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Global existence and optimal decay rates of solutions to the classical Timoshenko system in the framework of Besov spaces

Naofumi Mori
Graduate School of Mathematics, Kyushu University

Joint work with

Jiang Xu
Department of Mathematics, Nanjing University of Aeronautics and Astronautics

Shuichi Kawashima
Faculty of Mathematics, Kyushu University

1 Introduction

In this work, we consider the Timoshenko system (see [28, 29]), which is a set of two coupled wave equations, by introducing the nonlinear term and damping term:

$$\begin{cases} \varphi_{tt}-(\varphi_x-\psi)_x=0, \\ \psi_{tt}-\sigma(\psi_x)_x-(\varphi_x-\psi)+\gamma\psi_t=0. \end{cases} \tag{1.1}$$

The system (1.1) describes the transverse vibrations of a beam with shear deformations. Here, $t\geq 0$ is the time variable, $x\in\mathbb{R}$ is the spacial variable which denotes the point on the center line of the beam, $\varphi(t,x)$ denotes the transversal displacement of the beam from an equilibrium state, and $\psi(t,x)$ denotes the rotation angle of the filament of the beam. The smooth function $\sigma(\eta)$ satisfies $\sigma'(\eta)>0$ for any $\eta\in\mathbb{R}$, and γ is a positive constant. We focus on the Cauchy problem of (1.1). The initial data are supplemented as

$$(\varphi,\varphi_t,\psi,\psi_t)(x,0)=(\varphi_0,\varphi_1,\psi_0,\psi_1)(x). \tag{1.2}$$

Based on the change of variable introduced by Ide, Haramoto, and the third author [11]:

$$v=\varphi_x-\psi,\quad u=\varphi_t,\quad z=a\psi_x,\quad y=\psi_t, \tag{1.3}$$

with $a > 0$ being the sound speed defined by $a^2 = \sigma'(0)$, it is convenient to rewrite (1.1)-(1.2) as a Cauchy problem for the first-order hyperbolic system

$$\begin{cases} v_t - u_x + y = 0 \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - \sigma(z/a)_x - v + \gamma y = 0, \\ (v,u,z,y)(x,0) = (v_0,u_0,z_0,y_0)(x), \end{cases} \tag{1.4}$$

or

$$\begin{cases} U_t + A(U)U_x + LU = 0, \\ U(x,0) = U_0(x) \end{cases} \tag{1.5}$$

with $U = (v,u,z,y)^\top$ and $U_0(x) = (v_0,u_0,z_0,y_0)(x)$, where $v_0 = \varphi_{0,x} - \psi_0$, $u_0 = \varphi_1$, $z_0 = a\psi_{0,x}$, $y_0 = \psi_1$ and

$$A(U) = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & \frac{\sigma'(z/a)}{a} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}.$$

Note that $A(U)$ is a real symmetrizable matrix due to $\sigma'(z/a) > 0$, and the dissipative matrix L is nonnegative definite but not symmetric. Such degenerate dissipation forces (1.5) to go beyond the class of generally dissipative hyperbolic systems, so the recent global-in-time existence (see [31]) for hyperbolic systems with symmetric dissipation can not be applied directly, which is the motivation on studying the Timoshenko system (1.1).

2 Known results & Aim

Let us review several known results on (1.1). In a bounded domain, it is known that (1.1) is exponentially stable if the damping term φ_t is also present on the left-hand side of the first equation of (1.3) (see, e.g., [21]). Soufyane [27] showed that (1.1) could not be exponentially stable by considering only the damping term of the form ψ_t, unless for the case of $a = 1$ (equal wave speeds). A similar result was obtained by Rivera and Racke [23] with an alternative proof. In addition, Rivera and Racke [22] also investigated the Timoshenko system with the heat conduction, which is described by the classical Fourier law. In the whole space, the third author and his collaborators [11] considered the corresponding linearized form of (1.4):

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \gamma y = 0, \\ (v,u,z,y)(x,0) = (v_0,u_0,z_0,y_0)(x), \end{cases} \tag{2.6}$$

and showed that the dissipative structure could be characterized by

$$\begin{cases} \operatorname{Re}\lambda(i\xi) \le -c\eta_1(\xi) & \text{for} \quad a = 1, \\ \operatorname{Re}\lambda(i\xi) \le -c\eta_2(\xi) & \text{for} \quad a \neq 1, \end{cases} \tag{2.7}$$

where $\lambda(i\xi)$ denotes the eigenvalues of the system (2.6) in the Fourier space, $\eta_1(\xi) = \frac{\xi^2}{1+\xi^2}$, $\eta_2(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$, and $c > 0$ is some constant. Consequently, the following decay properties were established for $U = (v, u, z, y)^\top$ of (2.6) (see [11] for details):

$$\|\partial_x^k U(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + e^{-ct}\|\partial_x^k U_0\|_{L^2} \tag{2.8}$$

for $a = 1$, and

$$\|\partial_x^k U(t)\|_{L^2} \lesssim (1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{L^1} + (1+t)^{-\frac{l}{2}}\|\partial_x^{k+l} U_0\|_{L^2} \tag{2.9}$$

for $a \neq 1$. Recently, under the additional assumption $\int_{\mathbb{R}} U_0 dx = 0$, Racke and Said-Houari [24] strengthened (2.8)-(2.9) such that linearized solutions decay faster with a rate of $t^{-\gamma/2}$, by introducing the integral space $L^{1,\gamma}(\mathbb{R})$.

Remark 2.1. *Clearly, the high frequency part of (2.8) yields an exponential decay, whereas the corresponding part of (2.9) is of the regularity-loss type, since $(1+t)^{-\ell/2}$ is created by assuming the additional ℓ-th order regularity on the initial data. Consequently, extra higher regularity than that for global-in-time existence of classical solutions is imposed to obtain the optimal decay rates.*

In [12], Ide and the third author performed the time-weighted approach to establish the global existence and asymptotic decay of solutions to the nonlinear problem (1.5). To overcome the difficulty caused by the regularity-loss property, the spatially regularity $s \geq 6$ was needed. Denote by s_c the critical regularity for global existence of classical solutions. Actually, the local-in-time existence theory of Kato and Majda [13, 16] implies that $s_c = 2$ for the Timoshenko system (1.5), actually, the extra regularity is used to take care of optimal decay estimates. Consequently, some natural questions follow. Is $s = 6$ the minimal decay regularity for (1.5) with the regularity-loss? If not, which index characterises the minimal decay regularity? This motivates the following general definition.

Definition 2.1. *If the optimal decay rate of $L^1(\mathbb{R}^n)$-$L^2(\mathbb{R}^n)$ type is achieved under the lowest regularity assumption, then the lowest index is called the minimal decay regularity index for dissipative systems of regularity-loss, which is labelled as s_D.*

In this paper, we show the global existence and large-time behavior for (1.5) in spatially critical Besov spaces. To the best of our knowledge, there are few results available in this direction for the Timoshenko system, although the critical space has already been succeeded in the study of fluid dynamical equations, see [2, 7, 10, 19] for Navier-Stokes equations, [8, 35, 36, 37] for Euler equations and related models. In [31, 32], under the assumptions of dissipative entropy and Shizuta-Kawashima condition, the second and third authors have already investigated generally dissipative systems, however, the Timoshenko system admits the non-symmetric dissipation and goes beyond the class.

Hence, as a first step, we first constructed global solutions pertaining to data in the Besov space $B^{3/2}_{2,1}(\mathbb{R})$ in Section 4 by virtue of an elementary fact in Proposition 3.3 (also see [31]) that indicates the relation between homogeneous and inhomogeneous Chemin-Lerner spaces. Next, the optimal decay rate of solutions is shown in the space $B^{3/2}_{2,1}(\mathbb{R})\cap\dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ in Section 5. We shall overcome the difficulty of the weak dissipation due to the regularity-loss property and show $s_c = 3/2$ for global-in-time existence and $s_D = 3/2$ for the optimal decay estimate, which lead to reduce significantly the regularity requirements on the initial data in comparison with [12].

This paper is a summary of our two papers [18] and [34]. The interested reader, please refer to [18] and [34] for details.

Notations. Throughout the paper, $f \lesssim g$ denotes $f \leq Cg$, where $C > 0$ is a generic constant. $f \approx g$ means $f \lesssim g$ and $g \lesssim f$. Denote by $\mathcal{C}([0,T],X)$ (resp., $\mathcal{C}^1([0,T],X)$) the space of continuous (resp., continuously differentiable) functions on $[0,T]$ with values in a Banach space X. Also, $\|(f,g,h)\|_X$ means $\|f\|_X + \|g\|_X + \|h\|_X$, where $f, g, h \in X$.

