathbf{S}(\mathbf{v}) - \nabla K(\mathbf{v})) = \mathbf{g} + \gamma_1\nabla\theta + \gamma_1\nabla\psi & \text{in } \dot\Omega, \\ [[(\mathbf{S}(\mathbf{v}) - K(\mathbf{v})\mathbf{I})\mathbf{n}]] = [[\mathbf{h}]] - < [[\mathbf{h}]], \mathbf{n} > \mathbf{n} - [[f]]\mathbf{n}, \quad [[\mathbf{v}]] = 0 & \text{on } \Gamma, \\ (\mathbf{S}_-(\mathbf{v}_-) - K(\mathbf{v})\mathbf{I})\mathbf{n}_-|_{\Gamma_-} = (\mathbf{h}_- - < \mathbf{h}_-, \mathbf{n}_- > \mathbf{n}_- - f\mathbf{n}_-)|_{\Gamma_-} & \text{on } \Gamma_-. \end{cases} \tag{4.18}$$

Since $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{h}_-$ satisfy the orthogonal condition (1.6), we have

$$\begin{aligned} &(\gamma_1^{-1}(\mathbf{g} + \gamma_1\nabla\theta + \gamma_1\nabla\psi), \mathbf{p}_\ell)_{\dot\Omega} + ([[\mathbf{h}]] - < [[\mathbf{h}]], \mathbf{n} > \mathbf{n} - [[f]]\mathbf{n}, \mathbf{p}_\ell)_\Gamma \\ &\quad + (\mathbf{h}_- - < \mathbf{h}_-, \mathbf{n}_- > \mathbf{n}_- - f\mathbf{n}_-, \mathbf{p}_\ell)_{\Gamma_-} \\ &= ([[\theta]]\mathbf{n}, \mathbf{p}_\ell)_\Gamma + (\theta\mathbf{n}_-, \mathbf{p}_\ell)_{\Gamma_-} + ([[\psi]]\mathbf{n}, \mathbf{p}_\ell)_\Gamma + (\psi\mathbf{n}_-, \mathbf{p}_\ell)_{\Gamma_-} + (\theta + \psi, \operatorname{div}\mathbf{p}_\ell)_{\dot\Omega} \\ &\quad - (< [[\mathbf{h}]], \mathbf{n} >, \mathbf{p}_\ell)_\Gamma - (< \mathbf{h}_-, \mathbf{n}_- > \mathbf{n}_-, \mathbf{p}_\ell)_{\Gamma_-} - ([[f]]\mathbf{n}, \mathbf{p}_\ell)_\Gamma - (f\mathbf{n}_-, \mathbf{p}_\ell)_{\Gamma_-} \\ &= 0. \end{aligned}$$

Thus, by Theorem 4.2, problem (4.18) admits a unique solution $\mathbf{v} \in W_q^2(\dot\Omega)^N$ satisfying the orthogonal condition (1.7) and the estimate:

$$\|\mathbf{v}\|_{W_q^2(\dot\Omega)} \leq C\{\|\mathbf{g}\|_{L_q(\dot\Omega)} + \|\mathbf{h}\|_{W_q^1(\dot\Omega)} + \|\mathbf{h}_-\|_{W_q^1(\Omega_-)} + \|f\|_{W_q^1(\dot\Omega)}\},$$

where we have used the estimate:

$$\|\nabla\theta\|_{W_q^1(\dot\Omega)} + \|\nabla\psi\|_{W_q^1(\dot\Omega)} \leq C\{\|\mathbf{g}\|_{L_q(\dot\Omega)} + \|\mathbf{h}\|_{W_q^1(\dot\Omega)} + \|\mathbf{h}_-\|_{W_q^1(\Omega_-)} + \|f\|_{W_q^1(\dot\Omega)}\}.$$

Moreover, for any $\varphi \in W^1_{q',0}(\Omega)$, by (4.18) we have

$$
\begin{aligned}
(\gamma_1\nabla\psi, \nabla\varphi)_{\dot\Omega} &= (-\gamma_1 \mathrm{Div}\,(\mathbf{S}(\mathbf{v}) - K(\mathbf{v})\mathbf{I})), \nabla\varphi)_{\dot\Omega} \\
&= -(\gamma_1 \mathrm{Div}\,\mathbf{S}(\mathbf{v}) - \nabla \mathrm{div}\,\mathbf{v}, \nabla\varphi)_{\dot\Omega} + (\gamma_1 \nabla K(\mathbf{v}), \nabla\varphi)_{\dot\Omega} - (\nabla \mathrm{div}\,\mathbf{v}, \nabla\varphi)_{\dot\Omega},
\end{aligned}
$$

which, combined with (3.2) and (4.17), furnishes that

$$
(\nabla(f - \mathrm{div}\,\mathbf{v}), \nabla\varphi)_{\dot\Omega} = 0 \quad \text{for any } \varphi \in W^1_{q',0}(\Omega).
$$

Furthermore, by (3.2), (4.18) and (4.17)

$$
\begin{aligned}
&[[\mathrm{div}\,\mathbf{v}]] = -[[< \mathbf{S}(\mathbf{v})\mathbf{n}, \mathbf{n} > -K(\mathbf{v})]] = [[f]] && \text{on } \Gamma, \\
&\mathrm{div}\,\mathbf{v}|_{\Gamma_-} = -(< \mathbf{S}_-(\mathbf{v}_-)\mathbf{n}_-, \mathbf{n}_- > -K(\mathbf{v}))|_{\Gamma_-} = f|_{\Gamma_-} && \text{on } \Gamma_-,
\end{aligned}
$$

where the facts that $< [[\mathbf{h}]] - < [[\mathbf{h}]]\mathbf{n}, \mathbf{n} > \mathbf{n}, \mathbf{n} >= 0$ on Γ and $< \mathbf{h}_- - < \mathbf{h}_-\mathbf{n}_-, \mathbf{n}_- > \mathbf{n}_-, \mathbf{n}_- >= 0$ on Γ_- have been used. Thus, the uniqueness of the variational problem implies that $\mathrm{div}\,\mathbf{v} = f$ in $\dot\Omega$. Setting $\mathfrak{p} = -\theta + K(\mathbf{v}) - \psi$, we see that $\mathbf{v}$ and $\mathfrak{p}$ are required solutions of problem (1.5) with $\gamma_0 = 1$, which completes the proof of Theorem 1.4.

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