3 Tools

In this section, we present analysis properties in Besov spaces and Chemin-Lerner spaces in $\mathbb{R}^n (n \geq 1)$, which will be used in the sequence section. For the Littlewood–Paley decomposition and definitions for Besov spaces and Chemin-Lerner spaces in $\mathbb{R}^n (n \geq 1)$, see [5]. Firstly, we give an improved Bernstein inequality (see, *e.g.*, [30]), which allows the case of fractional derivatives.

Lemma 3.1. *Let $0 < R_1 < R_2$ and $1 \leq a \leq b \leq \infty$.*

(i) If $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq R_1\lambda\}$, *then*

$$\|\Lambda^\alpha f\|_{L^b} \lesssim \lambda^{\alpha+n(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a}, \quad \textit{for any} \quad \alpha \geq 0;$$

(ii) If $\mathrm{Supp}\mathcal{F}f \subset \{\xi \in \mathbb{R}^n : R_1\lambda \leq |\xi| \leq R_2\lambda\}$, *then*

$$\|\Lambda^\alpha f\|_{L^a} \approx \lambda^\alpha\|f\|_{L^a}, \quad \textit{for any} \quad \alpha \in \mathbb{R}.$$

Besov spaces obey various inclusion relations. Precisely,

Lemma 3.2. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then*

(1) If $s > 0$, then $B^s_{p,r} = L^p \cap \dot{B}^s_{p,r}$;

(2) If $\tilde{s} \leq s$, then $B^s_{p,r} \hookrightarrow B^{\tilde{s}}_{p,r}$. This inclusion relation is false for the homogeneous Besov spaces;

(3) If $1 \leq r \leq \tilde{r} \leq \infty$, then $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^s_{p,\tilde{r}}$ and $B^s_{p,r} \hookrightarrow B^s_{p,\tilde{r}}$;

(4) If $1 \leq p \leq \tilde{p} \leq \infty$, then $\dot{B}^s_{p,r} \hookrightarrow \dot{B}^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$ and $B^s_{p,r} \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},r}$;

(5) $\dot{B}^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0$, $\quad B^{n/p}_{p,1} \hookrightarrow \mathcal{C}_0 (1 \leq p < \infty)$;

where C_0 is the space of continuous bounded functions which decay at infinity.

Lemma 3.3. *Suppose that $\varrho > 0$ and $1 \leq p < 2$. It holds that*

$$\|f\|_{\dot{B}^{-\varrho}_{r,\infty}} \lesssim \|f\|_{L^p}$$

with $1/p - 1/r = \varrho/n$. In particular, this holds with $\varrho = n/2, r = 2$ and $p = 1$.

The global existence depends on a key fact, which indicates the connection between homogeneous Chemin-Lerner spaces and inhomogeneous Chemin-Lerner spaces, see [31] for the proof. Precisely,

Proposition 3.1. *Let $s \in \mathbb{R}$ and $1 \leq \theta, p, r \leq \infty$.*

(1) *It holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) \subset \widetilde{L}^\theta_T(B^s_{p,r});$$

(2) *Furthermore, as $s > 0$ and $\theta \geq r$, it holds that*

$$L^\theta_T(L^p) \cap \widetilde{L}^\theta_T(\dot{B}^s_{p,r}) = \widetilde{L}^\theta_T(B^s_{p,r})$$

for any $T > 0$.

Let us state the Moser-type product estimates, which plays an important role in the estimate of bilinear terms.

Proposition 3.2. *Let $s > 0$ and $1 \leq p, r \leq \infty$. Then $\dot{B}^s_{p,r} \cap L^\infty$ is an algebra and*

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}.$$

Let $s_1, s_2 \leq n/p$ such that $s_1 + s_2 > n\max\{0, \frac{2}{p} - 1\}$. Then one has

$$\|fg\|_{\dot{B}^{s_1+s_2-n/p}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}.$$

In the sequel we also need a estimate for commutator.

Proposition 3.3. *Let $1 < p < \infty, 1 \leq \theta \leq \infty$ and $s \in (-\frac{n}{p} - 1, \frac{n}{p}]$. Then there exists a generic constant $C > 0$ depending only on s, n such that*

$$\begin{cases} \|[f, \dot{\Delta}_q]g\|_{L^p} \leq Cc_q 2^{-q(s+1)}\|f\|_{\dot{B}^{\frac{n}{p}+1}_{p,1}}\|g\|_{\dot{B}^s_{p,1}}, \\ \|[f, \dot{\Delta}_q]g\|_{L^\theta_T(L^p)} \leq Cc_q 2^{-q(s+1)}\|f\|_{\widetilde{L}^{\theta_1}_T(\dot{B}^{\frac{n}{p}+1}_{p,1})}\|g\|_{\widetilde{L}^{\theta_2}_T(\dot{B}^s_{p,1})}, \end{cases}$$

with $1/\theta = 1/\theta_1 + 1/\theta_2$, where the commutator $[\cdot,\cdot]$ is defined by $[f, g] = fg - gf$ and $\{c_q\}$ denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$.

Finally, we state a continuity result for compositions (see, *e.g.*, [10]) to end this section.

Proposition 3.4. *Let $s>0$, $1\leq p,r,\theta\leq\infty$, $F\in W^{[s]+3,\infty}_{loc}(I;\mathbb{R})$ with $F(0)=0$, $T\in(0,\infty]$ and $f\in\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})\cap L^{\infty}_{T}(L^{\infty})$. Then there exists a function C depending only on s,p,r,n, and F such that*

$$\begin{cases}\|F(f)-F'(0)f\|_{\dot{B}^{s}_{p,r}}\leq C(\|f\|_{L^{\infty}})\|f\|^{2}_{\dot{B}^{s}_{p,r}},\\ \|F(f)-F'(0)f\|_{\widetilde{L}^{\theta}_{T}(\dot{B}^{s}_{p,r})}\leq C(\|f\|_{L^{\infty}_{T}(L^{\infty})})\|f\|^{2}_{\widetilde{L}^{\theta}_{T}(\dot{B}^{s}_{p,r})}.\end{cases}$$

In the analysis of decay estimates, we also need the general form of Moser-type product estimates, which was shown by Yong in [37].

Proposition 3.5. *Let $s>0$ and $1\leq p,r,p_1,p_2,p_3,p_4\leq\infty$. Assume that $f\in L^{p_1}\cap\dot{B}^{s}_{p_4,r}$ and $g\in L^{p_3}\cap\dot{B}^{s}_{p_2,r}$ with*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Then it holds that

$$\|fg\|_{\dot{B}^{s}_{p,r}}\lesssim\|f\|_{L^{p_1}}\|g\|_{\dot{B}^{s}_{p_2,r}}+\|g\|_{L^{p_3}}\|f\|_{\dot{B}^{s}_{p_4,r}}.$$

In [31], the first and third authors established a key fact, which indicates the connection between homogeneous Chemin-Lerner spaces and inhomogeneous Chemin-Lerner spaces.

Proposition 3.6. *Let $s\in\mathbb{R}$ and $1\leq\theta,p,r\leq\infty$.*

(1) *It holds that*

$$L^{\theta}_{T}(L^{p})\cap\widetilde{L}^{\theta}_{T}(\dot{B}^{s}_{p,r})\subset\widetilde{L}^{\theta}_{T}(B^{s}_{p,r});$$

(2) *Furthermore, as $s>0$ and $\theta\geq r$, it holds that*

$$L^{\theta}_{T}(L^{p})\cap\widetilde{L}^{\theta}_{T}(\dot{B}^{s}_{p,r})=\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})$$

for any $T>0$.

The property of continuity for product in $\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})$ is similar to in the stationary case (Proposition 3.1), whereas the time exponent θ behaves according to the Hölder inequality.

Proposition 3.7. *The following inequality holds:*

$$\|fg\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\lesssim(\|f\|_{L^{\theta_1}_{T}(L^{\infty})}\|g\|_{\widetilde{L}^{\theta_2}_{T}(B^{s}_{p,r})}+\|g\|_{L^{\theta_3}_{T}(L^{\infty})}\|f\|_{\widetilde{L}^{\theta_4}_{T}(B^{s}_{p,r})})$$

whenever $s>0,1\leq p\leq\infty,1\leq\theta,\theta_1,\theta_2,\theta_3,\theta_4\leq\infty$ and

$$\frac{1}{\theta}=\frac{1}{\theta_1}+\frac{1}{\theta_2}=\frac{1}{\theta_3}+\frac{1}{\theta_4}.$$

As a direct corollary, one has

$$\|fg\|_{\widetilde{L}^{\theta}_{T}(B^{s}_{p,r})}\lesssim\|f\|_{\widetilde{L}^{\theta_1}_{T}(B^{s}_{p,r})}\|g\|_{\widetilde{L}^{\theta_2}_{T}(B^{s}_{p,r})}$$

whenever $s\geq n/p$, $\frac{1}{\theta}=\frac{1}{\theta_1}+\frac{1}{\theta_2}$.

Finally, we state a continuity result for compositions (see [1]) to end this section.

Proposition 3.8. *Let $s > 0$, $1 \le p, r, \rho \le \infty$, $F \in W^{[s]+1,\infty}_{loc}(I;\mathbb{R})$ with $F(0) = 0$, $T \in (0,\infty]$ and $v \in \widetilde{L}^{\rho}_T(B^s_{p,r}) \cap L^{\infty}_T(L^{\infty})$. Then*

$$\|F(v)\|_{\widetilde{L}^{\rho}_T(B^s_{p,r})} \lesssim (1 + \|v\|_{L^{\infty}_T(L^{\infty})})^{[s]+1}\|v\|_{\widetilde{L}^{\rho}_T(B^s_{p,r})}.$$

In the recent decade, harmonic analysis tools, especially for techniques based on Littlewood-Paley decomposition and paradifferential calculus have proved to be very efficient in the study of partial differential equations. It is well-known that the frequency-localization operator $\dot{\Delta}_q f$ (or $\Delta_q f$) has a smoothing effect on the function f, even though f is quite rough. Moreover, the L^p norm of $\dot{\Delta}_q f$ can be preserved provided $f \in L^p(\mathbb{R}^n)$. To the best of our knowledge, so far there are few efforts about the decay property related to the operator $\dot{\Delta}_q f$. Here, the difficulty of regularity-loss mechanism forces us to develop the frequency-localization time-decay inequality. Precisely,

Proposition 3.9 ([33]). *Set $\eta(\xi) = \frac{|\xi|^2}{(1+|\xi|^2)^2}$. If $f \in \dot{B}^{\sigma+\ell}_{2,r}(\mathbb{R}^n) \cap \dot{B}^{-s}_{2,\infty}(\mathbb{R}^n)$ for $\sigma \in \mathbb{R}, s \in \mathbb{R}$ and $1 \le r \le \infty$ such that $\sigma + s > 0$, then it holds that*

$$\begin{aligned} &\Big\|2^{q\sigma}\|\widehat{\dot{\Delta}_q f}e^{-\eta(\xi)t}\|_{L^2}\Big\|_{l^r_q} \\ \lesssim\ & \underbrace{(1+t)^{-\frac{\sigma+s}{2}}\|f\|_{\dot{B}^{-s}_{2,\infty}}}_{Low-frequency\ Estimate} + \underbrace{(1+t)^{-\frac{\ell}{2}+\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|f\|_{\dot{B}^{\sigma+\ell}_{p,r}}}_{High-frequency\ Estimate}, \end{aligned} \tag{3.10}$$

for $\ell > n(\frac{1}{p} - \frac{1}{2})$ [1] *with $1 \le p \le 2$.*

4 Global-in-time existence

In this section, we give the global in time existence result for (1.5).

Theorem 4.1. *Let $a = 1$ or $a \neq 1$. Suppose that $U_0 \in B^{3/2}_{2,1}(\mathbb{R})$. There exists a positive constant δ_0 such that if*

$$\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})} \le \delta_0,$$

then the Cauchy problem (1.5) has a unique global classical solution $U \in \mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R})$ satisfying

$$U \in \widetilde{\mathcal{C}}(B^{3/2}_{2,1}(\mathbb{R})) \cap \widetilde{\mathcal{C}}^1(B^{1/2}_{2,1}(\mathbb{R}))$$

Moreover, the following energy inequality holds that

$$\begin{aligned} &\|U\|_{\widetilde{L}^{\infty}(B^{3/2}_{2,1}(\mathbb{R}))} + \Big(\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|(v, z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}\Big) \\ \le\ & C_0\|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})}, \end{aligned} \tag{4.11}$$

where $C_0 > 0$ is a constant.

[1]Let us remark that $\ell \ge 0$ in the case of $p = 2$.

Remark 4.1. *Theorem 4.1 exhibits the optimal critical regularity of global well-posedness for (1.5). Observe that there is 1-regularity-loss phenomenon for the dissipation rates due to the nonlinear influence in the case of not only $a \neq 1$ but also $a = 1$, which is totally different in comparison with the linearized system (2.6) with $a = 1$.*

Recently, the second and third authors [31] have already established a local existence theory for generally symmetric hyperbolic systems in spatially critical Besov spaces, which is viewed as the generalization of the basic theory of Kato and Majda [13, 16]. Fortunately, the new result can be applied to the current problem (1.5) directly, since the non-symmetric dissipation L has no influence on the local-in-time existence. Precisely,

Proposition 4.1. *Assume that $U_0 \in B^{3/2}_{2,1}$, then there exists a time $T_0 > 0$ (depending only on the initial data) such that*

(i) (Existence): system (1.5) has a unique solution $U(t,x) \in \mathcal{C}^1([0,T_0] \times \mathbb{R})$ satisfying $U \in \widetilde{\mathcal{C}}_{T_0}(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_{T_0}(B^{1/2}_{2,1})$;

(ii) (Blow-up criterion): if the maximal time $T^*(> T_0)$ of existence of such a solution is finite, then

$$\limsup_{t \to T^*} \|U(t,\cdot)\|_{B^{3/2}_{2,1}} = \infty$$

if and only if

$$\int_0^{T^*} \|\nabla U(t,\cdot)\|_{L^\infty} dt = \infty.$$

Furthermore, in order to show that classical solutions in Proposition 4.1 are globally defined, the next task is to construct a priori estimates according to the dissipative mechanism produced by the Tomoshenko system. To this end, we define by $E(T)$ the energy functional and by $D(T)$ the corresponding dissipation functional:

$$E(T) := \|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})}$$

and

$$D(T) := \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|(v, z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}$$

for any time $T > 0$.

The first lemma is related to the nonlinear a priori estimate for the dissipation for y.

Lemma 4.1. *(The dissipation for y) If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (1.5) for any $T > 0$, then*

$$E(T) + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \lesssim \|U_0\|_{B^{3/2}_{2,1}} + \sqrt{E(T)}D(T). \tag{4.12}$$

Proof. Firstly, we perform the usual energy method. Multiplying the first equation in (1.4) by v, the second one by u, the third one by $[\sigma(z/a) - \sigma(0)]/a$ and the last one by y, respectively, and then adding the resulting equalities, we get

$$\frac{1}{2}\frac{d}{dt}(v^2 + y^2 + u^2 + S(z)) - \Big(vu + [\sigma(z/a) - \sigma(0)]y\Big)_x + \gamma y^2 = 0, \tag{4.13}$$

where

$$S(z)=2\int_0^{z/a}\Big[\sigma(\eta)-\sigma(0)\Big]d\eta.$$

Note that $S(z)$ is equivalent to z^2, due to the fact $\sigma'(\eta)>0$ and the smallness assumption. Then we perform the integral to (4.13) with respect to x and obtain the basic energy equality

$$\frac{1}{2}\frac{d}{dt}E_0(U)+\gamma\|y\|_{L^2}^2=0, \tag{4.14}$$

where the energy functional $E_0(U)$ is defined by

$$E_0(U)=\|(v,u,y)\|_{L^2}^2+\int_{\mathbb{R}}S(z)dx\approx\|U\|_{L^2}^2.$$

By integrating in $t\in[0,T]$ and taking the square-root of the resulting inequality, we arrive at

$$\|U\|_{L_T^\infty(L^2)}+\sqrt{2\gamma}\|y\|_{L_T^2(L^2)}\leq\|U_0\|_{L^2} \tag{4.15}$$

for any $T>0$.

Next, we perform the frequency-localization estimate and get the dissipation rate from y in homogeneous Chemin-Lerner spaces. Applying the operator $\dot{\Delta}_q(q\in\mathbb{Z})$ to (1.5) gives

$$\begin{cases}\dot{\Delta}_qv_t-\dot{\Delta}_qu_x+\dot{\Delta}_qy=0,\\ \dot{\Delta}_qu_t-\dot{\Delta}_qv_x=0,\\ \dot{\Delta}_qz_t-a\dot{\Delta}_qy_x=0,\\ \dot{\Delta}_qy_t-\sigma'(z/a)\dot{\Delta}_q(z/a)_x-\dot{\Delta}_qv+\gamma\dot{\Delta}_qy=[\dot{\Delta}_q,\sigma'(z/a)](z/a)_x,\end{cases} \tag{4.16}$$

where the commutator is defined by $[f,g]:=fg-gf$. Multiplying (4.16) with $\dot{\Delta}_qv$, $\dot{\Delta}_qu$, $\sigma'(z/a)\dot{\Delta}_qz/a^2$ and $\dot{\Delta}_qy$, respectively, and then adding the resulting equalities, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(|\dot{\Delta}_qv|^2+|\dot{\Delta}_qy|^2+|\dot{\Delta}_qu|^2+\sigma'(z/a)|\dot{\Delta}_q(z/a)|^2\Big)\\&-\Big\{(\dot{\Delta}_qu\dot{\Delta}_qv)_x+\Big(\sigma'(z/a)\dot{\Delta}_q(z/a)\dot{\Delta}_qy\Big)_x\Big\}+\gamma|\dot{\Delta}_qy|^2\\=\;&\frac{1}{2}\sigma'(z/a)_t|\dot{\Delta}_q(z/a)|^2-\sigma'(z/a)_x\dot{\Delta}_q(z/a)\dot{\Delta}_qy+[\dot{\Delta}_q,\sigma'(z/a)](z/a)_x\dot{\Delta}_qy.\end{aligned} \tag{4.17}$$

Furthermore, by employing the integral with respect to x, with the aid of Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_0[\dot{\Delta}_qU]+\gamma\|\dot{\Delta}_qy\|_{L^2}^2\\\lesssim\;&\|\sigma'(z/a)_t\|_{L^\infty}\|\dot{\Delta}_qz\|_{L^2}^2+\|\sigma'(z/a)_x\|_{L^\infty}\|\dot{\Delta}_qz\|_{L^2}\|\dot{\Delta}_qy\|_{L^2}\\&+\|[\dot{\Delta}_q,\sigma'(z/a)]z_x\|_{L^2}\|\dot{\Delta}_qy\|_{L^2},\end{aligned} \tag{4.18}$$

where

$$E_0[\dot{\Delta}_qU]:=\|(\dot{\Delta}_qv,\dot{\Delta}_qy,\dot{\Delta}_qu)\|_{L^2}^2+\int_{\mathbb{R}}\sigma'(z/a)|\dot{\Delta}_q(z/a)|^2\,dx\approx\|\dot{\Delta}_qU\|_{L^2}^2.$$

From (1.4) and a priori assumption (5.23) below, we have

$$\|\sigma'(z/a)_t\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2 \lesssim \|z_t\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2 \lesssim \|y_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}^2. \tag{4.19}$$

Similarly,

$$\|\sigma'(z/a)_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}\|\dot{\Delta}_q y\|_{L^2} \lesssim \|z_x\|_{L^\infty}\|\dot{\Delta}_q z\|_{L^2}\|\dot{\Delta}_q y\|_{L^2}. \tag{4.20}$$

Together with (4.19)-(4.20), by integrating in $t \in [0,T]$, with the help of Young's inequality, we are led to

$$\begin{aligned}
&\sqrt{E_0[\dot{\Delta}_q U]} + \sqrt{2\gamma}\|\dot{\Delta}_q y\|_{L_T^2(L^2)} \\
\lesssim\ & \sqrt{E_0[\dot{\Delta}_q U_0]} + \sqrt{\|(y_x,z_x)\|_{L_T^\infty(L^\infty)}}\Big(\|\dot{\Delta}_q y\|_{L_T^2(L^2)} + \|\dot{\Delta}_q z\|_{L_T^2(L^2)}\Big) \\
&+\sqrt{\|[\dot{\Delta}_q, \sigma'(z/a)]z_x\|_{L_T^2(L^2)}\|\dot{\Delta}_q y\|_{L_T^2(L^2)}}.
\end{aligned} \tag{4.21}$$

It follows from the commutator estimate in Proposition 3.3 that

$$\|[\dot{\Delta}_q, \sigma'(z/a)]z_x\|_{L_T^2(L^2)} \lesssim c_q 2^{-\frac{3q}{2}}\|z\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{3/2})}\|z_x\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{1/2})}, \tag{4.22}$$

where $\{c_q\}$ denotes a sequence such that $\|c_q\|_{\ell^1} \leq 1$. Therefore, we obtain

$$\begin{aligned}
&2^{\frac{3q}{2}}\|\dot{\Delta}_q U\|_{L_T^\infty(L^2)} + \sqrt{2\gamma}2^{\frac{3q}{2}}\|\dot{\Delta}_q y\|_{L_T^2(L^2)} \\
\lesssim\ & \|\dot{\Delta}_q U_0\|_{L^2} + c_q\sqrt{\|(y_x,z_x)\|_{L_T^\infty(\dot{B}_{2,1}^{1/2})}}\Big(\|y\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{3/2})} + \|z_x\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{1/2})}\Big) \\
&+c_q\sqrt{\|z\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{3/2})}}\Big(\|y\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{3/2})} + \|z_x\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{1/2})}\Big).
\end{aligned} \tag{4.23}$$

Here, we would like to point out each $\{c_q\}$ has a possibly different form in (4.23) or in sequent inequalities, however, the bound $\|c_q\|_{\ell^1} \leq 1$ is well satisfied. Hence, summing up on $q \in \mathbb{Z}$, we arrive at

$$\begin{aligned}
&\|U\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{3/2})} + \sqrt{2\gamma}\|y\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{3/2})} \\
\lesssim\ & \|U_0\|_{\dot{B}_{2,1}^{3/2}} + \sqrt{\|(y,z)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{3/2})}}\Big(\|y\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{3/2})} + \|z_x\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{1/2})}\Big).
\end{aligned} \tag{4.24}$$

Finally, combining (4.15) and (4.24), we conclude that from Proposition 3.1

$$E(T) + \|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} \lesssim \|U_0\|_{B_{2,1}^{3/2}} + \sqrt{E(T)}D(T). \tag{4.25}$$

Therefore, the proof of Lemma 4.1 is complete. □

Lemma 4.2. *(The dissipation for v) If $U \in \tilde{\mathcal{C}}_T(B_{2,1}^{3/2}) \cap \tilde{\mathcal{C}}_T^1(B_{2,1}^{1/2})$ is a solution of (1.5) for any $T > 0$, then we have*

$$\|v\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} \lesssim E(T) + \|U_0\|_{B_{2,1}^{3/2}} + \|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + \sqrt{E(T)}D(T) \tag{4.26}$$

for $a = 1$, while in the case of $a \neq 1$, we have

$$\begin{aligned}
\|v\|_{\tilde{L}_T^2(B_{2,1}^{1/2})} \lesssim\ & E(T) + \|U_0\|_{B_{2,1}^{3/2}} + \varepsilon\|u_x\|_{\tilde{L}_T^2(B_{2,1}^{-1/2})} \\
&+(1 + C_\varepsilon)\|y\|_{\tilde{L}_T^2(B_{2,1}^{3/2})} + E(T)D(T)
\end{aligned} \tag{4.27}$$

for $\varepsilon > 0$, where C_ε is a position constant dependent on ε.

Note that the calculation for the dissipation of v in the case of $a \neq 1$ is a little different from $a = 1$. We would like to give the proof for $a \neq 1$ as follows.

Proof. We rewrite the system (1.4) as follows:

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \gamma y = g(z)_x, \end{cases} \tag{4.28}$$

where the smooth function $g(z)$ is defined by

$$g(z) = \sigma(z/a) - \sigma(0) - \sigma'(0)z/a = O(z^2)$$

satisfying $g(0) = 0$ and $g'(0) = 0$.

Firstly, applying the inhomogeneous frequency-localization operator $\Delta_q (q \geq -1)$ to (4.28) gives

$$\begin{cases} \Delta_q v_t - \Delta_q u_x + \Delta_q y = 0, \\ \Delta_q u_t - \Delta_q v_x = 0, \\ \Delta_q z_t - a\Delta_q y_x = 0, \\ \Delta_q y_t - a\Delta_q z_x - \Delta_q v + \gamma \Delta_q y = \Delta_q g(z)_x. \end{cases} \tag{4.29}$$

Next, multiplying the first equation in (4.29) by $-\Delta_q y$, the second one by $-a\Delta_q z$, the third one by $-a\Delta_q u$ and the fourth one by $-\Delta_q v$, respectively, then adding the resulting equalities, we have

$$\begin{aligned} & -(\Delta_q v \Delta_q y + a\Delta_q u \Delta_q z)_t + (a\Delta_q v \Delta_q z + a^2 \Delta_q u \Delta_q y)_x + |\Delta_q v|^2 \\ = \ & |\Delta_q y|^2 + (a^2 - 1)\Delta_q y \Delta_q u_x + \gamma \Delta_q y \Delta_q v - \Delta_q g(z)_x \Delta_q v. \end{aligned} \tag{4.30}$$

Integrating the equality (4.30) in $x \in \mathbb{R}$, with the aid of Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} E_1[\Delta_q U] + \frac{1}{2}\|\Delta_q v\|_{L^2}^2 \\ \lesssim \ & \|\Delta_q y\|_{L^2}^2 + |a^2 - 1| \|\Delta_q y\|_{L^2} \|\Delta_q u_x\|_{L^2} \\ & + \|\Delta_q g(z)_x\|_{L^2} \|\Delta_q v\|_{L^2}, \end{aligned} \tag{4.31}$$

where

$$E_1[\Delta_q U] := -\int_{\mathbb{R}} (\Delta_q v \Delta_q y + \Delta_q u \Delta_q z) dx.$$

By performing the integral with respect to $t \in [0, T]$, we are led to

$$\begin{aligned} & \|\Delta_q v\|_{L_t^2(L^2)}^2 \\ \lesssim \ & \|\Delta_q U\|_{L_T^\infty(L^2)}^2 + \|\Delta_q U_0\|_{L^2}^2 + \|\Delta_q y\|_{L_T^2(L^2)}^2 \\ & + \|\Delta_q y\|_{L_T^2(L^2)} \|\Delta_q u_x\|_{L_T^2(L^2)} + \|\Delta_q g(z)_x\|_{L_T^2(L^2)}^2, \end{aligned} \tag{4.32}$$

where we have noticed the case of $a \neq 1$. Furthermore, Young's inequality enables us to get

$$\begin{aligned}
& 2^{\frac{q}{2}}\|\Delta_q v\|_{L^2_T(L^2)} \\
\lesssim\ & c_q\|U\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})} + c_q\|U_0\|_{B^{1/2}_{2,1}} + \varepsilon c_q\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \\
& + c_q(1+C_\varepsilon)\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + c_q\|g(z)_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}
\end{aligned} \tag{4.33}$$

for $\varepsilon > 0$, where C_ε is a position constant dependent on ε and each $\{c_q\}$ has a possibly different form in (4.33), however, the bound $\|c_q\|_{\ell^1} \leq 1$ is well satisfied.

Recalling the fact $g'(0) = 0$, it follows from Propositions 3.7-3.8 that

$$\begin{aligned}
\|g(z)_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} &= \|g'(z)z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\
&\lesssim \|g'(z) - g'(0)\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\
&\lesssim \|z\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}.
\end{aligned} \tag{4.34}$$

Hence, together with (4.33)-(4.34), by summing up on $q \geq -1$, we deduce that

$$\begin{aligned}
& \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \\
\lesssim\ & \|U\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})} + \|U_0\|_{B^{1/2}_{2,1}} + \varepsilon\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \\
& + (1+C_\varepsilon)\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} + \|z\|_{\widetilde{L}^\infty_T(B^{1/2}_{2,1})}\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})},
\end{aligned} \tag{4.35}$$

which leads to the inequality (4.27) immediately. □

Lemma 4.3. *(The dissipation for z_x) If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (1.5) for any $T > 0$, then*

$$\begin{aligned}
\|z_x\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} \lesssim\ & E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \\
& + \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \sqrt{E(T)}D(T).
\end{aligned} \tag{4.36}$$

Proof. Multiplying the third equation in (4.28) by y_x and the fourth one by $-z_x$, respectively, and then integrating the resulting equalities over $\mathbb{R}$, we arrive at

$$\begin{aligned}
& \frac{d}{dt}E_2(U) + \|z_x\|^2_{L^2} \\
\lesssim\ & \|y_x\|^2_{L^2} + (\|v\|_{L^2} + \|y\|_{L^2})\|z_x\|_{L^2} + \|z\|_{L^\infty}\|z_x\|^2_{L^2},
\end{aligned} \tag{4.37}$$

where

$$E_2(U) := -\int_{\mathbb{R}} z_x y dx.$$

Therefore, we arrive at

$$\begin{aligned}
\|z_x\|_{L^2_T(L^2)} \lesssim\ & E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})} \\
& + \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \sqrt{E(T)}D(T).
\end{aligned} \tag{4.38}$$

On the other hand, from (4.29), we have

$$\begin{cases} \dot{\Delta}_q z_t - a\dot{\Delta}_q y_x = 0, \\ \dot{\Delta}_q y_t - a\dot{\Delta}_q z_x - \dot{\Delta}_q v + \gamma\dot{\Delta}_q y = \dot{\Delta}_q g(z)_x. \end{cases} \tag{4.39}$$

Then, by multiplying the first equation in (4.39) by $\dot{\Delta}_q y_x$ and the second one by $-\dot{\Delta}_q z_x$, respectively, and then employing the energy estimates on each block, we are led to

$$\begin{aligned} & 2^{\frac{q}{2}}\|\dot{\Delta}_q z_x\|_{L^2_T(L^2)} \\ \lesssim\ & c_q(\|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})} + \|U_0\|_{B^{3/2}_{2,1}}) + c_q\|y_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \\ & +c_q\epsilon\|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + c_q C_\epsilon(\|v\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}) \\ & +c_q\sqrt{\|z_x\|_{\widetilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}}\|g(z)_x\|^{\frac{1}{2}}_{\widetilde{L}^1_T(\dot{B}^{1/2}_{2,1})}. \end{aligned} \tag{4.40}$$

Consequently,

$$\begin{aligned} & \|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} \\ \lesssim\ & \|U\|_{\widetilde{L}^\infty_T(B^{3/2}_{2,1})} + \|U_0\|_{B^{3/2}_{2,1}} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{3/2}_{2,1})} \\ & +\|v\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})} + \sqrt{\|z\|_{\widetilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}}\|z_x\|_{\widetilde{L}^2_T(\dot{B}^{1/2}_{2,1})}, \end{aligned} \tag{4.41}$$

where we have chosen $0 < \epsilon \leq 1/2$.

Finally, by combining (4.38) and (4.41), we arrive at (4.36). □

Lemma 4.4. *(The dissipation for u_x) If $U \in \widetilde{\mathcal{C}}_T(B^{3/2}_{2,1}) \cap \widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (1.5) for any $T > 0$, then*

$$\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})} \lesssim E(T) + \|U_0\|_{B^{3/2}_{2,1}} + \|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})} + \|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}. \tag{4.42}$$

Proof. Applying the inhomogeneous operator $\Delta_q (q \geq -1)$ to the first equation and second one of (4.29) gives

$$\begin{cases} \Delta_q v_t - \Delta_q u_x + \Delta_q y = 0, \\ \Delta_q u_t - \Delta_q v_x = 0. \end{cases} \tag{4.43}$$

Multiplying the first equation in (4.43) by $-\Delta_q u_x$ and the second one by $\Delta_q v_x$, we can obtain

$$\frac{d}{dt}E_3[\Delta_q U] + \|\Delta_q u_x\|^2_{L^2} \leq \|\Delta_q v_x\|^2_{L^2} + \|\Delta_q u_x\|_{L^2}\|\Delta_q y\|_{L^2}, \tag{4.44}$$

where

$$E_3[\Delta_q U] := -\int_{\mathbb{R}} \Delta_q v \Delta_q u_x dx.$$

Then we integrate (4.44) with respect to $t \in [0, T]$ to get

$$\begin{aligned} \|\Delta_q u_x\|^2_{L^2_t(L^2)} \leq\ & \Big(|E_3[\Delta_q U]| + E_3[\Delta_q U_0]\Big) \\ & +\|\Delta_q v_x\|^2_{L^2_t(L^2)} + \|\Delta_q u_x\|_{L^2_t(L^2)}\|\Delta_q y\|_{L^2_t(L^2)}. \end{aligned} \tag{4.45}$$

By using Young's inequality and embedding properties in Lemma 3.2, we are led to

$$\begin{aligned}
&2^{-q/2}\|\Delta_q u_x\|_{L^2_T(L^2)}\\
\lesssim\ & c_q E(T) + c_q\|U_0\|_{B^{3/2}_{2,1}} + c_q\|v\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}\\
&+c_q\sqrt{\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}},
\end{aligned} \tag{4.46}$$

which leads to (4.42) immediately. □

Having Lemmas 4.1-4.4, we obtain the following a priori estimate for solutions. For brevity, we feel free to skip the details.

Proposition 4.2. *Let $a=1$ or $a\neq 1$. Suppose $U\in\widetilde{\mathcal{C}}_T(B^{3/2}_{2,1})\cap\widetilde{\mathcal{C}}^1_T(B^{1/2}_{2,1})$ is a solution of (1.5) for $T>0$. There exists $\delta_1>0$ such that if*

$$E(T)\le\delta_1, \tag{4.47}$$

then

$$E(T)+D(T)\lesssim\|U_0\|_{B^{3/2}_{2,1}}+\Big(\sqrt{E(T)}+E(T)\Big)D(T). \tag{4.48}$$

Furthermore, it holds that

$$E(T)+D(T)\lesssim\|U_0\|_{B^{3/2}_{2,1}}. \tag{4.49}$$

By using the standard boot-strap argument, Theorem 4.1 follows from the local existence result (Proposition 4.1) and a priori estimate (Proposition 4.2). Here, we give the outline for completeness.

The proof of Theorem 4.1. If the initial data satisfy $\|U_0\|_{B^{3/2}_{2,1}}\le\frac{\delta_1}{2}$, by Proposition 4.1, then we determine a time $T_1>0(T_1\le T_0)$ such that the local solutions of (1.5) exists in $\widetilde{\mathcal{C}}_{T_1}(B^{3/2}_{2,1})$ and $\|U\|_{\widetilde{L}^\infty_{T_1}(B^{3/2}_{2,1})}\le\delta_1$. Therefore from Proposition 4.2 the solutions satisfy the a priori estimate $\|U\|_{\widetilde{L}^\infty_{T_1}(B^{3/2}_{2,1})}\le C_1\|U_0\|_{B^{3/2}_{2,1}}\le\frac{\delta_1}{2}$ provided $\|U_0\|_{B^\sigma_{2,1}}\le\frac{\delta_1}{2C_1}$. Thus by Proposition 4.1 the system (1.5) for $t\ge T_1$ with the initial data $U(T_1)$ has again a unique solution U satisfying $\|U\|_{\widetilde{L}^\infty_{(T_1,2T_1)}(B^{3/2}_{2,1})}\le\delta_1$, further $\|U\|_{\widetilde{L}^\infty_{2T_1}(B^{3/2}_{2,1})}\le\delta_1$. Then by Proposition 4.2 we have $\|U\|_{\widetilde{L}^\infty_{2T_1}(B^{3/2}_{2,1})}\le C_1\|U_0\|_{B^{3/2}_{2,1}}\le\frac{\delta_1}{2}$. Subsequently, we continuous the same process for $0\le t\le nT_1, n=3,4,\ldots$ and finally get a global solution $U\in\widetilde{\mathcal{C}}(B^\sigma_{2,1})$ satisfying

$$\begin{aligned}
&\|U\|_{\widetilde{L}^\infty(B^{3/2}_{2,1})}+\Big(\|y\|_{\widetilde{L}^2_T(B^{3/2}_{2,1})}+\|(v,z_x)\|_{\widetilde{L}^2_T(B^{1/2}_{2,1})}+\|u_x\|_{\widetilde{L}^2_T(B^{-1/2}_{2,1})}\Big)\\
\le\ & C_1\|U_0\|_{B^{3/2}_{2,1}}\le\frac{\delta_1}{2}.
\end{aligned} \tag{4.50}$$

5 Optimal decay rates

In this section, with the aid of the new frequency-localization time-decay inequality in Proposition 4.1, we obtain the the optimal decay estimates by using the time-weighted energy approach in terms of high-frequency andlow-frequency decomposition.

Theorem 5.1. *Let $a = 1$ or $a \neq 1$ and $U(t,x) = (v,u,z,y)(t,x)$ be the global classical solution of Theorem 4.1. Assume that the initial data satisfy $U_0 \in B^{3/2}_{2,1}(\mathbb{R}) \cap \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$. Set $I_0 := \|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})\cap\dot{B}^{-1/2}_{2,\infty}(\mathbb{R})}$. If I_0 is sufficiently small, then the classical solution $U(t,x)$ of (1.5) admits the optimal decay estimate*

$$\|U\|_{L^2} \lesssim I_0(1+t)^{-\frac{1}{4}}. \tag{5.1}$$

Note that the embedding $L^1(\mathbb{R}) \hookrightarrow \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ in Lemma 3.3, as an immediate byproduct of Theorem 5.1, the usual optimal decay estimate of $L^1(\mathbb{R})$-$L^2(\mathbb{R})$ type is available.

Corollary 5.1. *Let $a = 1$ or $a \neq 1$ and $U(t,x) = (v,u,z,y)(t,x)$ be the global classical solutions of Theorem 4.1. If further the initial data $U_0 \in L^1(\mathbb{R})$ and $\widetilde{I}_0 := \|U_0\|_{B^{3/2}_{2,1}(\mathbb{R})\cap L^1(\mathbb{R})}$ is sufficiently small, then*

$$\|U\|_{L^2} \lesssim \widetilde{I}_0(1+t)^{-\frac{1}{4}}. \tag{5.2}$$

Remark 5.1. *Let us mention that Theorem 5.1 and Corollary 5.1 exhibit the optimal decay rate in the Besov space with $s_c = 3/2$, that is, $s_D = 3/2$, which implies that the minimal decay regularity coincides with the the critical regularity for global solutions, and the extra higher regularity is not necessary. In addition, it is worth noting that the present work opens a door for the study of dissipative systems of regularity-loss type, which encourages us to develop frequency-localization time-decay inequalities for other dissipative rates and investigate systems with the regularity-loss mechanism.*

Due to the better dissipative structure in the case of $a = 1$ (see [18]), we performed the Littlewood-Paley pointwise estimates for the linearized problem (2.6) and develop decay properties in the framework of Besov spaces. Furthermore, with the help of the frequency-localization Duhamel principle, the optimal decay estimates of (1.5) are shown by localized time-weighted energy approaches. For the case of $a \neq 1$, if the standard Duhamel principle is used, we need to deal with the weak mechanism of regularity-loss in the price of extra higher regularity, so it is impossible to achieve $s_D = 3/2$. Hence, we involve new observations. Actually, we perform "the square formula of the Duhamel principle" based on the Littlewood-Paley pointwise estimate in Fourier space for the linear system with right-hand side, see (5.5)-(5.6). Furthermore, we proceed the optimal decay estimate for (1.5) in terms of high-frequency and low-frequency decompositions, with the aid of the frequency-localization time-decay inequality first developed in [33].

To do this, we define the following energy functionals:

$$\mathcal{N}(t) = \sup_{0\leq\tau\leq t}(1+\tau)^{\frac{1}{4}}\|U(\tau)\|_{L^2}, \quad \mathcal{D}(t) = \|z_x(\tau)\|_{L^2_t(\dot{B}^{1/2}_{2,1})}.$$

The optimal decay estimate lies in a nonlinear time-weighted energy inequality, which is include in the following

Lemma 5.1. *Let $U = (v, u, z, y)^\top$ be the global classical solutions in Theorem 4.1. Additionally, if $U_0 \in \dot{B}^{-1/2}_{2,\infty}$, then it holds that*

$$\mathcal{N}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}} + \mathcal{N}(t)\mathcal{D}(t) + \mathcal{N}(t)^2. \tag{5.3}$$

Proof. As in [17], perform the energy method in Fourier spaces to get

$$\frac{d}{dt} E[\hat{U}] + c_3 \eta_1(\xi) |\hat{U}|^2 \lesssim \xi^2 |\hat{g}|^2, \tag{5.4}$$

with $\eta_1(\xi) = \frac{\xi^2}{(1+\xi^2)^2}$, where $E[\hat{U}] \approx |\hat{U}|^2$. As a matter of fact, following from the derivation of (5.4), we can obtain the corresponding Littlewood-Paley pointwise energy inequality

$$\frac{d}{dt} E[\widehat{\dot{\Delta}_q U}] + c_3 \eta_1 |\widehat{\dot{\Delta}_q U}|^2 \lesssim \xi^2 |\widehat{\dot{\Delta}_q g}|^2, \tag{5.5}$$

where $E[\widehat{\dot{\Delta}_q U}] \approx |\widehat{\dot{\Delta}_q U}|^2$. Gronwall's inequality implies that

$$|\widehat{\dot{\Delta}_q U}|^2 \lesssim e^{-c_3 \eta_1 t} |\widehat{\dot{\Delta}_q U_0}|^2 + \int_0^t e^{-c_3 \eta_1 (t-\tau)} \xi^2 |\widehat{\dot{\Delta}_q g}|^2 d\tau. \tag{5.6}$$

It follows from Fubini and Plancherel theorems that

$$\begin{aligned}
\|U\|^2_{L^2} &= \sum_{q \in \mathbb{Z}} \|\dot{\Delta}_q U\|^2_{L^2} \\
&\lesssim \sum_{q \in \mathbb{Z}} \|\widehat{\dot{\Delta}_q U_0} e^{-\frac{1}{2} c_3 \eta_1(\xi) t}\|^2_{L^2} \\
&\quad + \int_0^t \sum_{q \in \mathbb{Z}} \||\xi| \widehat{\dot{\Delta}_q g} e^{-\frac{1}{2} c_3 \eta_1(\xi)(t-\tau)}\|^2_{L^2} d\tau \\
&\triangleq I_1 + I_2.
\end{aligned} \tag{5.7}$$

For I_1, by taking $p = r = 2, \sigma = 0, s = 1/2$ and $\ell = 1$ in Proposition 4.1, we arrive at

$$\begin{aligned}
I_1 &= \Big(\sum_{q<0} + \sum_{q \geq 0}\Big)(\cdots) \\
&\lesssim \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty}} (1+t)^{-\frac{1}{2}} + \sum_{q \geq 0} 2^{2q} \|\dot{\Delta}_q U_0\|^2_{L^2} (1+t)^{-1} \\
&\lesssim \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty}} (1+t)^{-\frac{1}{2}} + \|U_0\|^2_{\dot{B}^1_{2,2}} (1+t)^{-1} \\
&\lesssim \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty} \cap B^{3/2}_{2,1}} (1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{5.8}$$

Next, we begin to bound the nonlinear term on the right-hand side of (5.7), which is written as the sum of low-frequency and high-frequency

$$I_2 = \int_0^t \Big(\sum_{q<0} + \sum_{q \geq 0}\Big)(\cdots) \triangleq I_{2L} + I_{2H}. \tag{5.9}$$

For I_{2L}, by taking $r = 2, \sigma = 1$ and $s = 1/2$ in Proposition 4.1, we have

$$\begin{aligned}
I_{2L} &\leq \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|g(z)\|^2_{\dot{B}^{-1/2}_{2,\infty}} d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|g(z)\|^2_{L^1} d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-\frac{3}{2}} \|z(\tau)\|^4_{L^2} d\tau \\
&\lesssim \mathcal{N}^4(t) \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+t)^{-1} d\tau \\
&\lesssim \mathcal{N}^4(t)(1+t)^{-1},
\end{aligned} \tag{5.10}$$

where we used the embedding $L^1(\mathbb{R}) \hookrightarrow \dot{B}^{-1/2}_{2,\infty}(\mathbb{R})$ in Lemma 3.3 and the fact $g(z) = O(z^2)$. For the high-frequency part I_{2H}, more elaborate estimates are needed. For the purpose, we write

$$I_{2H} = \Big(\int_0^{t/2} + \int_{t/2}^t\Big)\Big(\cdots\Big) \triangleq I_{2H1} + I_{2H2}.$$

For I_{2H1}, taking $p = r = 2, \sigma = 1$ and $\ell = 1/2$ in Proposition 4.1 gives

$$\begin{aligned}
I_{2H1} &= \int_0^{t/2} \sum_{q \geq 0} 2^{3q} \|\dot{\Delta}_q g(z)\|^2_{L^2} (1+t-\tau)^{-\frac{1}{2}} d\tau \\
&\leq \int_0^{t/2} (1+t-\tau)^{-\frac{1}{2}} \|g(z)\|^2_{\dot{B}^{3/2}_{2,2}} d\tau.
\end{aligned} \tag{5.11}$$

On the other hand, recalling $g(z) = O(z^2)$, Proposition 3.1 and Lemmas 3.1-3.2 enable us to get

$$\|g(z)\|_{\dot{B}^{3/2}_{2,2}} \lesssim \|g(z)\|_{\dot{B}^{3/2}_{2,1}} \lesssim \|z\|_{L^\infty} \|z_x\|_{\dot{B}^{1/2}_{2,1}}. \tag{5.12}$$

Combine (5.11) and (5.12) to arrive at

$$\begin{aligned}
I_{2H1} &\lesssim \int_0^{t/2} (1+t-\tau)^{-\frac{1}{2}} \|z(\tau)\|^2_{L^\infty} \|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}} d\tau \\
&\lesssim \sup_{0 \leq \tau \leq t/2} \Big\{(1+t-\tau)^{-\frac{1}{2}} \|z(\tau)\|^2_{L^\infty}\Big\} \int_0^{t/2} \|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}} d\tau \\
&\lesssim (1+t)^{-\frac{1}{2}} \|U_0\|^2_{B^{3/2}_{2,1}} \mathcal{D}^2(t) \\
&\lesssim (1+t)^{-\frac{1}{2}} \|U_0\|^2_{B^{3/2}_{2,1}}.
\end{aligned} \tag{5.13}$$

For the last step of (5.13), we would like to explain a little. It follows from Proposition 3.3 that

$$\mathcal{D}(t) \lesssim \|z_x\|_{\widetilde{L}^2_t(\dot{B}^{1/2}_{2,1})} \lesssim \|z_x\|_{\widetilde{L}^2_t(B^{1/2}_{2,1})} \lesssim \|U_0\|_{B^{3/2}_{2,1}}, \tag{5.14}$$

where we used the energy inequality (4.11) in Theorem 4.1. By choosing $r = 2, p = \sigma = 1$ and $\ell = 1/2$ in Proposition 4.1, I_{2H2} is proceeded as

$$\begin{aligned} I_{2H2} &= \int_{t/2}^{t} \sum_{q \geq 0} 2^{3q} \|\dot{\Delta}_q g(z)\|_{L^1}^2 d\tau \\ &\leq \int_{t/2}^{t} \|g(z)\|^2_{\dot{B}^{3/2}_{1,2}} d\tau. \end{aligned} \tag{5.15}$$

Thanks to $g(z) = O(z^2)$, it follows from Proposition 3.2 that

$$\|g(z)\|_{\dot{B}^{3/2}_{1,2}} \leq \|g(z)\|_{\dot{B}^{3/2}_{1,1}} \lesssim \|z\|_{L^2} \|z_x\|_{\dot{B}^{1/2}_{2,1}}. \tag{5.16}$$

Together with (5.15)-(5.16), we are led to

$$\begin{aligned} I_{2H2} &\lesssim \mathcal{N}^2(t) \int_{t/2}^{t} (1+\tau)^{-\frac{1}{2}} \|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}} d\tau \\ &\lesssim \mathcal{N}^2(t) \sup_{t/2 \leq \tau \leq t} (1+\tau)^{-\frac{1}{2}} \int_{t/2}^{t} \|z_x(\tau)\|^2_{\dot{B}^{1/2}_{2,1}} d\tau \\ &\lesssim (1+t)^{-\frac{1}{2}} \mathcal{N}^2(t) \mathcal{D}^2(t). \end{aligned} \tag{5.17}$$

Combine (5.13) and (5.17) to get

$$I_{2H} \lesssim (1+t)^{-\frac{1}{2}} \|U_0\|^2_{B^{3/2}_{2,1}} + (1+t)^{-\frac{1}{2}} \mathcal{N}^2(t) \mathcal{D}^2(t). \tag{5.18}$$

Therefore, it follows from (5.10) and (5.18) that

$$\begin{aligned} I_2 &\lesssim (1+t)^{-1} \mathcal{N}^4(t) + (1+t)^{-\frac{1}{2}} \|U_0\|^2_{B^{3/2}_{2,1}} \\ &\quad + (1+t)^{-\frac{1}{2}} \mathcal{N}^2(t) \mathcal{D}^2(t). \end{aligned} \tag{5.19}$$

Finally, noticing (5.7)-(5.8) and (5.19), we conclude that

$$\begin{aligned} \|U\|^2_{L^2} &\lesssim (1+t)^{-\frac{1}{2}} \|U_0\|^2_{\dot{B}^{-1/2}_{2,\infty} \cap B^{3/2}_{2,1}} + (1+t)^{-\frac{1}{2}} \mathcal{N}^2(t) \mathcal{D}^2(t) \\ &\quad + (1+t)^{-1} \mathcal{N}^4(t) \end{aligned} \tag{5.20}$$

which leads to (5.3) directly. □

Proof of Theorem 5.1. Note that (5.14), we arrive at

$$\mathcal{D}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1}} \lesssim \|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}}. \tag{5.21}$$

Thus, if the norm $\|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}}$ is sufficiently small, then we have

$$\mathcal{N}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}} + \mathcal{N}(t)^2 \tag{5.22}$$

which implies that $\mathcal{N}(t) \lesssim \|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}}$, provided that $\|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}}$ is sufficiently small. Consequently, the desired decay estimate in Theorem 5.1 follows

$$\|U\|_{L^2} \lesssim \|U_0\|_{B^{3/2}_{2,1} \cap \dot{B}^{-1/2}_{2,\infty}} (1+t)^{-\frac{1}{4}}. \tag{5.23}$$

Hence, the proof of Theorem 5.1 is complete eventually. □

References

[1] H. Abidi, Equation de Navier-Stokes avec densité et viscosité variables dans l'espace critique,*Revista Matemática Iberoamericana*, **23** (2007) 537–586.

[2] H. Abidi, G. L. Gui and P. Zhang, On the wellposedness of three-dimensional inhomogeneous Navier-Stokes equations in the critical spaces, *Arch. Ration. Mech. Anal.*, **204** (2012) 189–230.

[3] F. Ammar Khodja, A. Benabdallah, J. E. Muñoz Rivera and R. Racke, Energy decay for Timoshenko systems of memory type, *J. Differential Equations*, **194** (2003) 82–115.

[4] M. S. Alves, C. A Raposo, J. E. Muñoz Rivera, M. Sepulveda and O. Vera Villagran, Uniform stabilization for the transmission problem of the Timoshenko system with memory, *J. Math. Anal. Appl.*, **369** (2010) 323–345.

[5] H. Bahouri, J. Y. Chemin and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der Mathematischen Wissenschaften, Berlin: Springer-Verlag, 2011.

[6] J.-Y. Chemin and N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, *J. Differential Equations*, **121** (1995) 314–328.

[7] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, **141** (2000) 579–614.

[8] R. Danchin, On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework, *J. Differential Equations*, **248** (2010) 2130–2170.

[9] H. D. Fernández Sare and R. Racke, On the stability of damped Timoshenko systems: Cattaneo vs. Fourier law, *Arch. Rational Mech. Anal.*, **194** (2009) 221–251.

[10] B. Haspot, Existence of global strong solutions in critical spaces for barotropic viscous fluids, *Arch. Ration. Mech. Anal.*, **202** (2011) 427–460.

[11] K. Ide, K. Haramoto and S. Kawashima, Decay property of regularity-loss type for dissipative Timoshenko system, *Math. Models Meth. Appl. Sci.*, **18** (2008) 647–667.

[12] K. Ide and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system, *Math. Models Meth. Appl. Sci.*, **18** (2008) 1001–1025.

[13] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.*, **58** (1975) 181–205.

[14] Y. Liu and S. Kawashima, Decay property for the Timoshenko system with memory type dissipation, *Math. Models Methods. Appl. Sci.*, **22** (2012) 1–19.

[15] Z. Liu and C. Peng, Exponential stability of a viscoelastic Timoshenko beam, *Adv. Math. Sci. Appl.*, **8** (1998) 343–351.

[16] A. Majda, *Compressible Fluid Flow and Conservation laws in Several Space Variables*, Berlin/New York, Springer-Verlag: 1984.

[17] N. Mori and S. Kawashima, Decay property for the Timoshenko system with Fourier's type heat conduction, *J. Hyper. Differ. Equs.*, **11** (2014) 135–157.

[18] N. Mori, J. Xu and S. Kawashima, Global existence and optimal decay rates for the Timoshenko system: The case of equal wave speeds, *J. Differential Equations*, **258** (2015) 1494–1518.

[19] M. Paicu and P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier-Stokes system, *J. Funct. Anal.*, **262** (2012) 3556–3584.

[20] C. A. Raposo, W. D. Bastos and M. L. Santos, A transmission problem for Timoshenko system, *Comput. Appl. Math.*, **26** (2007) 215–234.

[21] C. A. Raposo, J. Ferreira, M. L. Santos and N. N. O. Castro, Exponential stability for the Timoshenko system with two weak damping, *Appl. Math. Lett.*, **18** (2005) 535–541.

[22] J. E. Muñoz Rivera and R. Racke, Mildy dissipative nonlinear Timoshenko systems: Global existence and exponential stability, *J. Math. Anal. Appl.*, **276** (2002) 248–278.

[23] J. E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, *Discrete Contin. Dyn. Syst.*, **9** (2003) 1625–1639.

[24] R. Racke and B. Said-Houari, Decay rates and global existence for semilinear dissipatie Timoshenko systems, *Quart. Appl. Math.*, **71** (2013) 229–266.

[25] M. L. Santos, D. S. Almeida Júnior and J. E. Muñoz Rivera, The stability number of the Timoshenko system with second sound, *J. Differential Equations*, **253** (2012) 2715–2733.

[26] B. Said-Houari and A. Kasimov, Decay property of Timoshenko system in thermoelasticity, *Math. Methods Appl. Sci.*, **35** (2012) 314–333.

[27] A. Soufyane, Stabilisation de la poutre de Timoshenko, *C. R. Acad. Sci. Paris Sér. I Math.*, **328** (1999) 731–734.

[28] S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philos. Mag.*, **41** (1921) 744–746.

[29] S. P. Timoshenko, On the transverse vibrations of bars of uniform cross-section, *Philos. Mag.*, **43** (1922) 125–131.

[30] J. H. Wu, Lower bounds for an integral involving fractional Laplacians and the generalized Navier-Stokes equations in Besov spaces, *Commun. Math. Phys.*, **263** (2005) 803–831.

[31] J. Xu and S. Kawashima, Global classical solutions for partially dissipative hyperbolic system of balance laws, *Arch. Rational Mech. Anal.*, **211** (2014) 513–553.

[32] J. Xu and S. Kawashima, The optimal decay estimates on the framework of Besov spaces for generally dissipative systems, Preprint (2014).

[33] J.Xu, N. Mori and S.Kawashima, L^p-L^q-L^r and minimal decay regularity for compressible Euler-Maxwell equations, J. Math. Pures Appl. (2015).

[34] J. Xu, N. Mori and S. Kawashima, Global existence and minimal decay regularity for the Timoshenko system: The case of non-equal wave speeds, J. Differential Equations, **259** (2015) 5533– 5553.

[35] J. Xu and Z. J. Wang, Relaxation limit in Besov spaces for compressible Euler equations, *J. Math. Pures Appl.*, **99** (2013) 43–61.

[36] J.Xu and Wen-A.Yong, Relaxation-time limits of non-isentropic hydrodynamic models for semiconductors, *J. Differential Equations*, **247** (2009) 1777–1795.

[37] Y.Zhou, Local well-posedness for the incompressible Euler equations in the critical Besov spaces, *Ann. Inst. Fourier*, **54** (2004) 773–786.

Naofumi Mori
Graduate School of Mathematics, Kyushu University
Fukuoka 819-0395, Japan
n-mori@math.kyushu-u.ac.jp

Jiang Xu
Department of Mathematics, Nanjing University of Aeronautics and Astronautics
Nanjing 211106, P.R.China
jiangxu_79@nuaa.edu.cn

Shuichi Kawashima
Faculty of Mathematics, Kyushu University
Fukuoka 819-0395, Japan
kawashim@math.kyushu-u.ac.jp

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